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Error estimation for the reproducing kernel method to solve linear boundary value problems

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ABSTRACT

In the previous works, the authors presented the reproducing kernel method(RKM) for solving various boundary value problems. However, an effective error estimation for this method has not yet been discussed. The aim of this paper is to fill this gap. In this paper, we shall give the error estimation for the reproducing kernel method to solve linear boundary value problems.

1. Introduction

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, learning theory and so on. Boundary value problems have been investigated in many application areas. However, these problems are difficult to solve analytically. Recently, reproducing kernel methods for solving a variety of boundary value problems were presented by Cui and Geng [1–6], Lin and Zhou [7,8], Yao, Chen and Jiang [9,10], Wang, Li and Wu [11,12], Mohammadi and Mokhtari [13], Akram and Ur Rehman [14].

In this paper, we consider the error estimation for the reproducing kernel method applied to the following second order two-point boundary value problems:

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = f(x), & 0 < x < 1, \\ u(0) = 0, \qquad u(1) = 0, \end{cases} \tag{1.1}$$

where $p(x), q(x)$ are continuous and f is given such that (1.1) satisfies the existence and uniqueness of the solutions. There is no loss of generality in considering only homogeneous boundary conditions in (1.1) because it is always possible to reduce nonhomogeneous problems to the treated cases, by means of suitable transformations.

The rest of the paper is organized as follows. In the next section, the reproducing kernel method for solving (1.1) is introduced. The error estimation is presented in Section 3. Numerical examples are provided in Section 4. Section 5 ends this paper with a brief conclusion.

2. Reproducing kernel method for (1.1)

In this section, we introduce the RKM for solving linear two-point boundary value problems.

To solve (1.1), first, we construct reproducing kernel spaces $W^m[0, 1]$, $(m \geq 3)$ in which every function satisfies the boundary conditions of (1.1).

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Definition 2.1. $W^m[0, 1] = \{u(x) \mid u^{(m-1)}(x)$ is an absolutely continuous real value function, $u^{(m)}(x) \in L^2[0, 1]$, $u(0) = 0$, $u(1) = 0\}$. The inner product and norm in $W^m[0, 1]$ are given respectively by

$$(u, v)_m = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)dx$$

and

$$\|u\|_m = \sqrt{(u, u)_m}, \quad u, v \in W^m[0, 1].$$

By [1,6], $W^m[0, 1]$ is a reproducing kernel space and its reproducing kernel $k(x, y)$ can be obtained.

In [6], Cui and Lin defined a reproducing kernel space $W^1[0, 1]$ and gave its reproducing kernel

$$\overline{k}(x, y) = \begin{cases} 1 + y, & y \le x, \\ 1 + x, & y > x. \end{cases}$$

In (1.1), put $Lu(x) = u''(x) + p(x)u'(x) + q(x)u(x)$, it is clear that $L : W^m[0, 1] \to W^1[0, 1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{k}(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $\overline{k}(x_i, x)$ is the reproducing kernel of $W^1[0, 1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^m[0, 1]$ can be derived from the Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots). \tag{2.1}$$

According to [1,6], we have the following theorem:

Theorem 2.1. *For (1.1), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^m[0, 1]$ and $\psi_i(x) = L_s k(x, s)|_{s=x_i}$.*

Theorem 2.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$ and the solution of (1.1) is unique, then the solution of (1.1) is*

$$u(x) = \sum_{j=1}^{\infty} A_j\overline{\psi}_j(x), \tag{2.2}$$

where $A_j = \sum_{l=1}^{j} \beta_{jl} f(x_l)$.

Now, the approximate solution $u(x)$ can be obtained by taking finitely many terms in the series representation of $u(x)$ and

$$u_N(x) = \sum_{j=1}^{N} A_j\overline{\psi}_j(x). \tag{2.3}$$

Remark. Since $W^m[0, 1]$ is a Hilbert space, it is clear that $\sum_{i=1}^{\infty}(\sum_{k=1}^{i} \beta_{ik} f(x_k))^2 < \infty$. Therefore, the sequence u_N is convergent in the sense of norm $\| \cdot \|_m$.

Lemma 2.1. *If $u(x) \in W^m[0, 1]$, then there exists a constant c such that $|u(x)| \le c\|u(x)\|_m$, $|u^{(k)}(x)| \le c\|u(x)\|_m$ $1 \le k \le m - 1$.*

Proof. Since

$$|u(x)| = |(u(y), k(x, y))_m| \le \|u(y)\|_m\|k(x, y)\|_m,$$

there exists a constant c_0 such that

$$|u(x)| \le c_0\|u\|_m.$$

Note that

$$\begin{aligned} |u^{(i)}(x)| &= \left| \left(u(y), \frac{\partial^i k(x, y)}{\partial x^i} \right)_m \right| \\ &\le \|u\|_4 \left\| \frac{\partial^i k(x, y)}{\partial x^i} \right\|_m \\ &\le c_i\|u\|_m, \quad (i = 0, 1, 2, \ldots, m - 1), \end{aligned}$$

where c_i are constants.

Putting $c = \max_{0 \le i \le m-1}\{c_i\}$ and the proof of the lemma is complete. $\square$

From above lemma, by convergence of $u_n(x)$ in the sense of norm, it is easy to obtain the following theorem.

Theorem 2.3. *The approximate solution $u_n(x)$ and its derivatives $u_n^{(k)}(x)$, $1 \le k \le m-1$ are all uniformly convergent.*

3. Error estimation

Theorem 3.1. *Let $u_N(x)$ be the approximate solution of (1.1) in space $W^4[0,1]$ and $u(x)$ be the exact solution of (1.1). If $0 = x_1 < x_2 < \cdots < x_N = 1$, and if $p(x), q(x), f(x) \in C^2[0,1]$, then*

$$\|u(x) - u_N(x)\|_\infty \le d_1\, h, \qquad \|u^{(k)}(x) - u_N^{(k)}(x)\|_\infty \le d_1\, h, \quad 1 \le k \le 2$$

where $\|u(x)\|_\infty = \max_{x\in[0,1]} |u(x)|$, d_1 is a constant, $h = \max_{1\le i\le N-1} |x_{i+1} - x_i|$.

Proof. Note here that

$$Lu_N(x) = \sum_{i=1}^{N} A_i L\overline{\psi}_i(x)$$

and

$$(Lu_N)(x_n) = \sum_{i=1}^{N} A_i (L\overline{\psi}_i, \varphi_n) = \sum_{i=1}^{N} A_i (\overline{\psi}_i, L^*\varphi_n) = \sum_{i=1}^{N} A_i (\overline{\psi}_i, \psi_n).$$

Therefore,

$$\sum_{j=1}^{n} \beta_{nj} (Lu_N)(x_j) = \sum_{i=1}^{N} A_i \left(\overline{\psi}_i, \sum_{j=1}^{n} \beta_{nj}\psi_j \right) = \sum_{i=1}^{N} A_i (\overline{\psi}_i, \overline{\psi}_n) = A_n. \tag{3.1}$$

If $n = 1$, then $(Lu_N)(x_1) = f(x_1)$.

If $n = 2$, then $\beta_{21}(Lu_N)(x_1) + \beta_{22}(Lu_N)(x_2) = \beta_{21} f(x_1) + \beta_{22} f(x_2)$.

It is clear that $(Lu_N)(x_2) = f(x_2)$.

Moreover, it is easy to see by induction that

$$(Lu_N)(x_j) = f(x_j), \quad j = 1, 2, \ldots, N. \tag{3.2}$$

Put $R_N(x) = f(x) - Lu_N(x)$. Obviously, $R_N(x) \in C^2[0,1]$ and $R_N(x_j) = 0$, $j = 1, 2, \ldots, N$. Suppose that $l(x)$ is a polynomial of degree $= 1$ that interpolates the function $R_N(x)$ at x_i, x_{i+1}. It is clear that $l(x) = 0$. Also, for $\forall\, x \in [x_i, x_{i+1}]$,

$$R_N(x) = R_N(x) - l(x) = \frac{R_N''(\xi_i)}{2!}(x - x_i)(x - x_{i+1}), \quad \xi_i \in [x_i, x_{i+1}]. \tag{3.3}$$

Hence, for $\forall\, x \in [x_i, x_{i+1}]$,

$$|R_N(x)| \le \frac{|R_N''(\xi_i)|}{8} h_i^2 = c_i h_i^2, \quad c_i = \frac{|R_N''(\xi_i)|}{8}, \; h_i = |x_{i+1} - x_i|.$$

Putting $c = \max_{1\le i\le N-1} c_i$ and $h = \max_{1\le i\le N-1} h_i$, we have

$$\|R_N(x)\|_\infty = \max_{x\in[0,1]} |R_N(x)| \le c\, h^2.$$

Obviously,

$$R_N^2(0) \le c^2\, h^4. \tag{3.4}$$

From (3.3), one obtains

$$\int_0^1 (R_N')^2 dx = \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} (R_N')^2 dx \le cc\, h^2, \tag{3.5}$$

where cc is a constant.

In view of (3.4) and (3.5), there exists a constant $\overline{c}$ such that

$$\begin{aligned} \|R_N(x)\|_1 &= \left[R_N^2(0) + \int_0^1 (R_N')^2 dx \right]^{\frac{1}{2}} \\ &\le \overline{c} h. \end{aligned}$$

Noting that

$$u(x) - u_N(x) = L^{-1}R_N(x),$$

there exists a constant d_1 such that

$$\|u(x) - u_N(x)\|_4 = \|L^{-1}R_N(x)\|_4 \le \|L^{-1}\| \, \|R_N(x)\|_1 \le d_1 h.$$

According to Lemma 2.1, it is easy to see that

$$\|u(x) - u_N(x)\|_\infty \le d_1 h, \qquad \|u^{(k)}(x) - u_N^{(k)}(x)\|_\infty \le d_1 h, \quad 1 \le k \le 2. \quad \square$$

Theorem 3.2. *Let $u_N(x)$ be the approximate solution of (1.1) in space $W^5[0,1]$ and $u(x)$ be the exact solution of (1.1). If $0 = x_1 < x_2 < \cdots < x_N = 1$, and if $p(x), q(x), f(x) \in C^4[0,1]$, then*

$$\|u(x) - u_N(x)\|_\infty \le d_2 h^3, \qquad \|u^{(k)}(x) - u_N^{(k)}(x)\|_\infty \le d_2 h^3, \quad 1 \le k \le 2$$

where d_2 is a constant, $h = \max_{1\le i\le N-1} |x_{i+1} - x_i|$.

Proof. From the proof of Theorem 3.1, we have

$$Lu_N(x_j) = f(x_j), \quad j = 1, 2, \ldots, N.$$

Put

$$R_N(x) = f(x) - Lu_N(x).$$

Obviously,

$$R_N(x) \in C^4[0,1], \qquad R_N(x_j) = 0, \quad j = 1, 2, \ldots, N.$$

On interval $[x_i, x_{i+1}]$, the application of Roll's theorem to $R_N(x)$ yields

$$R_N'(y_i) = 0, \quad y_i \in (x_i, x_{i+1}), \; i = 1, 2, \ldots, N-1.$$

On interval $[y_i, y_{i+1}]$, the application of Roll's theorem to $R_N'(x)$ yields

$$R_N''(z_i) = 0, \quad z_i \in (y_i, y_{i+1}), \; i = 1, 2, \ldots, N-2.$$

Putting

$$h = \max_{1\le i\le N-1}\{|x_{i+1} - x_i|\}, \qquad h_y = \max_{1\le i\le N-2}\{|y_{i+1} - y_i|\}, \qquad h_z = \max_{1\le i\le N-3}\{|z_{i+1} - z_i|\},$$

clearly,

$$h_y \le 2h, \qquad h_z \le 4h.$$

Suppose that $l_1(x)$ is a polynomial of degree $= 1$ that interpolates the function $R_N''(x)$ at z_1, z_2. It is clear that $l_1(x) = 0$. Also, for $\forall\, x \in [x_1, z_2]$, there exist $\eta_1 \in [x_1, z_2]$ and a constant b_1 such that

$$R_N''(x) = R_N''(x) - l_1(x) = \frac{R_N^{(4)}(\eta_1)}{2!}(x - z_1)(x - z_2) \le b_1 h^2.$$

In a similar way, there exist constants c_i, b_2 such that

$$R_N''(x) \le c_i h^2, \quad x \in [z_i, z_{i+1}], \; i = 2, 3, \ldots, N-3,$$

and

$$R_N''(x) \le b_2 h^2, \quad x \in [z_{N-2}, x_N].$$

Hence, there exist a constant d_2 such that

$$\|R_N''(x)\|_\infty \le d_2 h^2.$$

On interval $[x_i, x_{i+1}]$ $i = 1, 2, \ldots, N-1$, noting that

$$R_N'(x) = \int_{y_i}^{x} R_N''(s)ds,$$

there exist constants $\bar{a}_i$

$$|R_N'(x)| \le \|R_N''(x)\|_\infty |x - y_i| \le \bar{a}_i h^3.$$

It turns out that

$$\|R_N'(x)\|_\infty \le a_0\, h^3, \quad x \in [x_1, x_N] = [0, 1] \tag{3.6}$$

where a_0 is a constant.

In a similar way, there exists a constant a_1 such that

$$\|R_N(x)\|_\infty \le a_1\, h^4, \quad x \in [0, 1].$$

Obviously,

$$|R_N^2(0)| \le a_1^2\, h^8. \tag{3.7}$$

From (3.4), one obtains

$$\int_0^1 (R_N')^2 dx \le a_2\, h^6, \tag{3.8}$$

where a_2 is a constant.

From (3.7) and (3.8), there exists a constant a_3 such that

$$\begin{aligned}\|R_N(x)\|_1 &= \left[R_N^2(0) + \int_0^1 (R_N')^2 dx\right]^{\frac{1}{2}} \\ &\le a_3\, h^3.\end{aligned}$$

Noting that

$$u(x) - u_N(x) = L^{-1} R_N(x),$$

there exists a constant d_2 such that

$$\|u(x) - u_N(x)\|_4 = \|L^{-1} R_N(x)\|_4 \le \|L^{-1}\|\, \|R_N(x)\|_1 \le d_2\, h^3.$$

According to Lemma 2.1, it is easy to see that

$$\|u(x) - u_N(x)\|_\infty \le d_2\, h^3, \qquad \|u^{(k)}(x) - u_N^{(k)}(x)\|_\infty \le d_2\, h^3, \quad 1 \le k \le 2. \quad \square$$

Theorem 3.3. *Let $u_N(x)$ be the approximate solution of (1.1) in space $W^6[0, 1]$ and $u(x)$ be the exact solution of (1.1). If $0 = x_1 < x_2 < \cdots < x_N = 1$, and if $p(x), q(x), f(x) \in C^6[0, 1]$, then*

$$\|u(x) - u_N(x)\|_\infty \le d_2\, h^5, \qquad \|u^{(k)}(x) - u_N^{(k)}(x)\|_\infty \le d_3\, h^5, \quad 1 \le k \le 2$$

where d_3 is a constant, $h = \max_{1 \le i \le N-1} |x_{i+1} - x_i|$.

Proof. The proof of this theorem is similar to Theorem 3.2. $\square$

4. Numerical examples

Example 4.1. Consider the following linear two-point boundary value problem

$$\begin{cases} u''(x) + 200e^x u'(x) + 300 \sin(x) u(x) = f(x), & 0 < x < 1, \\ u(0) = 1, \qquad u(1) = \dfrac{\sqrt{3}}{2}, \end{cases}$$

where $f(x)$ is given such that the exact solution is $u(x) = \sinh(x)$.

Using the method presented in Section 2, taking $N = 11$, $x_i = 0.1(i-1)$, $i = 1, 2, \ldots, N$, the numerical results of $u_N(x)$ in different reproducing kernel spaces are shown in Figs. 1–3.

5. Conclusion

In this paper, we give firstly an error estimation for the reproducing kernel method applied to linear two point boundary value problems. The error estimation presented in this paper can be extended to more general linear boundary values with linear boundary conditions.

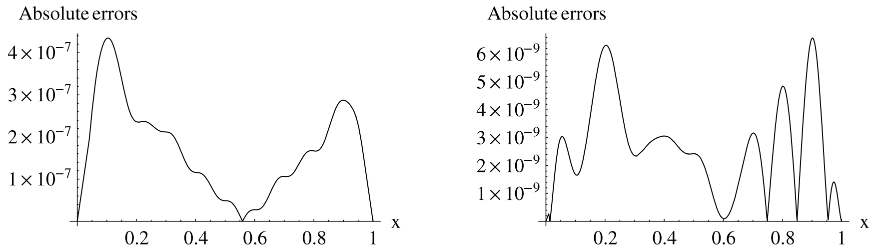

Fig. 1. Absolute errors of $u_{11}(x)$ in W^4 and W^5.

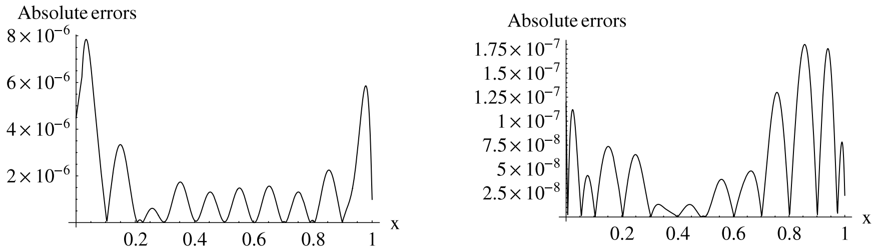

Fig. 2. Absolute errors of $u'_{11}(x)$ in W^4 and W^5.

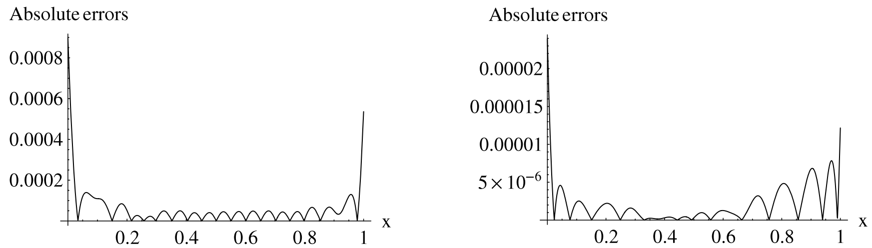

Fig. 3. Absolute errors of $u''_{11}(x)$ in W^4 and W^5.

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