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Numeric and mesh algorithms for the Coxeter spectral study of positive edge-bipartite graphs and their isotropy groups

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ABSTRACT

We develop algorithmic techniques for the Coxeter spectral analysis of the class $\mathcal{UB}igr_n$ of connected loop-free positive edge-bipartite graphs Δ with $n \geq 2$ vertices (i.e., signed graphs). In particular, we present numerical and graphical algorithms allowing us a computer search in the study of such graphs Δ by means of their Gram matrix $\check{G}_\Delta$, the (complex) spectrum $\mathbf{specc}_\Delta \subseteq \mathbb{C}$ of the Coxeter matrix $\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr}$, and the geometry of Weyl orbits in the set $\mathbf{Mor}_{D\Delta}$ of matrix morsifications $A \in \mathbb{M}_n(\mathbb{Z})$ of a simply laced Dynkin diagram $D\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ associated with Δ and mesh root systems of type $D\Delta$. Our algorithms construct the Coxeter–Gram polynomials $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$ and mesh geometries of root orbits of small connected loop-free positive edge-bipartite graphs Δ. We apply them to the study of the following Coxeter spectral analysis problem: *Does the $\mathbb{Z}$-congruence $\Delta \approx_\mathbb{Z} \Delta'$ hold (i.e., the matrices $\check{G}_\Delta$ and $\check{G}_{\Delta'}$ are $\mathbb{Z}$-congruent), for any pair of connected positive loop-free edge-bipartite graphs Δ, Δ' in $\mathcal{UB}igr_n$ such that $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$?* The problem if any square integer matrix $A \in \mathbb{M}_n(\mathbb{Z})$ is $\mathbb{Z}$-congruent with its transpose A^{tr} is also discussed. We present a solution for graphs in $\mathcal{UB}igr_n$, with $n \leq 6$.

1. Preliminaries and notation

One of the aims of the paper is to develop algorithmic techniques for the study of Coxeter spectral analysis problems formulated in [1–3] for loop-free edge-bipartite graphs $\Delta = (\Delta_0, \Delta_1)$, with $n \geq 2$ vertices. Here we keep the terminology and notation introduced in [3,4]. In particular, we denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ the rational, the real, and the complex number field, respectively. We view $\mathbb{Z}^n$, with $n \geq 1$, as a free abelian group, and we denote by $e_1, \ldots, e_n$ the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$. We denote by $\mathbb{M}_n(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square n by n matrices, by $E \in \mathbb{M}_n(\mathbb{Z})$ the identity matrix. We also use the right action $* : \mathbb{M}_n(\mathbb{Q}) \times \mathrm{Gl}(n, \mathbb{Z}) \longrightarrow \mathbb{M}_n(\mathbb{Q})$, $(A, C) \mapsto A * C := C^{tr} \cdot A \cdot C$ of the general $\mathbb{Z}$-linear group $\mathrm{Gl}(n, \mathbb{Z}) := \{A \in \mathbb{M}_n(n, \mathbb{Z}),\ \det A \in \{-1, 1\}\}$ on $\mathbb{M}_n(\mathbb{Q})$.

Following [3,4], by an *edge-bipartite graph* (bigraph, for short), we mean a pair $\Delta = (\Delta_0, \Delta_1)$, where Δ_0 is a finite non-empty set of vertices and Δ_1 is a finite set of edges equipped with a *bipartition* $\Delta_1 = \Delta_1^- \cup \Delta_1^+$ such that the set $\Delta_1(i, j) = \Delta_1^-(i, j) \cup \Delta_1^+(i, j)$ of edges connecting the vertices i and j does not contain edges lying in $\Delta_1^-(i, j) \cap \Delta_1^+(i, j)$, for each pair of vertices $i, j \in \Delta_0$, and either $\Delta_1(i, j) = \Delta_1^-(i, j)$ or $\Delta_1(i, j) = \Delta_1^+(i, j)$. Obviously, edge-bipartite graphs can be viewed as signed multi-graphs satisfying a separation property, see [3,5]. We call $\Delta = (\Delta_0, \Delta_1)$ *loop-free* if it has no loops, that is, $\Delta_1(j, j) = \emptyset$, for all $j \in \Delta_0$. We denote by $\mathcal{B}igr_n$ the category of finite edge-bipartite graphs, with $n \geq 2$ vertices, and the usual edge-bipartite graph maps as morphisms, see [3] for details. The full subcategory of $\mathcal{B}igr_n$ consisting of all loop-free graphs is denoted by $\mathcal{UB}igr_n$.

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We visualize Δ as a graph in a Euclidean space $\mathbb{R}^m$, $m \geq 2$, with the vertices $a_1, \ldots, a_n$ numbered by the integers $1, \ldots, n$; usually we simply write $\Delta_0 = \{1, \ldots, n\}$. An edge in $\Delta_1^-(a_i, a_j)$ is visualized as a continuous one a_i——— a_j, and an edge in $\Delta_1^+(a_i, a_j)$ is visualized as a dotted one a_i- - -a_j.

We view any finite graph $\Delta = (\Delta_0, \Delta_1)$ as an edge-bipartite one by setting $\Delta_1^-(a_i, a_j) = \Delta_1(a_i, a_j)$ and $\Delta_1^+(a_i, a_j) = \emptyset$, for each pair of vertices $a_i, a_j \in \Delta_0$. We study the loop-free edge-bipartite graphs $\Delta \in \mathcal{UB}igr_n$ by means of the *non-symmetric Gram matrix*

$$\check{G}_\Delta = [d_{ij}^\Delta] \in \mathbb{M}_n(\mathbb{Z}),$$

where $d_{ij}^\Delta = -|\Delta_1^-(a_i, a_j)|$, if there is an edge a_i——— a_j and $i \leq j$, $d_{ij}^\Delta = |\Delta_1^+(a_i, a_j)|$, if there is an edge a_i- - -a_j and $i \leq j$. We set $d_{ij}^\Delta = 0$, if $\Delta_1(a_i, a_j)$ is empty or $j < i$. The matrix $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr})$ is called the *symmetric Gram matrix*. We call $\Delta = (\Delta_0, \Delta_1)$ *positive* (resp. non-negative), if the rational symmetric Gram matrix $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr})$ of Δ is positive definite (resp. positive semi-definite). Two graphs $\Delta, \Delta' \in \mathcal{B}igr_n$ are defined to be $\mathbb{Z}$-*equivalent* (resp. $\mathbb{Z}$-*bilinear equivalent*) if there exists $B \in \mathrm{Gl}(n, \mathbb{Z})$ such that $G_{\Delta'} = B^{tr} \cdot G_\Delta \cdot B$ (resp. $\check{G}_{\Delta'} = B^{tr} \cdot \check{G}_\Delta \cdot B$). In this case, we write $\Delta \sim_\mathbb{Z} \Delta'$ (resp. $\Delta \approx_\mathbb{Z} \Delta'$) and we say that $B \in \mathrm{Gl}(n, \mathbb{Z})$ defines the $\mathbb{Z}$-equivalence $\Delta \sim_\mathbb{Z} \Delta'$ (resp. $\Delta \approx_\mathbb{Z} \Delta'$).

Following [3] (see also [6]), we associate with any edge-bipartite graph Δ in $\mathcal{UB}igr_n$, with $n \geq 2$, the *Coxeter spectrum* $\mathbf{specc}_\Delta \subseteq \mathbb{C}$, i.e., the spectrum of the *Coxeter(–Gram) matrix* and of the *Coxeter(–Gram) polynomial*

$$\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr} \in \mathbb{M}_n(\mathbb{Z}), \qquad \mathrm{cox}_\Delta(t) := \det(t \cdot E - \mathrm{Cox}_\Delta) \in \mathbb{Z}[t]. \tag{1.1}$$

The Coxeter transformation $\Phi_\Delta : \mathbb{Z}^n \to \mathbb{Z}^n$ of Δ is defined by $\Phi_\Delta(v) := v \cdot \mathrm{Cox}_\Delta$ and the Coxeter number $\mathbf{c}_\Delta$ of Δ is a minimal integer $c \geq 2$ such that Φ_Δ^c is the identity map on $\mathbb{Z}^n$. By the *integral quadratic form* of Δ we mean

$$q_\Delta(x) := b_\Delta(x, x) = x_1^2 + \cdots + x_n^2 + \sum_{i<j} d_{ij}^\Delta x_i x_j = x \cdot G_\Delta \cdot x^{tr} = x \cdot \check{G}_\Delta \cdot x^{tr}. \tag{1.2}$$

We recall from [3, Lemma 2.1] that the Coxeter spectrum $\mathbf{specc}_\Delta \subseteq \mathbb{C}$ lies on the unit circle $\mathcal{S}^1 := \{z \in \mathbb{C};\ |z| = 1\}$ and all points in $\mathbf{specc}_\Delta$ are roots of unity, if Δ is non-negative. If, in addition, Δ is positive then $1 \notin \mathbf{specc}_\Delta$ and the set $\mathcal{R}_\Delta := \{v \in \mathbb{Z}^n;\ v \cdot G_\Delta \cdot v^{tr} = 1\} \subseteq \mathbb{Z}^n$ of *roots of* Δ is finite.

One of our aims of the paper is to present an algorithmic technique for a computer search of the following Coxeter spectral analysis problems stated in [3] and discussed in [2–4,7].

Problem 1.3. Given $n \geq 2$, compute the set $\mathcal{CG}pol_n^+$ of all Coxeter(–Gram) polynomials $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$, with positive connected loop-free edge-bipartite graphs Δ in $\mathcal{UB}igr_n$.

Problem 1.4. Show that, given a pair of connected positive loop-free edge-bipartite graphs Δ and Δ' in $\mathcal{UB}igr_n$, the equality $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$ is equivalent to the existence of a $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ such that $\check{G}_{\Delta'} = B^{tr} \cdot \check{G}_\Delta \cdot B$. Construct an algorithm computing such a matrix $B \in \mathrm{Gl}(n, \mathbb{Z})$.

Problem 1.5. For any matrix $A \in \mathbb{M}_n(\mathbb{Z})$, with $\det A = 1$, find a matrix $C \in \mathrm{Gl}(n, \mathbb{Z})$ such that $A^{tr} = C^{tr} \cdot A \cdot C$ and $C^2 = E$, see [8].

Problem 1.6. Given a connected positive loop-free edge-bipartite graph Δ in $\mathcal{UB}igr_n$, construct a minimal Φ_Δ-mesh geometry of roots of Δ (that is, a Φ_Δ-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ satisfying the conditions of [2, Definition 1.11], see Section 2) such that, for any a pair of connected positive edge-bipartite graphs Δ and Δ' in $\mathcal{UB}igr_n$, the equality $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$ implies the existence of a group automorphism $\mathbb{Z}^n \cong \mathbb{Z}^n$ that restricts to a Φ_Δ-mesh translation quiver isomorphism $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta) \cong \Gamma(\widehat{\mathcal{R}}_{\Delta'}, \Phi_{\Delta'})$.

The main results of the paper are the following two theorems (proved in Section 3) that contain a partial solution of Problems 1.3–1.6 for edge-bipartite graphs Δ, Δ' in $\mathcal{UB}igr_n$, with $n \leq 6$.

Theorem 1.7. *Assume that Δ, Δ' are positive connected loop-free edge-bipartite graphs in $\mathcal{UB}igr_n$, with $2 \leq n \leq 6$ and $D\Delta, D\Delta'$ are the simply laced Dynkin diagrams associated in Theorem 2.1, with $\Delta \sim_\mathbb{Z} D\Delta$ and $\Delta' \sim_\mathbb{Z} D\Delta'$.*

(a) *$(\mathrm{cox}_\Delta(t), \mathbf{c}_\Delta)$ is one of the pairs $(F_{D\Delta}^{(j)}(t), \mathbf{c}_{D\Delta}^{(j)})$ listed in Table 1.8, see also [3, Figure 3].*
(b) *$\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$ if and only if $\Delta \approx_\mathbb{Z} \Delta'$.*
(c) *Given Δ, there exists a matrix $C \in \mathrm{Gl}(n, \mathbb{Z})$ such that $\check{G}_\Delta^{tr} = C^{tr} \cdot \check{G}_\Delta \cdot C$ and $C^2 = E$.*
(d) *Given Δ, there is a minimal Φ_Δ-mesh geometry of roots $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ of Δ satisfying the conditions listed in 1.6.*

The following theorem contains a complete classification of positive connected loop-free edge-bipartite graphs with at most six vertices, up to the congruence $\Delta \approx_\mathbb{Z} \Delta'$.

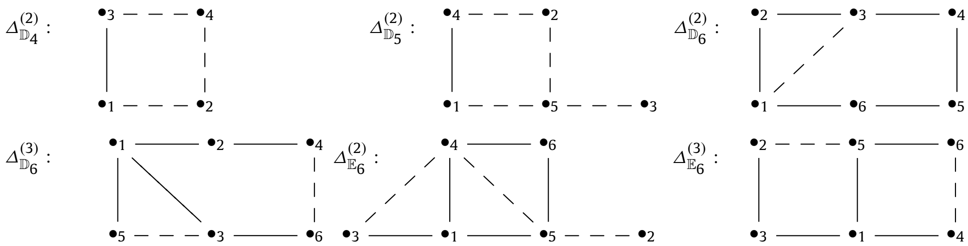
Theorem 1.9. *Assume that Δ is a positive connected loop-free edge-bipartite graph in $\mathcal{UB}igr_n$, with $2 \leq n \leq 6$. Under the notation in Theorem 1.7, we have*

(a) *If $\mathrm{cox}_\Delta(t) = F_{D\Delta}^{(1)}(t)$ then $\Delta \approx_\mathbb{Z} D\Delta$.*

Table 1.8
Coxeter polynomials $\mathrm{cox}_\Delta(t)$ and Coxeter numbers $\mathbf{c}_\Delta$ of positive connected bigraphs $\Delta \in \mathcal{UB}igr_n$, with $n \le 6$ and $D = D\Delta$.

$D = D\Delta$	$\mathcal{CP}ol_\Delta^+ = \{F_D^{(j)}(t)\}_{j \le \mathbf{s}_\Delta}$	$\mathbf{c}_\Delta^{(j)}$	$\#\Phi_\Delta$-orbits in $\mathcal{R}_\Delta$
$\mathbb{A}_n$	$F_{\mathbb{A}_n}^{(1)}(t) = t^n + t^{n-1} + \cdots + t + 1$	$n+1$	$no \times n+1$
$\mathbb{D}_4$	$F_{\mathbb{D}_4}^{(1)}(t) = t^4 + t^3 + t + 1$	6	$4o \times 6$
	$F_{\mathbb{D}_4}^{(2)}(t) = t^4 + 2t^2 + 1$	4	$6o \times 4$
$\mathbb{D}_5$	$F_{\mathbb{D}_5}^{(1)}(t) = t^5 + t^4 + t + 1$	8	$5o \times 8$
	$F_{\mathbb{D}_5}^{(2)}(t) = t^5 + t^3 + t^2 + 1$	12	$2o \times 12 + 2o \times 6 + 1o \times 4$
$\mathbb{D}_6$	$F_{\mathbb{D}_6}^{(1)}(t) = t^6 + t^5 + t + 1$	10	$6o \times 10$
	$F_{\mathbb{D}_6}^{(2)}(t) = t^6 + t^4 + t^2 + 1$	8	$7o \times 8 + 1o \times 4$
	$F_{\mathbb{D}_6}^{(3)}(t) = t^6 + 2t^3 + 1$	6	$10o \times 6$
$\mathbb{E}_6$	$F_{\mathbb{E}_6}^{(1)}(t) = t^6 + t^5 - t^3 + t + 1$	12	$6o \times 12$
	$F_{\mathbb{E}_6}^{(2)}(t) = t^6 + t^3 + 1$	9	$8o \times 9$
	$F_{\mathbb{E}_6}^{(3)}(t) = t^6 - t^5 + 2t^4 - t^3 + 2t^2 - t + 1$	6	$12o \times 6$

(b) *If* $\mathrm{cox}_\Delta(t) = F_{D\Delta}^{(j)}(t)$ *and* $j \ge 2$ *then* $D\Delta \in \{\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6\}$ *and* $\Delta \approx_\mathbb{Z} \Delta_{D\Delta}^{(j)}$, *where*



Our study is inspired on one hand by the spectral graph theory, a graph coloring technique and algebraic methods in graph theory (see [3,6,9–13]), and on the other hand by application in the representation theory of posets, finite groups, finite-dimensional algebras over a field, cluster algebras, Lie theory, and elementary Diophantine geometry, where various problems of extremal graph theory, discrete mathematics technique, numeric and symbolic algorithms, and computer algebra tools are successfully applied, see [1–7,9,10,12–37].

We recall that the main idea of the classical spectral analysis of signed graphs Δ (and edge-bipartite graphs) is to study them by means of the properties of the symmetric adjacency matrix Ad_Δ, its (real) spectrum $\mathbf{spec}_\Delta \subset \mathbb{R}$, and in terms of the Laplacian matrix of Δ. In particular, Laplacian spectral space and the signed graphs Δ with the Laplacian matrix positive semi-definite are studied in [31], where application to spectral machine learning methods, electrical networks, to link sign prediction in signed unipartite and bipartite networks, and community partition in social networks are discussed, see also [3,10,25,31,38].

Main results of the paper were presented at the International Conference on Applied and Computational Mathematics (ICACM) organized by the Institute of Applied Mathematics, Middle East Technical University, Ankara, October 3–6, 2012.

2. On mesh root systems and mesh geometries of root orbits

Our Coxeter spectral analysis of positive connected edge-bipartite graphs in $\mathcal{UB}igr_n$ essentially uses the inflation algorithm $\Delta \mapsto D\Delta$ defined in [11, Algorithm 5.9] and [3, Algorithm 3.1] that associates with any positive edge-bipartite graph in $\mathcal{UB}igr_n$ a simply laced Dynkin diagram $D\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ by means of two types of inflation operators $\Delta \mapsto \mathbf{t}_a^- \Delta$, $a \in \Delta_0$, and $\Delta \mapsto \mathbf{t}_{ab}^- \Delta$, with a — — — b in Δ^+. It successively reduces Δ to an edge-bipartite graph in $\mathcal{UB}igr_n$ having no dashed edges. The effect of the inflation algorithm is described in the following theorem proved in [3] (see also [11]).

Theorem 2.1. *Assume that Δ is a positive connected loop-free edge-bipartite graph in $\mathcal{UB}igr_n$, with $n \ge 2$. There exists a simply laced Dynkin diagram $D\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, called the Dynkin type of Δ (uniquely determined by Δ, up to permutation of vertices) and a sequence $\mathbf{t}_s, \mathbf{t}_{s-1}, \ldots, \mathbf{t}_1$ of inflations such that $D\Delta = \mathbf{t}_s \circ \mathbf{t}_{s-1} \circ \cdots \circ \mathbf{t}_1(\Delta)$ and $G_{D\Delta} = C^{tr} \cdot G_\Delta \cdot C = G_\Delta * C$, for some $C \in \mathrm{Gl}(n, \mathbb{Z})$.* □

It is pointed out in [3,18–20,35,39] that the study of Problems 1.3–1.6 is related to the classification of irreducible root systems in the sense of Bourbaki (see [36]). We recall that a (reduced) root system is a finite subset $\mathcal{R} \subset \mathbb{R}^n$ generating the Euclidean space $\mathbb{R}^n$, $n \ge 1$, satisfying:

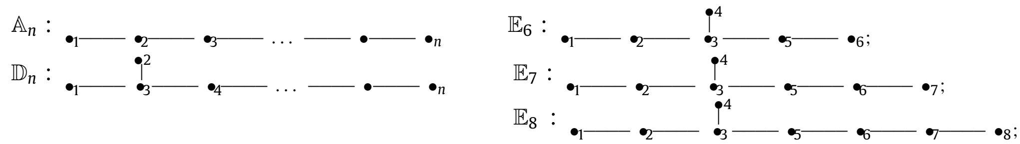
(i) $0 \notin \mathcal{R}$,

(ii) for every $v \in \mathcal{R}$ there exists an $\mathbb{R}$-linear map $v^{\vee} : \mathbb{R}^n \to \mathbb{R}$ such that $v^{\vee}(v) = 2$ and $\mathcal{R}$ is invariant under the reflection $\mathbf{s}_v(x) = x - v^{\vee}(x) \cdot v$, and

(iii) if $v, w \in \mathcal{R}$ then $\mathbf{s}_v(w) - w$ is an integer multiple of v.

With any such an irreducible root systems $\mathcal{R}$ a Coxeter–Dynkin $\mathbf{C}_{\mathcal{R}}$ diagram is associated.

It is observed in [3,20] that, given a positive connected graph Δ in $\mathcal{UB}igr_n$, its set of roots $\mathcal{R}_\Delta$ is finite and conditions (i)–(iii) are satisfied, where the $\mathbb{R}$-linear map $v^{\vee} : \mathbb{R}^n \to \mathbb{R}$ is defined by the formula $v^{\vee}(x) = 2 \cdot v \cdot G_\Delta \cdot x^{tr}$. Moreover, by Theorem 2.1, the inflation algorithm $\Delta \mapsto D\Delta$ [3, Algorithm 3.1] reduces Δ to a uniquely determined Dynkin graph $D\Delta \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ such that $\Delta \sim_{\mathbb{Z}} D\Delta$, that is, $G_{D\Delta} = G_\Delta * B$, for some $B \in \mathrm{Gl}(n, \mathbb{Z})$. It follows that the Coxeter–Dynkin diagram $\mathbf{C}_{\mathcal{R}}$ associated with the root system $\mathcal{R} = \mathcal{R}_\Delta$ equals $D\Delta$ and is one of the following simply laced Dynkin diagrams:



By [35, Proposition 4.1], the set $\mathcal{R}_\Delta = \{v \in \mathbb{Z}^n;\ q_\Delta(v) = 1\}$ of roots of the quadratic form $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$ (1.1) is finite. We frequently use the following well-known result (see [3,28,32]).

Lemma 2.2. *If Δ is a simply laced Dynkin diagram, viewed as an edge-bipartite graph and $\mathbf{c}_\Delta$ is the Coxeter number of Δ then* $\mathrm{cox}_\Delta(t) = F_\Delta(t)$, *where*

$$F_\Delta(t) := \begin{cases} t^n + t^{n-1} + \cdots + t^2 + t + 1, & \mathbf{c}_\Delta = n+1, \quad \textit{for } \Delta = \mathbb{A}_n, n \geq 1,\\ t^n + t^{n-1} + t + 1, & \mathbf{c}_\Delta = 2(n-1), \quad \textit{for } \Delta = \mathbb{D}_n, n \geq 4,\\ t^6 + t^5 - t^3 + t + 1, & \mathbf{c}_\Delta = 12, \quad \textit{for } \Delta = \mathbb{E}_6,\\ t^7 + t^6 - t^4 - t^3 + t + 1, & \mathbf{c}_\Delta = 18, \quad \textit{for } \Delta = \mathbb{E}_7,\\ t^8 + t^7 - t^5 - t^4 - t^3 + t + 1, & \mathbf{c}_\Delta = 30, \quad \textit{for } \Delta = \mathbb{E}_8. \quad \square \end{cases}$$

Let Δ be a connected positive loop-free edge-bipartite graph in $\mathcal{UB}igr_n$, $n \geq 2$ and let $\Phi_\Delta : \mathbb{Z}^n \to \mathbb{Z}^n$ be the Coxeter transformation of Δ. By [35, Proposition 4.1], $\mathcal{R}_\Delta$ is a Φ_Δ-invariant subset of $\mathbb{Z}^n$ and Φ_Δ restricts to the bijection $\Phi_\Delta : \mathcal{R}_\Delta \to \mathcal{R}_\Delta$. Then $\mathcal{R}_\Delta$ splits into finitely many Φ_Δ-orbits Φ_Δ-$\mathcal{O}(v)$, with $v \in \mathcal{R}_\Delta$, each of length at most $\mathbf{c}_\Delta$. We visualize Φ_Δ-$\mathcal{O}(v)$ as an infinite planar graph

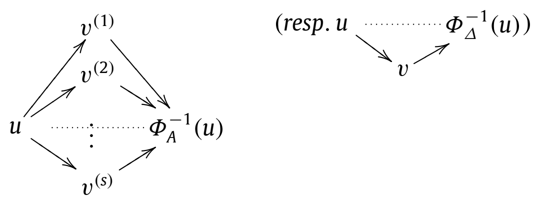
$$\Phi_\Delta\text{-}\mathcal{O}(v) : \cdots --- \Phi^3(v) --- \Phi^2(v) --- \Phi(v) --- v --- \Phi^{-1}(v) --- \Phi^{-2}(v) --- \Phi^{-3}(v) --- \quad \cdots$$

in the Euclidean plane $\mathbb{R}^2$. Since $\Phi_\Delta^j(v) = \Phi_\Delta^{j+\mathbf{c}_\Delta}(v)$, the orbit lies on a circle, if we make the identification $\Phi_\Delta^j(v) \equiv \Phi_\Delta^{j+\mathbf{c}_\Delta}(v)$, for any $j \in \mathbb{Z}$. It is shown in [1–3,35] that for an edge-bipartite graph Δ, with the Coxeter polynomial $\mathrm{cox}_\Delta(t) \in \mathbb{Z}[t]$ of a special form, the set of all Φ_Δ-orbits Φ_Δ-$\mathcal{O}(v)$, with $v \in \mathcal{R}_\Delta$, admits a geometrical structure $\Gamma(\mathcal{R}_\Delta, \Phi_\Delta)$ of a Φ_Δ-mesh translation quiver (digraph) in the sense of [35]. We recall from [35] that the vectors $u, v^{(1)}, \ldots, v^{(s)}, w \in \mathbb{Z}^n$ form a *Φ_Δ-mesh* starting from u and terminating at w, if the following two conditions are satisfied:

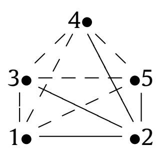
(a) $u = \Phi(w)$ and $u + w = v^{(1)} + \cdots + v^{(s)}$, and

(b) the vectors $v^{(1)}, \ldots, v^{(s)}$ are pairwise different and none of them lies in the Φ_Δ-orbit of u.

If each of the vectors $u, v^{(1)}, \ldots, v^{(s)}$ is non-zero, we say that the mesh is of *width* $s \geq 1$. If the Φ_Δ-mesh is of width 1 and lies in $\mathcal{R}_\Delta$, we say that Φ_Δ-$\mathcal{O}(v)$ is the *border Φ_Δ-orbit* in $\mathcal{R}_\Delta$, that is, the vector $v + \Phi_\Delta(v)$ is a root. We visualize the Φ_Δ-mesh (resp. the border Φ_Δ-mesh) as the following quivers in $\mathbb{R}^2$:



Example 2.3. Consider the following connected edge-bipartite graph Δ in $\mathcal{UB}igr_5$:



with $\check{G}_\Delta = \begin{bmatrix} 1 & -1 & 1 & 1 & 1\\ 0 & 1 & -1 & -1 & -1\\ 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $\mathrm{Cox}_\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0 & 1\\ 0 & -1 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.

One shows that Δ is positive, the Coxeter–Dynkin graph $D\Delta$ is the Dynkin diagram $\mathbb{A}_5$ (apply the inflation algorithm [3, Algorithm 3.1]), $\mathrm{cox}_\Delta(t) = t^5 + t^4 + t^3 + t^2 + t + 1 = \mathrm{cox}_{\mathbb{A}_5}(t)$, and $\mathbf{c}_\Delta = 6$. The set $\mathcal{R}_\Delta$ of roots of Δ has 30 vectors and splits into five Φ_Δ-orbits $\mathcal{O}^{(1)}, \mathcal{O}^{(2)}, \mathcal{O}^{(3)}, \mathcal{O}^{(4)}, \mathcal{O}^{(5)}$. It is easy to see that the orbits form the following Φ_Δ-mesh root system $\Gamma(\mathcal{R}_\Delta, \Phi_\Delta)$ (that is, Φ_Δ-mesh translation quiver) lying on a cylinder, with two border orbits:

$$\begin{array}{rccccccccccccccc}
\mathcal{O}^{(1)}: & & 001\hat{1}0 & \cdots & 0001\hat{1} & \cdots & 00001 & \cdots & \hat{1}0000 & \cdots & 11000 & \cdots & 0\hat{1}\hat{1}00 & \cdots & 001\hat{1}0 & \cdots \\
& \nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \\
\mathcal{O}^{(2)}: & 0\hat{1}0\hat{1}0 & \cdots & 0010\hat{1} & \cdots & 00010 & \cdots & \hat{1}0001 & \cdots & 01000 & \cdots & 10\hat{1}00 & \cdots & 0\hat{1}0\hat{1}0 & \cdots & \\
& \searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \\
\Gamma(\mathcal{R}_\Delta, \Phi_\Delta): \mathcal{O}^{(3)}: & & 0\hat{1}00\hat{1} & \cdots & 00100 & \cdots & \hat{1}0010 & \cdots & 01001 & \cdots & 00\hat{1}00 & \cdots & 100\hat{1}0 & \cdots & 0\hat{1}00\hat{1} & \cdots \\
& \nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \searrow\nearrow & & \\
\mathcal{O}^{(4)}: & 1000\hat{1} & \cdots & 0\hat{1}000 & \cdots & \hat{1}0100 & \cdots & 01010 & \cdots & 00\hat{1}01 & \cdots & 000\hat{1}0 & \cdots & 1000\hat{1} & \cdots & \\
& \searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \nearrow\searrow & & \\
\mathcal{O}^{(5)}: & & 10000 & \cdots & \hat{1}\hat{1}000 & \cdots & 01100 & \cdots & 00\hat{1}10 & \cdots & \boxed{000\hat{1}1} & \cdots & 0000\hat{1} & \cdots & 10000 & \cdots
\end{array}$$

where $\widehat{1} = -1$. One shows that $\Delta \approx_{\mathbb{Z}} \mathbb{A}_5$, that is, the non-symmetric Gram matrices $\check{G}_\Delta$ and $\check{G}_{\mathbb{A}_5}$ are $\mathbb{Z}$-congruent (as is claimed in Problem 1.6). The congruence is defined by the matrix $C = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, because $\check{G}_{\mathbb{A}_5} = \check{G}_\Delta * C^{tr} = C \cdot \check{G}_\Delta \cdot C^{tr}$.

The matrix C is constructed by applying the algorithmic procedure described in the following theorem that contains a partial solution of Problem 1.4. In Example 2.3, we apply Theorem 2.4 to the framed vector $v = [0, 0, 0, -1, 1]$.

Theorem 2.4. *Assume that Δ is a connected positive edge-bipartite graph in $\mathcal{UB}igr_n$, $2 \le n \le 6$, such that $\mathrm{cox}_\Delta(t) = \mathrm{cox}_{\mathbb{A}_n}(t) = t^n + t^{n-1} + \cdots + t + 1$ (i.e., $\mathbf{specc}_\Delta = \mathbf{specc}_{\mathbb{A}_n}$).*

(a) *The simply laced Dynkin graph $D\Delta$ of Δ equals $\mathbb{A}_n$, $\mathbf{c}_\Delta = n + 1$, $\mathcal{R}_\Delta$ splits into $n\Phi_\Delta$-orbits, each of them is of length $\mathbf{c}_\Delta = n + 1$, and there are exactly two different border Φ_Δ-orbits in $\mathcal{R}_\Delta$.*

(b) *$\Delta \approx_{\mathbb{Z}} D\Delta = \mathbb{A}_n$ and we have $\check{G}_{\mathbb{A}_n} = \check{G}_\Delta * C^{tr} = C \cdot \check{G}_\Delta \cdot C^{tr}$, where $C \in \mathrm{Gl}(n, \mathbb{Z})$ is the matrix with rows $v, \Phi_\Delta(v), \ldots, \Phi_\Delta^{n-1}(v)$ and $v \in \mathcal{R}_\Delta$ is a fixed root lying on a border Φ_Δ-orbit, that is, v is a root such that $v + \Phi_\Delta(v)$ is a root, too.*

(c) *The set of Φ_Δ-orbits in $\mathcal{R}_\Delta$ admits a unique structure of Φ_Δ-mesh root system $\Gamma(\mathcal{R}_\Delta, \Phi_\Delta)$, with two border Φ_Δ-orbits, such that every root v is a source (and a sink) of a unique Φ_Δ-mesh of width at most two. If $n = 5$, the Φ_Δ-mesh root system $\Gamma(\mathcal{R}_\Delta, \Phi_\Delta)$, viewed as a mesh translation quiver, is isomorphic with the mesh root system constructed in Example 2.3.*

The proof is outlined in Section 3. It essentially uses a reduction from a positive edge-bipartite graph in $\mathcal{UB}igr_n$ to the Coxeter spectral analysis of orbits of matrix morsifications under the action (3.1) of the $\mathrm{Gl}(n, \mathbb{Z})_D$ isotropy group, for the simply laced Dynkin diagrams D.

3. A matrix morsification technique

It is shown in [2] that a key role in the Coxeter spectral classification of positive edge-bipartite graphs Δ in $\mathcal{UB}igr_n$ is played by the right action

$$* : \mathbb{M}_n(\mathbb{Q}) \times \mathrm{Gl}(n, \mathbb{Z})_\Delta \longrightarrow \mathbb{M}_n(\mathbb{Q}), \qquad (A, B) \mapsto A * B := B^{tr} \cdot A \cdot B \tag{3.1}$$

of the isotropy group $\mathrm{Gl}(n, \mathbb{Z})_\Delta = \{B \in \mathrm{Gl}(n, \mathbb{Z});\ G_\Delta * B = G_\Delta\}$ of Δ. By [2, 2.12], the matrix Weyl group $\mathbb{W}_\Delta$ of Δ is a subgroup of $\mathrm{Gl}(n, \mathbb{Z})_\Delta$. Our main aim in this section is to show that the computation of the set $\mathcal{CG}pol_n^+$ of all polynomials $\mathrm{cox}_\Delta(t)$, with connected edge-bipartite graphs Δ in $\mathcal{UB}igr_n$, $n \ge 2$, and solutions of Problems 1.3–1.6 reduce to analogous problems for matrix morsifications $A, A' \in \mathbf{Mor}_{D\Delta}$, with a simply laced Dynkin diagram $D\Delta = D\Delta'$ associated with Δ and Δ' in Theorem 2.1. Using this idea, we construct algorithmic procedures for the Coxeter spectral analysis and we partially solve Problems 1.3–1.6, by applying symbolic computer algebra computations and numerical algorithmic computations in Linux, Maple and C++, with GNU Scientific Library. Here we apply the technique and results given in [2,3,18–22].

We recall that matrix morsifications of Dynkin diagrams are introduced in [35] and applied in [1] in the mesh geometric study of solutions of a class of Diophantine equations. Following [1,2,35], an integral (resp. rational) *matrix morsification* of an edge-bipartite graph Δ in $\mathcal{UB}igr_n$ is a non-singular matrix $A \in \mathbb{M}_n(\mathbb{Z})$ (resp. $A \in \mathbb{M}_n(\mathbb{Q})$) such that

(i) $A + A^{tr} = 2 \cdot G_\Delta$, and

(ii) the $\mathbb{Z}$-invertible *Coxeter matrix* $\mathrm{Cox}_A := -A \cdot A^{-tr}$ has integer coefficients.

We denote by $\mathbf{Mor}_\Delta \subseteq \widehat{\mathbf{Mor}}_\Delta$ the sets of all integral and all rational morsifications of Δ. Note that $\check{G}_\Delta$ lies in $\mathbf{Mor}_\Delta$, and G_Δ lies in $\widehat{\mathbf{Mor}}_\Delta$ in the case when Δ is positive. It is not clear (at the moment) if, given Δ in $\mathcal{UB}igr_n$, the sets $\mathbf{Mor}_\Delta \subseteq \widehat{\mathbf{Mor}}_\Delta$ and $\{\mathrm{cox}_A(t);\ A \in \mathbf{Mor}_\Delta\}$ are finite.

It is shown in [2] that $\det A \ge 1$, for any $A \in \mathbf{Mor}_\Delta$, if Δ is positive, and the map

$$* : \widehat{\mathbf{Mor}}_\Delta \times \mathrm{Gl}(n, \mathbb{Z})_\Delta \to \widehat{\mathbf{Mor}}_\Delta, \qquad (A, B) \mapsto A * B := B^{tr} \cdot A \cdot B, \tag{3.2}$$

is a right action of the isotropy group $\mathrm{Gl}(n,\mathbb{Z})_\Delta$ of Δ, and of the matrix Weyl group $\mathbb{W}_\Delta \subseteq \mathrm{Gl}(n,\mathbb{Z})_\Delta$. The subset $\mathbf{Mor}_\Delta \subseteq \widehat{\mathbf{Mor}}_\Delta$ is $\mathrm{Gl}(n,\mathbb{Z})_\Delta$-invariant. Moreover, $\det A \in \mathbb{Q}$, the *Coxeter number* $\mathbf{c}_A \geq 2$ (i.e., a minimal integer $r \geq 2$ such that $\mathrm{Cox}_A^r = E$), and the *Coxeter polynomial* $\mathrm{cox}_A(t) := \det(t\cdot E - \mathrm{Cox}_A) \in \mathbb{Z}[t]$ of A are $\mathrm{Gl}(n,\mathbb{Z})_\Delta$-invariant, see [1,2,35]. The *Coxeter spectrum* of A is the set $\mathbf{specc}_A$ of all n complex roots of the polynomial $\mathrm{cox}_A(t)$. The group automorphism $\Phi_A : \mathbb{Z}^n \to \mathbb{Z}^n$, $v \mapsto v\cdot \mathrm{Cox}_A$, is called the *Coxeter transformation* of A.

Example 3.3. The matrices

$$A = \begin{bmatrix} 1 & -1 & 1 & -3 \\ 1 & 1 & -1 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & -1 & -1 & 1 \end{bmatrix}, \quad \text{with } \det A = 4, \qquad A' = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & -1 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{bmatrix}, \quad \text{with } \det A' = \frac{4}{9},$$

and

$$A_5 = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & -1 & 1 \end{bmatrix}, \quad \det A_5 = 2,$$

are morsifications of the Dynkin diagrams $\mathbb{D}_4$ and $\mathbb{D}_5$, respectively, with

$$\mathrm{cox}_A(t) = t^4 - t^2 + 1, \qquad \mathbf{c}_A = 12, \qquad \mathrm{cox}_{A'}(t) = t^4 + 2t^3 + 3t^2 + 2t + 1, \qquad \mathbf{c}_{A'} = 3,$$
$$\mathrm{cox}_{A_5}(t) = t^5 + 1, \qquad \mathbf{c}_{A_5} = 10, \qquad \mathrm{cox}_{A'_5}(t) = t^5 + 3t^4 + 4t^3 + 4t^2 + 3t + 1, \qquad \mathbf{c}_{A'_5} = 4.$$

The following theorem (implicitly applied in the proof of [3, Theorem 3.3]) defines an important relation between the Coxeter spectral study of positive edge-bipartite graphs in $\mathcal{UB}igr_n$ and the Coxeter spectral study of matrix morsifications in $\mathbf{Mor}_{D\Delta}$, where $D\Delta$ is the simply laced Dynkin diagram associated with Δ.

Theorem 3.4. *Let Δ be a connected positive edge-bipartite graph in $\mathcal{UB}igr_n$ and let $D\Delta$ be a unique simply laced Dynkin diagram such that $\Delta \sim_\mathbb{Z} D\Delta$, as in Theorem 2.1. Fix a matrix $C \in \mathrm{Gl}(n,\mathbb{Z})$ defining the congruence $D\Delta \sim_\mathbb{Z} \Delta$, that is, the equality $G_{D\Delta} = C^{tr}\cdot G_\Delta \cdot C$ holds.*

(a) *The non-symmetric Gram matrix $\check{G}_\Delta$ of Δ lies in $\mathbf{Mor}_\Delta$, the matrix $A_\Delta := C^{tr}\cdot \check{G}_\Delta \cdot C$ lies $\mathbf{Mor}_{D\Delta}$, $\mathrm{cox}_{A_\Delta}(t) = \mathrm{cox}_\Delta(t)$, and $\mathbf{c}_{A_\Delta} = \mathbf{c}_\Delta$.*

(b) *The correspondence $A \mapsto h_C A := C^{tr}\cdot A \cdot C = A * C$ defines the bijection*

$$h_C : \mathbf{Mor}_\Delta \to \mathbf{Mor}_{D\Delta} \tag{3.5}$$

such that $\det h_C A = \det A$, $\mathrm{cox}_{h_C A}(t) = \mathrm{cox}_A(t)$, and the Coxeter numbers $\mathbf{c}_A$ and $\mathbf{c}_{h_C A}$ of the morsifications A and $h_C A$ coincide.

(c) *The group automorphism $h'_C : \mathbb{Z}^n \to \mathbb{Z}^n$, $v \mapsto v\cdot C^{tr}$, makes the following diagram commutative, with $A_\Delta := \check{G}_\Delta * C$,*

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "\Phi_{A_\Delta}"] \arrow[d, "h'_C"', "\simeq"] & \mathbb{Z}^n \arrow[d, "h'_C"', "\simeq"] \\ \mathbb{Z}^n \arrow[r, "\Phi_\Delta"] & \mathbb{Z}^n \end{tikzcd} \tag{3.6}$$

(d) *The group automorphism $h'_C : \mathbb{Z}^n \to \mathbb{Z}^n$ restricts to the bijection $h'_C : \mathcal{R}_{D\Delta} \to \mathcal{R}_\Delta$, carries Φ_{A_Δ}-orbits of length $\ell \geq 1$ in $\mathcal{R}_{D\Delta}$ to Φ_Δ-orbits of length ℓ in $\mathcal{R}_\Delta$, carries Φ_{A_Δ}-meshes of width $s \geq 1$ in $\mathcal{R}_{D\Delta}$ to Φ_Δ-meshes of width $s \geq 1$ in $\mathcal{R}_\Delta$, and carries any Φ_{A_Δ}-mesh geometry of roots $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_{A_\Delta})$ in the sense of [35] to a Φ_Δ-mesh geometry of roots.*

Proof. (a) is a consequence of (b), because $\check{G}_\Delta \in \mathbf{Mor}_\Delta$.

(b) By our assumption, we have $G_{D\Delta} = G_\Delta * C$. Then, given $A \in \mathbf{Mor}_\Delta$, we get

$$h_C A + (h_C A)^{tr} = A * C + (A * C)^{tr} = (A + A^{tr}) * C = 2G_\Delta * C = 2G_{D\Delta}.$$

Furthermore, one shows that the matrices Cox_{A*C} and Cox_A are adjoint by C^{tr}. Hence Cox_{A*C} has integer coefficients and (b) follows, compare with [2, Proposition 2.8]. To prove (c), we note that $\mathrm{Cox}_{\check{G}_\Delta * C} = C^{tr}\cdot \mathrm{Cox}_\Delta \cdot C^{-tr}$. Hence it easily follows that the diagram (3.6) is commutative. Since statement (d) is a consequence of (c), the proof is complete. □

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Example 3.7. Assume that $D \in \{\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6\}$ and let $\Delta = \Delta_D^{(j)}$ be one of the six edge-bipartite graphs $\Delta_{\mathbb{D}_4}^{(2)}$, $\Delta_{\mathbb{D}_5}^{(2)}$, $\Delta_{\mathbb{D}_6}^{(2)}$, $\Delta_{\mathbb{D}_6}^{(3)}$, $\Delta_{\mathbb{E}_6}^{(2)}$, $\Delta_{\mathbb{E}_6}^{(3)}$ of Theorem 1.9(b). By applying the inflation algorithm (see Theorem 2.1) to $\Delta_D^{(j)}$ we get:

(a) $D\Delta_D^{(j)} = D$, $\mathrm{cox}_{\Delta_D^{(j)}}(t) = F_D^{(j)}(t)$, $\Delta_D^{(j)} \sim_{\mathbb{Z}} D$, and the congruence is defined by the matrix $C_D^{(j)}$, where

$$C_{\mathbb{D}_4}^{(2)} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{\mathbb{D}_5}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$C_{\mathbb{D}_6}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C_{\mathbb{D}_6}^{(3)} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_{\mathbb{E}_6}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad C_{\mathbb{E}_6}^{(3)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The bijection $h_C : \mathbf{Mor}_{\Delta_D^{(j)}}^{F^{(j)}} \to \mathbf{Mor}_D^{F^{(j)}}$ (3.5) defined by the matrix $C := C_D^{(j)}$ carries the non-symmetric Gram matrix $\check{G}_{\Delta_D^{(j)}} \in \mathbf{Mor}_{\Delta_D^{(j)}}^{F^{(j)}}$ to the morsification $A_D^{(j)} \in \mathbf{Mor}_D$, where

$$A_{\mathbb{D}_4}^{(2)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \qquad A_{\mathbb{D}_5}^{(2)} = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{\mathbb{D}_6}^{(2)} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}, \qquad A_{\mathbb{D}_6}^{(3)} = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_{\mathbb{E}_6}^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A_{\mathbb{E}_6}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{bmatrix}$$

It follows from Theorem 3.4 that the bijection (3.5) reduces our original problems for graphs in $\mathcal{UB}igr_n$ to analogous problems for matrix morsifications $A \in \mathbf{Mor}_D$, with $\det A = 1$, where D is one of the Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. We can construct efficient computational algorithms for the Coxeter spectral analysis of matrices $A \in \mathbf{Mor}_D$ by using the set

$$\mathcal{C}ox_D := \left\{ C \in \mathrm{Gl}(n, \mathbb{Z})_D^{tr};\ \det(E - C) \neq 0 \text{ and } G_D = G_D \cdot (E - C)^{-tr} + (E - C)^{-1} \cdot G_D \right\},$$

where G_D is the symmetric Gram matrix of the Dynkin diagram D, and the pair of bijections

$$\mathcal{C}ox_D \underset{\Theta_D}{\overset{\xi_D}{\rightleftarrows}} \widehat{\mathbf{Mor}}_\Delta \subseteq \mathbb{M}_n(\mathbb{Q}), \tag{3.8}$$

(inverse to each other) defined by $\Theta_\Delta(A) := \mathrm{Cox}_A = -A \cdot A^{-tr}$ and $\xi_\Delta(C) := 2G_\Delta \cdot (E - C)^{-tr}$, for any $A \in \widehat{\mathbf{Mor}}_\Delta$ and any $C \in \mathcal{C}ox_\Delta$, see [2, Note Added in Proof]. It was shown in [2] that, given $C \in \mathrm{Gl}(n, \mathbb{Z})_D$ and a root $v \in \mathcal{R}_D$ of q_D, the vector $v \cdot C^{tr}$ lies in $\mathcal{R}_D$. Hence we conclude the following important facts:

Fact A. *The sets $\mathcal{C}ox_D^{tr} \subseteq \mathrm{Gl}(n, \mathbb{Z})_D$ are finite and the rows of the matrices $C \in \mathrm{Gl}(n, \mathbb{Z})_D$ lie in the finite set $\mathcal{R}_D \subseteq \mathbb{Z}^n$ of roots of D.*

Fact B. *A $\mathbb{Z}$-invertible matrix $C \in \mathbb{M}_n(\mathbb{Z})$ is of the form $C = \mathrm{Cox}_A$, with $A \in \widehat{\mathbf{Mor}}_\Delta$ if and only if 1 is not an eigenvalue of C, the rows of C lie in the finite set $\mathcal{R}_\Delta$ of roots of Δ, and the equality $G_\Delta = G_\Delta \cdot (E - C)^{-tr} + (E - C)^{-1} \cdot G_\Delta$ holds.*

Details of the proof will be presented in a subsequent paper. By applying Facts A and B, we construct the following algorithms computing the finite set $\mathcal{C}ox_D$ (and the finite set $\widehat{\mathbf{Mor}}_D$), the finite group $\mathrm{Gl}(n, \mathbb{Z})_D$, the $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits of the action (3.2), the finite set of all Coxeter polynomials $\mathrm{cox}_A(t) \in \mathbb{Z}[t]$, the Coxeter numbers $\mathbf{c}_A$ and the determinants $\det A$, with $A \in \widehat{\mathbf{Mor}}_D$.

Algorithm 3.9. *Input*: A simply laced Dynkin diagram D, with $n \geq 2$ vertices enumerated as in the table shown in Section 2, and its symmetric Gram matrix $G_D \in \mathbb{M}_n(\mathbb{Q})$.

Output: The set $\widehat{\mathbf{Mor}}_D$ of all rational morsifications $A \in \mathbb{M}_n(\mathbb{Q})$, together with their Coxeter numbers $\mathbf{c}_A$, Coxeter polynomials $\mathrm{cox}_A(t)$, and the determinant $\det A$.

STEP 1: Compute the set $\mathcal{R} := \mathcal{R}_D$.

STEP 2: Construct a generic matrix $C = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, with the jth row $w_j = [w_{j1}, \dots, w_{jn}]$.

STEP 3: Given $i \in \{1, \dots, n\}$, form the sequential subset of $\mathcal{R} = \mathcal{R}_D$:

$$_i\mathcal{R} = \{(r_1, \dots, r_n) \in \mathcal{R};\ r_1 \geq 0, \dots, r_{i-1} \geq 0,\ r_i > 0,\ r_{i+1} = \cdots = r_n = 0\}.$$

STEP 4: For $i = 1, \dots, n$, substitute the unknown matrix row w_i, sequentially by vectors in the set $\mathcal{R} \setminus \{w_1, \dots, w_{i-1}, -w_1, \dots, -w_{i-1}, e_i\}$. After the subsequent i-substitution, check if $r \cdot C \in \mathcal{R}$, for any $r \in_i \mathcal{R}$. If the condition is not fulfilled, take a sequent i-substitution.

STEP 5: If $\det(E - C) \neq 0$ and $G_D = G_D \cdot (E - C)^{-tr} + (E - C)^{-1} \cdot G_D$ then we add the matrix $A := 2G_D \cdot (E - C)^{-1}$ to the set $\widehat{\mathbf{Mor}}_D$, we calculate the Coxeter number $\mathbf{c}_A$, the determinant $\det A$, and the Coxeter polynomial $\mathrm{cox}_A(t)$, next we go to Step 4 with a new substitution.

Remark 3.10. By applying Algorithm 3.9, we easily compute the set $\widehat{\mathbf{Mor}}_D$, for $D \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7\}_{n \leq 7}$. If $D = \mathbb{A}_n$, the cardinality $|\widehat{\mathbf{Mor}}_D|$ equals 5, 15, 69, 345, 2.295, 16.065, for $n = 2, 3, 4, 5, 6, 7$, respectively. If $D = \mathbb{D}_n$ and $n = 4, 5, 6, 7$, the cardinality $|\widehat{\mathbf{Mor}}_D|$ equals 385, 945, 10.395, 135.135. Finally, if $D = \mathbb{E}_6$ and $D = \mathbb{E}_7$, the cardinality $|\widehat{\mathbf{Mor}}_D|$ equals 28.163 and 740.340, respectively.

Algorithm 3.11. *Input*: A simply laced Dynkin diagram D, with $n \geq 2$ vertices enumerated as in the table shown in Section 2, and its symmetric Gram matrix $G_D \in \mathbb{M}_n(\mathbb{Q})$.

Output: The isotropy group $\mathrm{Gl}(n, \mathbb{Z})_D$.

STEP 1: Compute the set $\mathcal{R} := \mathcal{R}_D$.

STEP 2: Construct a generic matrix $B = [\kappa_1 \cdots \kappa_n]$, with columns $\kappa_1 \dots \kappa_n$ and n rows.

STEP 3: Given $i \in \{1, \dots, n\}$, form the sequential subset of $\mathcal{R} = \mathcal{R}_D$:

$$_i\mathcal{R} = \{(r_1, \dots, r_n) \in \mathcal{R};\quad r_1 \geq 0, \dots,\ r_{i-1} \geq 0,\ r_i > 0, r_{i+1} = \cdots = r_n = 0\}.$$

STEP 4: For $i = 1, \dots, n$, substitute the unknown matrix column κ_i, sequentially by a vector in the set $\mathcal{R} \setminus \{w_1, \dots, w_{i-1}, -w_1, \dots, -w_{i-1}\}$. After the subsequent i-substitution, check if $r \cdot B^{\mathrm{tr}} \in \mathcal{R}$, for any $r \in_i \mathcal{R}$. If the condition is not fulfilled, take a sequent i-substitution.

STEP 5: If $G_D = B^{tr} \cdot G_D \cdot B$, then add the matrix B to the set $\mathrm{Gl}(n, \mathbb{Z})_D$, next go to Step 4 with a new substitution.

Algorithm 3.12. *Input*: A simply laced Dynkin diagram D, with $n \geq 2$ vertices enumerated as in the table shown in Section 2, and its symmetric Gram matrix $G_D \in \mathbb{M}_n(\mathbb{Q})$.

Output: The $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits of the action (3.2), the finite set $\mathbf{CPol}_D$ of all Coxeter polynomials $\mathrm{cox}_A(t) \in \mathbb{Z}[t]$, the Coxeter numbers $\mathbf{c}_A$ and the determinants $\det A$.

STEP 1: Compute the set $\mathcal{M} := \widehat{\mathbf{Mor}}_D$ (from Algorithm 3.9).

STEP 2: Compute $\mathrm{Gl}(n, \mathbb{Z})_D$ (from Algorithm 3.11).

STEP 3: Given $A \in \mathcal{M}$, calculate $A * \mathrm{Gl}(n, \mathbb{Z})_D$, the Coxeter number $\mathbf{c}_A$, the determinant $\det A$ and the Coxeter polynomial $\mathrm{cox}_A(t)$.

STEP 4: $\mathcal{M} := \mathcal{M} \setminus A * \mathrm{Gl}(n, \mathbb{Z})_D$. If $\mathcal{M} \neq \emptyset$ then go to Step 3.

By applying Algorithms 3.9–3.12, a routine computer calculation yields the following result.

Proposition 3.13. *Assume that D is a simply laced Dynkin diagram, with $n \le 6$ vertices.*

(a) *The set $\mathbf{CPol}_D$ is finite and coincides with that presented in [4, Table 3.11]. Moreover, a polynomial $F(t) \in \mathbb{Z}[t]$ is of the form $F(t) = \mathrm{cox}_A(t)$, for some $A \in \mathbf{Mor}_D$ with $\det A = 1$, if and only if there exists a connected positive $\Delta \in \mathcal{UB}igr_n$ such that $D\Delta = D$ and $F(t) = \mathrm{cox}_\Delta(t)$.*

(b) *For any $F(t) \in \mathbb{Z}[t]$ the $\mathrm{Gl}(n,\mathbb{Z})_D$-invariant subset $\mathbf{Mor}_D^F := \{A \in \mathbf{Mor};\ \mathrm{cox}_A(t) = F(t)\}$ of $\mathbf{Mor}_D$ is of the form $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n,\mathbb{Z})_D$, for some $A_F \in \mathbf{Mor}_D$.*

(c) *If D and D' are a simply laced Dynkin diagram with at most six vertices and $\mathrm{cox}_A(t) = \mathrm{cox}_{A'}(t)$, for some $A \in \mathbf{Mor}_D$ and $A' \in \mathbf{Mor}_{D'}$ then $D \cong D'$.* □

Outline of proof of Theorem 1.7. (a) If Δ is positive and connected in $\mathcal{UB}igr_n$ and $D\Delta = D$ and $C \in \mathrm{Gl}(n,\mathbb{Z})$ is such that $G_D = G_\Delta * C$ then, by Theorem 3.4, the matrix $A_\Delta := \check{G}_\Delta * C$ lies in $\mathbf{Mor}_D$, $\det A_\Delta = 1$, and $(\mathrm{cox}_{A_\Delta}(t), \mathbf{c}_{A_\Delta}) = (\mathrm{cox}_\Delta(t), \mathbf{c}_\Delta)$. This proves the sufficiency of (a). The necessity follows from the fact that $\mathrm{cox}_\Delta(t) = F_D^{(j)}(t)$, if $2 \le j \le 3$, $D \in \{\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6\}$ and $\Delta = \Delta_D^{(j)}$ is one of the edge-bipartite graphs of Theorem 1.9(b). Note also that $\mathrm{cox}_D(t) = F_D^{(1)}(t)$, for arbitrary D.

(b) Assume that $\Delta, \Delta' \in \mathcal{UB}igr_n$ are positive, connected, and $n \le 6$. If $\Delta \approx_\mathbb{Z} \Delta'$ then [1, Proposition 4.8] and [3, Lemma 2.1(e)] yield $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$. Conversely, assume that $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$. Then, by Theorem 3.4, $\mathrm{cox}_{A_\Delta}(t) = \mathrm{cox}_\Delta(t) = \mathrm{cox}_{\Delta'}(t) = \mathrm{cox}_{A_{\Delta'}}(t)$, $\det A_{\Delta'} = 1 = \det A_\Delta$ and $A_\Delta \in \mathbf{Mor}_{D\Delta}$, $A'_\Delta \in \mathbf{Mor}_{D\Delta}$. Since $n \le 6$, Proposition 3.13(c) yields $D\Delta = D\Delta'$.

To show that $\Delta \approx_\mathbb{Z} \Delta'$, we set $D := D\Delta = D\Delta'$ and we use the isotropy group $\mathrm{Gl}(n,\mathbb{Z})_D$ of the Dynkin diagram D, where G_D is the symmetric Gram matrix of D. Consider the right action (3.2) of $\mathrm{Gl}(n,\mathbb{Z})_D$ on $\widehat{\mathbf{Mor}}_D$. Denote by $F(t) \in \mathbb{Z}[t]$ the Coxeter polynomial $F(t) := \mathrm{cox}_\Delta(t) = \mathrm{cox}_{\Delta'}(t) = \mathrm{cox}_{A_\Delta}(t) = \mathrm{cox}_{A_{\Delta'}}(t)$. It follows from Proposition 3.13(b) that $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n,\mathbb{Z})_D$, for some $A_F \in \mathbf{Mor}_D^F$. Since, by Theorem 3.4, the matrices $\check{G}_\Delta$ and $\check{G}_{\Delta'}$ lie in $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n,\mathbb{Z})_D$ then there is a matrix $B \in \mathrm{Gl}(n,\mathbb{Z})_D$ such that $A_\Delta = B^{tr} \cdot A_{\Delta'} \cdot B = A_\Delta * B$, that is, $\check{G}_\Delta * C = (\check{G}_{\Delta'} * C') * B$. It follows that $\check{G}_\Delta = \check{G}_{\Delta'} * (C'BC^{-1})$, that is, $\Delta \approx_\mathbb{Z} \Delta'$.

(c) It is easy to see that each of the matrices $\check{G}_\Delta, \check{G}_\Delta^{tr}$ lies in $\mathbf{Mor}_\Delta$ and they have the common Coxeter polynomial $F(t) := \mathrm{cox}_\Delta(t)$. It follows from Theorem 3.4 and Proposition 3.13(b) that the images $A := h_C\check{G}_\Delta$ and $A' := h_C\check{G}_\Delta^{tr} = A^{tr}$ of $\check{G}_\Delta$ and $\check{G}_\Delta^{tr}$ under the map (3.5) lie in $\mathbf{Mor}_D^F = A_F * \mathrm{Gl}(n,\mathbb{Z})_D$, where $D = D\Delta$. Then there is a matrix $B \in \mathrm{Gl}(n,\mathbb{Z})_D$ such that $A = B^{tr} \cdot A' \cdot B = A' * B$. Hence we conclude as above that $\check{G}_\Delta = \check{G}_{\Delta'} * (C \cdot B \cdot C^{-1}) = \check{G}_{\Delta'} * C_1$, where $C_1 = C \cdot B \cdot C^{-1}$ and $C_1^2 = C \cdot B^2 \cdot C^{-1}$. Then the first part of (c) follows. To get such a matrix C_1 that $C_1^2 = E$, we need a more advanced technique analogous to that applied in [22]. We present it in a subsequent paper. For Δ, with $D\Delta = \mathbb{D}_4$, the result is proved in [2]. For the edge-bipartite graph $\Delta := \Delta_{\mathbb{D}_5}^{(2)}$ of Dynkin type $D\Delta = \mathbb{D}_5$ shown in Theorem 1.9(b) we can use the mesh quiver technique presented below. By applying the symmetry of the triangle marked on the third part of the mesh quiver presented below we find the matrix (compare with the method illustrated in [22, Figures 7.7–7.8] and [23, Algorithm 5.6])

$$C = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

such that $C^{tr} \cdot \check{G}_\Delta \cdot C = \check{G}_\Delta^{tr}$ and $C^2 = E$.

(d) Let $F(t) := \mathrm{cox}_\Delta(t)$ and let $D = D\Delta$ be a unique simply laced Dynkin diagram such that $\Delta \sim_\mathbb{Z} D\Delta$, as in Theorem 2.1. Fix a matrix $C \in \mathrm{Gl}(n,\mathbb{Z})$ defining the congruence $D = D\Delta \sim_\mathbb{Z} \Delta$, that is, the equality $G_D = C^{tr} \cdot G_\Delta \cdot C$ holds. By Theorem 3.4, the bijection $h_C : \mathbf{Mor}_\Delta^F \to \mathbf{Mor}_D^F$ reduces the existence of a Φ_Δ-mesh geometry $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ of roots of Δ to the existence of such a Φ_A-mesh geometry $\Gamma(\widehat{\mathcal{R}}_D, \Phi_A)$, for a fixed $A \in \mathbf{Mor}_D^F$, with $\det A = 1$. Indeed, if $A' \in \mathbf{Mor}_D^F$, with $\det A' = 1$, is arbitrary then $A' = A * B$, for some $B \in \mathrm{Gl}(n,\mathbb{Z})_D$ (by 3.13(b)). It follows that the group automorphism $h'_B : \mathbb{Z}^n \to \mathbb{Z}^n$, $v \mapsto v \cdot B^{tr}$, makes the following diagram commutative:

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "\Phi_{A'}"] \arrow[d, "h'_B"', "\simeq"] & \mathbb{Z}^n \arrow[d, "h'_B"', "\simeq"] \\ \mathbb{Z}^n \arrow[r, "\Phi_A"] & \mathbb{Z}^n \end{tikzcd} \tag{3.14}$$

Moreover, $h'_B : \mathbb{Z}^n \to \mathbb{Z}^n$ restricts to the bijection $h'_B : \mathcal{R}_D \to \mathcal{R}_D$, carries $\Phi_{A'}$-orbits of length $\ell \ge 1$ in $\mathcal{R}_D$ to Φ_D-orbits of length ℓ in $\mathcal{R}_D$, carries $\Phi_{A'}$-meshes of width $s \ge 1$ in $\mathcal{R}_D$ to Φ_A-meshes of width $s \ge 1$ in $\mathcal{R}_D$ and, consequently, carries any $\Phi_{A'}$-mesh geometry of roots $\Gamma(\widehat{\mathcal{R}}_D, \Phi_{A'})$ to a Φ_A-mesh geometry of roots.

In the case when $D = D\Delta = \mathbb{A}_n$, $n \ge 2$, Theorem 2.4 and its proof given below apply. If $D = D\Delta \ne \mathbb{A}_n$ and $n \ge 2$, then $D = \mathbb{D}_n$ and $4 \le n \le 6$ or $D = \mathbb{E}_6$, by our assumption in 1.7. For $D = \mathbb{D}_4$, the result is proved in [2,7]. Now we construct the geometry of root orbits for the edge-bipartite graph $\Delta := \Delta_{\mathbb{D}_5}^{(2)}$ of Dynkin type $D\Delta = \mathbb{D}_5$ presented in Theorem 1.9(b).

Note that

$$\check{G}_\Delta = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{with } \mathrm{Cox}_\Delta = \begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix} \qquad \mathbf{c}_\Delta = 12,$$
$$\mathrm{cox}_\Delta(t) = t^5 + t^3 + t^2 + 1,$$

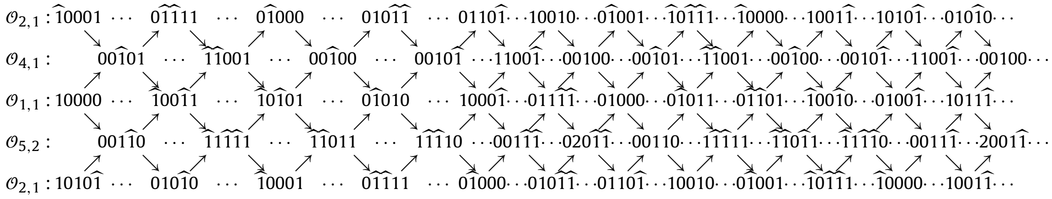
and $q_\Delta(x) = x \cdot A \cdot x^{tr} = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_1x_4 + x_2x_4 + x_2x_5 + x_1x_5 + x_3x_5$, where $\mathbf{c}_\Delta = 12$ is the Coxeter number of Δ. We define the Coxeter transformation $\Phi_\Delta : \mathbb{Z}^5 \to \mathbb{Z}^5$ by the formula $\Phi_\Delta(v) = v \cdot \mathrm{Cox}_\Delta$. The set $\mathcal{R}_\Delta = \{v \in \mathbb{Z}^5;\ q_\Delta(v) = 1\}$ of roots of Δ consists of 40 vectors and is Φ_Δ-invariant. An easy computation shows that $\mathcal{R}_\Delta$ splits into the following five Φ_Δ-orbits (two of length $\mathbf{c}_\Delta = 12$, two of length 6, and one of length 4):

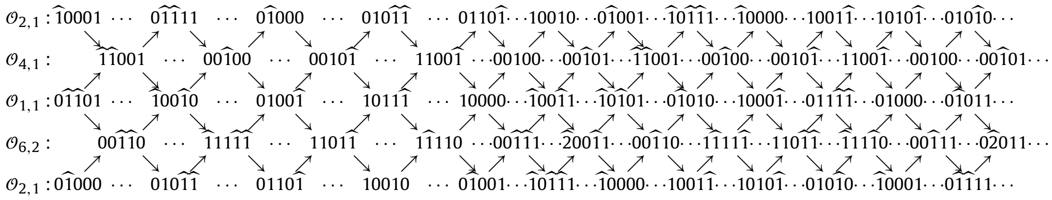
$$\begin{array}{ll}
\mathcal{O}_{1,1}: & 10000 \cdots \hat{1}00\hat{1}\hat{1} \cdots \hat{1}0\hat{1}01 \cdots 0\hat{1}010 \cdots 1000\hat{1} \cdots 011\hat{1}\hat{1} \cdots 01000 \cdots 0\hat{1}011 \cdots 0\hat{1}\hat{1}01 \cdots \hat{1}00\hat{1}0 \cdots 0100\hat{1} \cdots 1011\hat{1} \cdots 10000 \\
\mathcal{O}_{2,1}: & 0\hat{1}000 \cdots 010\hat{1}\hat{1} \cdots 0110\hat{1} \cdots 10010 \cdots 0\hat{1}001 \cdots \hat{1}0\hat{1}\hat{1}1 \cdots \hat{1}0000 \cdots 1001\hat{1} \cdots 1010\hat{1} \cdots 010\hat{1}0 \cdots \hat{1}0001 \cdots 0\hat{1}\hat{1}11 \cdots 0\hat{1}000 \\
\mathcal{O}_{3,1}: & 00001 \cdots \hat{1}\hat{1}\hat{1}02 \cdots \hat{1}\hat{1}\hat{1}01 \cdots 0000\hat{1} \cdots 1110\hat{2} \cdots 1110\hat{1} \cdots 00001 \\
\mathcal{O}_{4,1}: & 00100 \cdots 00\hat{1}01 \cdots \hat{1}\hat{1}001 \cdots 00\hat{1}00 \cdots 0010\hat{1} \cdots 1100\hat{1} \cdots 00100 \\
\mathcal{O}_{5,1}: & 00010 \cdots 1\hat{1}010 \cdots 000\hat{1}0 \cdots \hat{1}10\hat{1}0 \cdots 00010
\end{array}$$

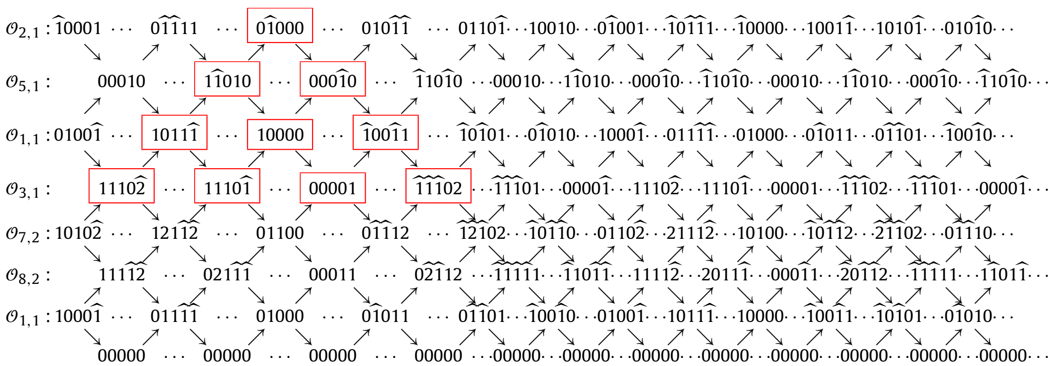
where we set $\widehat{a} = -a$, for $a \in \mathbb{N}$. Note that $\mathcal{O}_{4,1} = -\mathcal{O}_{4,1}$ is a unique border orbit, $\mathcal{O}_{1,1} = -\mathcal{O}_{2,1}$, $\mathcal{O}_{3,1} = -\mathcal{O}_{3,1}$, $\mathcal{O}_{5,1} = -\mathcal{O}_{5,1}$, and we have the following Φ_Δ-mesh in $\mathcal{R}_\Delta$ of width three (see [1,35])

$$\begin{tikzcd}
 & w \arrow[dr] & \\
 & w' \arrow[r] & \\
u \arrow[uur] \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & \Phi_\Delta^{-1}(u) \\
 & v \arrow[ur] &
\end{tikzcd} \tag{3.15}$$

(i.e., $w + w' + v = u + \Phi_\Delta^{-1}(u)$), where $w = [0010\hat{1}] \in \mathcal{O}_{4,1}$, $w' = [000\hat{1}0] \in \mathcal{O}_{5,1}$, $u = [0\hat{1}000] \in \mathcal{O}_{2,1}$, $\Phi_\Delta^{-1}(u) = [010\hat{1}\hat{1}] \in \mathcal{O}_{2,1}$, and $v = [00\hat{1}00] \in \mathcal{O}_{4,1}$. We construct the following three Φ_Δ-mesh translation quivers in $\mathbb{Z}^n$:







We can do it by a direct manipulations or by applying the toroidal mesh algorithm presented in [1,2]. The constructed Φ_Δ-mesh translation quivers contain all Φ_Δ-orbits in $\mathcal{R}\Delta$ and four Φ_Δ-orbits $\mathcal{O}_{5,2}, \mathcal{O}_{6,2}, \mathcal{O}_{7,2}$, and $\mathcal{O}_{8,2}$ (each of length $\mathbf{c}_\Delta = 12$) consisting of vectors $v \in \mathbb{Z}^n$ such that $q_\Delta(v) = 2$. Further, following the technique applied in [35, Case 4, p. 26] and [7], we glue the three Φ_Δ-mesh translation quivers along the Φ_Δ-orbits defined by the Φ_Δ-mesh $(*)$ of width three and, consequently, we obtain the Φ_Δ-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$, which is the Φ_Δ-mesh geometry of roots required in (d). Its perpendicular section appears as follows

$$\begin{tikzcd}
& \mathcal{O}_{1,1} \arrow[dl, dash] \arrow[dr, dash] & & & & & & & \\
\mathcal{O}_{6,2} \arrow[dr, dash] & & \mathcal{O}_{4,1} \arrow[dl, dash] & & & & & & \\
& \mathcal{O}_{2,1} \arrow[rrr, dash] \arrow[dl, dash] \arrow[dr, dash] & & & \mathcal{O}_{5,1} \arrow[r, dash] & \mathcal{O}_{1,1} \arrow[r, dash] & \mathcal{O}_{3,1} \arrow[r, dash] & \mathcal{O}_{7,2} \arrow[r, dash] & \mathcal{O}_{8,2} \arrow[r, dash] & \mathcal{O}_{1,1} \\
\mathcal{O}_{5,2} \arrow[dr, dash] & & \mathcal{O}_{4,1} \arrow[dl, dash] & & & & & & \\
& \mathcal{O}_{1,1} & & & & & & &
\end{tikzcd}.$$

It contains $88 = 40 + 4 \times 12$ vectors. This finishes the proof in the case when $D\Delta = \mathbb{D}_5$. In remaining cases when $D\Delta \in \{\mathbb{D}_6, \mathbb{E}_6\}$, the proof is analogous but more technical. We shall present it in a future paper. This finishes our outline of the proof of Theorem 1.7 for $n \le 6$. The idea of the proof for $n \ge 7$ is analogous, but leads to computer calculations of rather high complexity. □

Proof of Theorem 1.9. Assume that $\Delta \in \mathcal{UB}igr_n$ is positive, connected, and $2 \le n \le 6$. Let $D := D\Delta$ be a unique simply laced Dynkin diagram such that $\Delta \sim_\mathbb{Z} D\Delta$, as in Theorem 2.1. By Theorem 1.7(a), there exists $j \ge 1$ such that $\mathrm{cox}_\Delta(t) = F_D^{(j)}(t)$, where $F_D^{(j)}(t)$ is one of the polynomials of Table 1.8.

Note that $\mathrm{cox}_D(t) = F_D^{(1)}(t)$. If the polynomial $\mathrm{cox}_\Delta(t)$ has the form $\mathrm{cox}_\Delta(t) = F_D^{(1)}(t)$ then $\mathbf{specc}_\Delta = \mathbf{specc}_D$ and Theorem 1.7(b) yields $\Delta \sim_\mathbb{Z} D = D\Delta$, that is, (a) follows.

Assume that $\mathrm{cox}_\Delta(t) = F_D^{(j)}(t)$ and $j \ge 2$. In view of Table 1.8, we have $n \ge 2$ and $D = D\Delta \in \{\mathbb{D}_4, \mathbb{D}_5, \mathbb{D}_6, \mathbb{E}_6\}$. It is shown in Example 3.7 that $\mathrm{cox}_{\Delta_D^{(i)}}(t) = F_D^{(i)}(t)$, if $\Delta_D^{(i)}$ is one of the edge-bipartite graphs of Theorem 1.9(b). Hence $\mathrm{cox}_\Delta(t) = \mathrm{cox}_{\Delta_D^{(i)}}(t)$, $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta_D^{(i)}}$, and Theorem 1.7(b) yields $\Delta \approx_\mathbb{Z} \Delta_{D\Delta}^{(j)}$. This finishes the proof of Theorem 1.9. □

Outline of proof of Theorem 2.4. Let $F(t) = F_{\mathbb{A}_n}(t)$. By Theorem 3.4, the non-symmetric Gram matrix $\check{G}_\Delta$ of Δ lies in $\mathbf{Mor}_\Delta^F$, there exists a matrix $A_\Delta \in \mathbf{Mor}_{D\Delta}^F$ such that $A_\Delta := C^{tr} \cdot \check{G}_\Delta \cdot C$, for some $C \in \mathrm{Gl}(n, \mathbb{Z})$, and $\mathrm{cox}_{A_\Delta}(t) = \mathrm{cox}_\Delta(t) = F(t)$. It follows that $\Delta = \mathbb{A}_n$ (by Proposition 3.13(d)) and $A_\Delta, \check{G}_{\mathbb{A}_n} \in \mathbf{Mor}_{D\Delta}^F = \mathbf{Mor}_{\mathbb{A}_n}^F$. Then, by Proposition 3.13(c), the matrices $A_\Delta, \check{G}_{\mathbb{A}_n}$ are $\mathbb{Z}$-congruent. It follows that the matrices $\check{G}_\Delta, \check{G}_{\mathbb{A}_n}$ are $\mathbb{Z}$-congruent, that is, $\Delta \approx_\mathbb{Z} D\Delta = \mathbb{A}_n$ and (b) follows. Hence we conclude that the diagram (3.6) is commutative, with A_Δ and $\check{G}_{\mathbb{A}_n}$, interchanged. Consequently, the proof of (a) and (c) reduces to the case when $\Delta = \mathbb{A}_n$. For $n = 5$, the proof is given in Example 2.3; in the remaining cases it is analogous, see [18–23]. □

4. Concluding remarks and problems

We study the class $\mathcal{UB}igr_n$ of loop-free edge-bipartite (multi)graphs Δ, with $n \ge 2$ vertices (i.e., signed graphs), by means of:

(a) their Gram matrices $\check{G}_\Delta$ and $\widehat{G}_\Delta = \check{G}_\Delta + \check{G}_\Delta^{tr} \in \mathbb{M}_n(\mathbb{Z})$,

(b) the complex Coxeter spectrum $\mathbf{specc}_\Delta$ of the Coxeter matrix $\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr}$, and

(c) the right action $* : \mathbb{M}_n(\mathbb{Z}) \times \mathrm{Gl}(n, \mathbb{Z}) \longrightarrow \mathbb{M}_n(\mathbb{Z})$, $(A, C) \mapsto A * C := C^{tr} \cdot A \cdot C$ of the general $\mathbb{Z}$-linear group $\mathrm{Gl}(n, \mathbb{Z}) := \{A \in \mathbb{M}_n(n, \mathbb{Z}),\ \det A \in \{-1, 1\}\}$ on $\mathbb{M}_n(\mathbb{Z})$.

One of our aims is to classify the equivalence classes of connected loop-free positive edge-bipartite graphs Δ (i.e., the symmetric matrix $\widehat{G}_\Delta$ is positive definite) with respect to one of the two $\mathbb{Z}$-congruences $\sim_\mathbb{Z}$ and $\approx_\mathbb{Z}$, where

(i) $\Delta \sim_\mathbb{Z} \Delta' \iff$ the symmetric Gram matrices $\widehat{G}_{\Delta'}$ and $\widehat{G}_\Delta$ lie in the same $\mathrm{Gl}(n, \mathbb{Z})$-orbit, and

(ii) $\Delta \approx_\mathbb{Z} \Delta' \iff$ the non-symmetric Gram matrices $\check{G}_{\Delta'}$ and $\check{G}_\Delta$ lie in the same $\mathrm{Gl}(n, \mathbb{Z})$-orbit.

We develop an algorithmic technique for the Coxeter spectral analysis of the class $\mathcal{UB}igr_n$. In particular, we show that $\Delta \approx_\mathbb{Z} \Delta'$ implies $\Delta \sim_\mathbb{Z} \Delta'$ and $\mathbf{specc}_\Delta = \mathbf{specc}_{\Delta'}$. If Δ is connected and positive, we have $\Delta \sim_\mathbb{Z} D\Delta$, where $D\Delta$ is a simply laced Dynkin diagram of one of the types $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. We study the Coxeter spectral analysis question, whether the Coxeter spectrum $\mathbf{specc}_\Delta$ determines $\Delta \in \mathcal{UB}igr_n$ uniquely, up to the congruence $\approx_\mathbb{Z}$. For $n \le 6$ and Δ positive and connected, an affirmative answer to the question is given and, up to the congruence $\approx_\mathbb{Z}$, a complete classification of such edge-bipartite graphs is presented in Theorem 1.9.

Remark 4.1. It follows from the computational technique developed in this paper and in [1–3,28,32,35] that the solution of the Coxeter spectral analysis problems stated in Section 1 and studied in [2,3,19], for the connected positive edge-bipartite graphs in $\mathcal{UB}igr_n$, reduces to analogous computational problems for the $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits on the sets $\mathbf{Mor}_D$ of matrix morsifications of simply laced Dynkin diagrams $D \in \mathcal{UB}igr_n$ and to an algorithmic description of mesh geometries of roots in the sense of [35], see also [40]. It is shown in Section 3 that the reduction leads to an algorithmic computer search of rather high complexity, see Remark 3.10. For instance, the computation time of the sets $\widehat{\mathbf{Mor}}_{\mathbb{E}_6}$ and $\widehat{\mathbf{Mor}}_{\mathbb{E}_7}$ equals 27 min and 31 h and 24 min, respectively. We estimate that the computation time of $\widehat{\mathbf{Mor}}_D$, with $|D_0| = n \geq 8$ grows at least as $|\mathcal{R}_D|^n$, where $\mathcal{R}_D = \{v \in \mathbb{Z}^n;\ v \cdot G_D \cdot v^{tr} = 1\}$ is the (finite) set of roots of D. For example, for the computation of the set $\widehat{\mathbf{Mor}}_{\mathbb{D}_9}$ we need about 36 days and nights. Unfortunately, at the moment we are not able to compute the set $\widehat{\mathbf{Mor}}_D$, with $|D_0| = n \geq 11$. To calculate the set $\widehat{\mathbf{Mor}}_D$ and its $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits, for $|D| = n \geq 10$, we need a modification of our algorithm and more advanced computational tools that should be developed.

Recently, an efficient computation technique of the isotropy group $\mathrm{Gl}(n, \mathbb{Z})_D$ and the $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits in the sets $\widehat{\mathbf{Mor}}_D$ has been developed by Gąsiorek [21]. Results of a similar nature for finite posets and connected loop-free non-negative edge-bipartite graphs of finite corank are presented in the two recent articles [24,25].

In computing $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits in the set $\widehat{\mathbf{Mor}}_D$, for a simply laced Dynkin diagram D with $|D_0| = n$, and in computing connected positive edge-bipartite graphs Δ in $\mathcal{UB}igr_n$, with $n \geq 2$ sufficiently small, their Coxeter polynomials $\mathrm{cox}_\Delta(t)$, the Coxeter numbers $\mathbf{c}_\Delta$, and to get more experimental data, we use our dedicated programs written in Maple and C++. The problem is computationally hard, because it can be shown that it is NP-hard and also co-NP-hard, see [3]. Nevertheless, by a brute-force approach, with a precomputation and suitable heuristic, we compute the positive connected edge-bipartite graphs $\Delta \in \mathcal{UB}igr_n$ and the $\mathrm{Gl}(n, \mathbb{Z})_{D\Delta}$-orbits in $\widehat{\mathbf{Mor}}_{D\Delta}$, for n small.

Remark 4.2. Although we have rather efficient algorithms for the isotropy group $\mathrm{Gl}(n, \mathbb{Z})_D$ and the $\mathrm{Gl}(n, \mathbb{Z})_D$-orbits in the matrix morsification sets $\mathbf{Mor}_D$, where D is a simply laced Dynkin diagram, we have no such algorithms in the case when D is a Euclidean diagram.

Remark 4.3. Algorithms constructing mesh geometries of roots for connected positive edge-bipartite graphs Δ in $\mathcal{UB}igr_n$ are described only for $n \leq 6$. In the case $n \geq 7$, we shall construct such algorithms and mesh geometries of roots $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ in future papers.

Remark 4.4. We recall from Theorem 2.4 that we have such an algorithm for Δ of Dynkin type $D\Delta = \mathbb{A}_n$, where $n \leq 6$; fortunately, the technique we have applied there generalizes to $n \geq 7$. For $\Delta \in \mathcal{UB}igr_5$ positive of Dynkin type $D\Delta = \mathbb{D}_5$ and of Coxeter type $F^{(2)}_{\mathbb{D}_5}(t) = t^5 + t^3 + t^2 + 1$, the algorithm describing a mesh geometry of roots $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ is implicitly presented in the proof of Theorem 1.7. We outline it as follows.

Step 1 Compute the Coxeter matrix Cox_Δ, the Coxeter transformation $\Phi_\Delta : \mathbb{Z}^5 \to \mathbb{Z}^5$, the list of roots $\mathcal{R}_\Delta \subseteq \mathbb{Z}^5$. The set of cardinality 40 can be listed by using algorithm [35, Algorithm 4.2].

Step 2 Split the set $\mathcal{R}_\Delta$ into the union of five Φ_Δ-orbits: a unique border orbit $\mathcal{O}_{4,1} = -\mathcal{O}_{4,1}$ of length six (such that $z + \Phi_\Delta(u) \in \mathcal{R}_\Delta$, for some $z \in \mathcal{O}_{4,1}$), a non-border orbit $\mathcal{O}_{3,1} = -\mathcal{O}_{3,1}$ of length six, a unique orbit $\mathcal{O}_{5,1} = -\mathcal{O}_{5,1}$ of length four, and two orbits $\mathcal{O}_{1,1}$ and $\mathcal{O}_{2,1} = -\mathcal{O}_{1,1}$ of length $\mathbf{c}_\Delta = 12$.

Step 3 Find a pair w, v of vectors in the border orbit $\mathcal{O}_{4,1}$ of length six, a vector w' in the shortest orbit $\mathcal{O}_{5,1}$ of length four, and a vector u in the longest orbit $\mathcal{O}_{1,1}$ such that $w + w' + v = u + \Phi_\Delta^{-1}(u)$, that is, the vectors $u, \Phi_\Delta^{-1}(u), w, w', v$ form the Φ_Δ-mesh (3.15) in $\mathcal{R}_\Delta \subseteq \mathbb{Z}^5$.

Step 4 Construct the three Φ_Δ-mesh translation quivers in $\mathbb{Z}^5$ of the shape shown in the proof of Theorem 1.7 in Section 3.

Step 5 Construct the Φ_Δ-mesh translation quiver $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ by glueing the three Φ_Δ-mesh translation quivers along the common Φ_Δ-orbits with respect to the Φ_Δ-mesh (3.15).

Remark 4.5. Algorithms describing a mesh geometry of roots $\Gamma(\widehat{\mathcal{R}}_\Delta, \Phi_\Delta)$ for positive edge-bipartite graphs $\Delta \in \mathcal{UB}igr_6$ of the Dynkin types $D\Delta = \mathbb{D}_6$ and $D\Delta = \mathbb{E}_6$, and of Coxeter types $F^{(2)}_{\mathbb{D}_6}(t)$, $F^{(3)}_{\mathbb{D}_6}(t)$, $F^{(2)}_{\mathbb{E}_6}(t)$, and $F^{(3)}_{\mathbb{E}_6}(t)$ (see Table 1.8), have a similar character and will be presented in a future publication.

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