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Determination of a solely time-dependent source in a semilinear parabolic problem by means of boundary measurements☆

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ABSTRACT

A semilinear parabolic problem of second order with an unknown solely time-dependent source term is studied. The missing source is recovered from an additional integral measurement over the boundary. The global in time existence, uniqueness as well as the regularity of a solution are addressed. A new numerical scheme based on Rothe's method is designed and convergence of iterates towards the exact solution is shown.

1. Introduction

Inverse problems (IPs) usually lead to mathematical models that are ill-posed in the sense of Hadamard. Many of them do not have a solution in the strict classical sense, or if there is a solution, it might not be unique or might not depend continuously on the data. To prove global (in time) existence and uniqueness of a solution turn out to be a serious task. Another important goal in IPs is their solvability and description of a constructive algorithm for finding a solution. The standard algorithms for IPs start with suitable parametrization and they involve continuous dependence of a parametrized solution on the parameter. A cost functional capturing the error between parametrized and exact solutions at a given measurement place is minimized in appropriate function spaces. The common disadvantage of this approach is lack of convexity of the cost functional, which can be remediated by an appropriate regularization – cf. e.g. [1–3] – based on adding a suitable term to the functional in order to guarantee its convexity, ensuring the existence of a unique solution to the minimization problem by means of the theory of monotone operators [4,5]. This later problem can be solved numerically by adequate approximation techniques, such as the steepest descend, Ritz or Newton or Levenberg–Marquardt method, see e.g. [6,7].

We are interested in determining the unknown couple (u, h) obeying the following semilinear parabolic problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = h(t)f(x) + \alpha(u(x,t)) + \beta(x,t) & \text{in } \Omega \times (0,T), \\ \nabla u(x,t) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{R}^d,\ d \geq 1$ is a bounded domain with a sufficiently smooth boundary Γ. The symbol $\boldsymbol{\nu}$ denotes the outer normal vector associated with Γ. The data functions u_0, f, α, β are given and $T > 0$. The unknown purely time-dependent source term $h(t)$ will be recovered from the following (non-invasive) measurement on the boundary

$$m(t) = \int_\Gamma u(x,t)\mathrm{d}x, \quad t \in [0,T]. \tag{2}$$

Recovery of an unknown source belongs to hot topics in inverse problems. If the source solely depends on the space variable, one needs an additional space measurement (e.g. solution at the final time), cf. [8–17]. For purely time-dependent source a supplementary time-dependent measurement is needed, cf. [18–20]. This means that both kinds of inverse source problems (ISPs) need totally different additional data. The integral overdetermination is frequently used in various IPs for evolutionary problems, cf. [8,21,22] and the references therein. The integral is usually taken over the whole domain (or over a sub-domain). To get such a measurement is not always obvious. We consider the integration just over the boundary Γ in (2).

The goal of this paper is to address the well-posedness of the ISP, to study the regularity of a solution and to describe a constructive way for finding it. The added value of this paper relies on reformulating the ISP into an appropriate direct formulation. This means that we are looking on the ISP as on a system with two unknowns (u, h). We eliminate h from (1) by (2). The proposed numerical scheme involves the semi-discretization in time by Rothe's method cf. [23]. We prove the existence of approximations at each time step of the time partitioning and we establish some stability results. The convergence of iterates towards the exact solution is obtained by arguments of functional analysis. Finally, we discuss uniqueness of the ISP.

Notations. Denote by $(\cdot,\cdot)$ the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working at the boundary Γ we use a similar notation, namely $(\cdot,\cdot)_\Gamma$, $L^2(\Gamma)$ and $\|\cdot\|_\Gamma$. By $C([0,T],X)$ we denote the set of abstract functions $w : [0,T] \to X$ equipped with the usual norm $\max_{t\in[0,T]} \|\cdot\|_X$ and $L^p((0,T),X)$ is furnished with the norm $\left(\int_0^T \|\cdot\|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}$ with $p > 1$, cf. [24]. The symbol X^* stands for the dual space to X. As is usual in papers of this sort, C, ε and C_ε will denote generic positive constants depending only on a priori known quantities, where ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$ is large.

Take any function $\varphi \in H^1(\Omega)$, and derive from (1) after integration over Ω and involving the Green theorem that

$$(\partial_t u, \varphi) + (\nabla u, \nabla\varphi) = h(f,\varphi) + (\alpha(u),\varphi) + (\beta,\varphi). \tag{P}$$

Integrating (1) over Γ and taking into account the measurement (2) we have

$$m' - \int_\Gamma \Delta u = h\int_\Gamma f + \int_\Gamma \alpha(u) + \int_\Gamma \beta. \tag{MP}$$

The relations (P) and (MP) represent the variational formulation of (1) and (2).

2. Time discretization

In Rothe's method [23], a time-dependent problem is approximated by a sequence of elliptic tasks which have to be solved successively with increasing time step. Rothe's method can be also used for determination of the unknown time dependent source h. For ease of explanation we consider an equidistant time-partitioning of the time frame $[0,T]$ with a step $\tau = T/n$, for any $n \in \mathbb{N}$. We use the notation $t_i = i\tau$ and for any function z we write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

Consider a system with unknowns (u_i, h_i) for $i = 1,\ldots,n$. At time t_i we infer from (P) by the backward Euler scheme

$$(\delta u_i, \varphi) + (\nabla u_i, \nabla\varphi) = h_i(f,\varphi) + (\alpha(u_{i-1}),\varphi) + (\beta_i,\varphi). \tag{DPi}$$

Considering u_{i-1} in the right-hand side makes (DPi) linear in u_i. From (MP) we obtain

$$m_i' - \int_\Gamma \Delta u_{i-1} = h_i\int_\Gamma f + \int_\Gamma \alpha(u_{i-1}) + \int_\Gamma \beta_i. \tag{DMPi}$$

The decoupling of u_i and h_i has been achieved by considering u_{i-1} in (DMPi). Note that for a given $i \in \{1,\ldots,n\}$ we solve first equation (DMPi) and then (DPi). Further we increase i to $i+1$.

Let us introduce the following space of test functions

$$\boldsymbol{V} = \{\varphi : \Omega \to \mathbb{R};\ \ \|\varphi\| + \|\nabla\varphi\| + \|\Delta\varphi\| + \|\nabla\Delta\varphi\| < \infty\},$$

which is suitable for our purposes. We will seek u within this space. Let us note that we have to work in sufficiently regular function space in order to keep h depending on $\int_\Gamma \Delta u$ under control, cf. (MP).

Lemma 2.1. *Let $m' \in C([0,T])$, $\beta \in C\left([0,T], H^1(\Omega)\right)$, $f \in H^1(\Omega)$, $\int_\Gamma f \neq 0$, α is global Lipschitz continuous. Assume that $u_0 \in \boldsymbol{V}$. Then for each $i \in \{1,\ldots,n\}$ there exists a unique couple $(u_i, h_i) \in \boldsymbol{V} \times \mathbb{R}$ solving* (DPi) *and* (DMPi).

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Proof. From (DMPi) we see that

$$h_i = \frac{-m_i' + \int_\Gamma \Delta u_{i-1} + \int_\Gamma \alpha(u_{i-1}) + \int_\Gamma \beta_i}{\int_\Gamma f}.$$

If $u_{i-1} \in \boldsymbol{V}$ then $h_i \in \mathbb{R}$ by the trace theorem and Lipschitz continuity of α.

The relation (DPi) for $\varphi \in H^1(\Omega)$ can we rewritten as follows

$$\frac{1}{\tau}(u_i, \varphi) + (\nabla u_i, \nabla \varphi) = \frac{1}{\tau}(u_{i-1}, \varphi) + h_i(f, \varphi) + (\alpha(u_{i-1}), \varphi) + (\beta_i, \varphi).$$

The left-hand side represents a continuous, elliptic and bilinear form in $H^1(\Omega) \times H^1(\Omega)$ and the right-hand side is a linear bounded functional on $H^1(\Omega)$. The Lax–Milgram lemma ensures existence and uniqueness of $u_i \in H^1(\Omega)$. Inspecting the relation (DPi) we may write for any $\varphi \in H^1(\Omega)$ that

$$(-\Delta u_i, \varphi) = h_i(f, \varphi) + (\alpha(u_{i-1}), \varphi) + (\beta_i, \varphi) - (\delta u_i, \varphi).$$

The term $-\Delta u_i$ has to be understood in the sense of duality, as a functional on $H^1(\Omega)$. The right-hand side can be estimated by $C_i(\tau)\,\|\varphi\|$. Thus there exists an extension of $-\Delta u_i$ to $L^2(\Omega)$ according to Hahn–Banach theorem, cf. [25, p. 173]. This extension will have the same norm as the functional on $H^1(\Omega)$ and

$$-\Delta u_i = h_i f + \alpha(u_{i-1}) + \beta_i - \delta u_i \in L^2(\Omega).$$

Applying the gradient operator to this relation we see that

$$-\nabla \Delta u_i = h_i \nabla f + \nabla \alpha(u_{i-1}) + \nabla \beta_i - \delta \nabla u_i \in L^2(\Omega) \tag{3}$$

taking into account the assumptions of this lemma, which concludes the proof. □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be fulfilled. Then there exists a positive constant C such that*

$$\max_{1\le j\le n} \left\|u_j\right\|^2 + \sum_{i=1}^n \|\nabla u_i\|^2 \tau + \sum_{i=1}^n \|u_i - u_{i-1}\|^2 \le C\left(1 + \sum_{i=1}^n h_i^2 \tau\right).$$

Proof. Set $\varphi = u_i \tau$ in (DPi), sum it up for $i = 1, \dots, j$ to get

$$\sum_{i=1}^j (\delta u_i, u_i)\,\tau + \sum_{i=1}^j \|\nabla u_i\|^2 \tau = \sum_{i=1}^j h_i(f, u_i)\,\tau + \sum_{i=1}^j (\alpha(u_{i-1}), u_i)\,\tau + \sum_{i=1}^j (\beta_i, u_i)\,\tau.$$

The summation by parts formula says that

$$\sum_{i=1}^j (\delta u_i, u_i)\tau = \sum_{i=1}^j (u_i - u_{i-1}, u_i) = \frac{1}{2}\left(\left\|u_j\right\|^2 - \|u_0\|^2 + \sum_{i=1}^j \|u_i - u_{i-1}\|^2\right).$$

Making use of the Cauchy and Young inequalities we deduce in a standard way that

$$\left\|u_j\right\|^2 + \sum_{i=1}^j \|u_i - u_{i-1}\|^2 + \sum_{i=1}^j \|\nabla u_i\|^2 \tau \le C\left(1 + \sum_{i=1}^j h_i^2 \tau + \sum_{i=1}^j \|u_i\|^2 \tau\right) \tag{4}$$

as α is Lipschitz continuous. We conclude the proof by Grönwall's argument, cf. [26]. □

Lemma 2.3. *Let the assumptions of Lemma 2.1 be fulfilled. Then there exists a positive constant C such that*

$$\max_{1\le j\le n} \left\|\nabla u_j\right\|^2 + \sum_{i=1}^n \|\delta u_i\|^2 \tau + \sum_{i=1}^n \|\nabla u_i - \nabla u_{i-1}\|^2 \le C\left(1 + \sum_{i=1}^n h_i^2 \tau\right).$$

Proof. Putting $\varphi = \delta u_i \tau$ into (DPi), summing it up for $i = 1, \dots, j$ we obtain

$$\sum_{i=1}^j (\nabla u_i, \delta \nabla u_i)\,\tau + \sum_{i=1}^j \|\delta u_i\|^2 \tau = \sum_{i=1}^j h_i(f, \delta u_i)\,\tau + \sum_{i=1}^j (\alpha(u_{i-1}), \delta u_i)\,\tau + \sum_{i=1}^j (\beta_i, \delta u_i)\,\tau.$$

The Abel summation gives

$$\sum_{i=1}^j (\nabla u_i, \nabla \delta u_i)\tau = \sum_{i=1}^j (\nabla u_i, \nabla u_i - \nabla u_{i-1}) = \frac{1}{2}\left(\left\|\nabla u_j\right\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^j \|\nabla u_i - \nabla u_{i-1}\|^2\right).$$

Making use of the Cauchy and Young inequalities we deduce in a standard way that

$$\begin{aligned}\left\|\nabla u_j\right\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2+\sum_{i=1}^{j}\|\delta u_i\|^2\tau &\le C_\varepsilon\left(1+\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\|u_i\|^2\tau\right)+\varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau\\ &\le C_\varepsilon\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+\varepsilon\sum_{i=1}^{j}\|\delta u_i\|^2\tau \end{aligned}\tag{5}$$

using Lemma 2.2. Setting a sufficiently small $\varepsilon > 0$ we complete the proof. □

Lemma 2.4. *Let the assumptions of Lemma 2.1 be fulfilled. Then there exists a positive constant C such that*

$$\max_{1\le j\le n}\left\|\Delta u_j\right\|^2+\sum_{i=1}^{n}\|\nabla\Delta u_i\|^2\tau+\sum_{i=1}^{n}\|\Delta u_i-\Delta u_{i-1}\|^2\le C\left(1+\sum_{i=1}^{n}h_i^2\tau\right).$$

Proof. We start from (3). We multiply this by $-\nabla\Delta u_i\tau$, integrate over Ω and sum the result up for $i = 1,\dots,j$ to find

$$-\sum_{i=1}^{j}(\delta\nabla u_i,\nabla\Delta u_i)\,\tau+\sum_{i=1}^{j}\|\nabla\Delta u_i\|^2\tau=-\sum_{i=1}^{j}h_i\,(\nabla f,\nabla\Delta u_i)\,\tau-\sum_{i=1}^{j}(\nabla\alpha(u_{i-1}),\nabla\Delta u_i)\,\tau-\sum_{i=1}^{j}(\nabla\beta_i,\nabla\Delta u_i)\,\tau.$$

Integration by parts formula together with Abel's summation yields

$$-\sum_{i=1}^{j}(\delta\nabla u_i,\nabla\Delta u_i)\,\tau=\sum_{i=1}^{j}(\delta\Delta u_i,\Delta u_i)\,\tau=\frac{1}{2}\left(\left\|\Delta u_j\right\|^2-\|\Delta u_0\|^2+\sum_{i=1}^{j}\|\Delta u_i-\Delta u_{i-1}\|^2\right).$$

Making use of the Cauchy and Young inequalities we deduce in a standard way that

$$\begin{aligned}\left\|\Delta u_j\right\|^2+\sum_{i=1}^{j}\|\Delta u_i-\Delta u_{i-1}\|^2+\sum_{i=1}^{j}\|\nabla\Delta u_i\|^2\tau &\le C_\varepsilon\left(1+\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\|\nabla u_i\|^2\tau\right)+\varepsilon\sum_{i=1}^{j}\|\nabla\Delta u_i\|^2\tau\\ &\le C_\varepsilon\left(1+\sum_{i=1}^{j}h_i^2\tau\right)+\varepsilon\sum_{i=1}^{j}\|\nabla\Delta u_i\|^2\tau\end{aligned}\tag{6}$$

using Lemma 2.2. Fixing a sufficiently small $\varepsilon > 0$ we close the proof. □

Theorem 2.1 (*Stability of Approximations*)**.** *Let the assumptions of Lemma 2.1 be fulfilled. Then there exists a positive constant C such that*

$$\max_{1\le j\le n}\left\|u_j\right\|^2+\max_{1\le j\le n}\left\|\nabla u_j\right\|^2+\max_{1\le j\le n}\left\|\Delta u_j\right\|^2+\sum_{i=1}^{n}\|\delta u_i\|^2\tau+\sum_{i=1}^{n}\|\nabla\Delta u_i\|^2\tau\le C,$$

$$\sum_{i=1}^{n}\|u_i-u_{i-1}\|^2+\sum_{i=1}^{n}\|\nabla u_i-\nabla u_{i-1}\|^2+\sum_{i=1}^{n}\|\Delta u_i-\Delta u_{i-1}\|^2\le C$$

and

$$\sum_{i=1}^{j}h_i^2\tau\le C.$$

Proof. Putting the relations (4)–(6) together, and fixing a suitable $\varepsilon > 0$ we have

$$\begin{aligned}&\left\|u_j\right\|^2+\left\|\nabla u_j\right\|^2+\left\|\Delta u_j\right\|^2+\sum_{i=1}^{j}\|\delta u_i\|^2\tau+\sum_{i=1}^{j}\|\nabla\Delta u_i\|^2+\sum_{i=1}^{j}\|u_i-u_{i-1}\|^2+\sum_{i=1}^{j}\|\nabla u_i-\nabla u_{i-1}\|^2\\ &\quad+\sum_{i=1}^{j}\|\Delta u_i-\Delta u_{i-1}\|^2\le C\left(1+\sum_{i=1}^{j}h_i^2\tau+\sum_{i=1}^{j}\|u_i\|^2\tau\right).\end{aligned}$$

In virtue of (DMPi) we deduce that

$$\begin{aligned}|h_i| &\le\left|\frac{-m_i'+\int_\Gamma\Delta u_{i-1}+\int_\Gamma\alpha(u_{i-1})+\int_\Gamma\beta_i}{\int_\Gamma f}\right|\le C\left(\left|m_i'\right|+\|\Delta u_{i-1}\|_\Gamma+\|\alpha(u_{i-1})\|_\Gamma+\|\beta_i\|_\Gamma\right)\\ &\le C\left(1+\|\Delta u_{i-1}\|_\Gamma+\|u_{i-1}\|_\Gamma\right),\end{aligned}$$

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which implies by the Nečas inequality [27]

$$\|z\|_\Gamma^2 \le \varepsilon \|\nabla z\|^2 + C_\varepsilon \|z\|^2, \quad \forall z \in H^1(\Omega),\ 0 < \varepsilon < \varepsilon_0 \tag{7}$$

that

$$\begin{aligned}\sum_{i=1}^{j} h_i^2 \tau &\le C\left(1 + \sum_{i=1}^{j} \|\Delta u_{i-1}\|_\Gamma^2 \tau + \sum_{i=1}^{j} \|u_{i-1}\|_\Gamma^2 \tau\right)\\ &\le C_\varepsilon\left(1 + \sum_{i=1}^{j}\left[\|\Delta u_{i-1}\|^2 + \|\nabla u_{i-1}\|^2 + \|u_{i-1}\|^2\right]\tau\right) + \varepsilon \sum_{i=1}^{j} \|\nabla \Delta u_{i-1}\|^2 \tau.\end{aligned} \tag{8}$$

Putting things together and setting a small $\varepsilon > 0$ we get

$$\begin{aligned}&\|u_j\|^2 + \|\nabla u_j\|^2 + \|\Delta u_j\|^2 + \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \sum_{i=1}^{j} \|\nabla \Delta u_i\|^2 + \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j} \|\nabla u_i - \nabla u_{i-1}\|^2\\ &\quad + \sum_{i=1}^{j} \|\Delta u_i - \Delta u_{i-1}\|^2 \le C\left(1 + \sum_{i=1}^{j}\left[\|u_i\|^2 + \|\nabla u_i\|^2 + \|\Delta u_i\|^2\right]\tau\right).\end{aligned}$$

An application of Grönwall's lemma gives the estimates for u_i. Relation (8) implies the bound for h_i in $L^2(0, T)$. □

3. Well-posedness

Now, let us introduce the following piecewise linear function in time

$$u_n : [0, T] \to L^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0\\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 0 \le i \le n,$$

and a step function

$$\overline{u}_n : [0, T] \to L^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0\\ u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 0 \le i \le n.$$

Similarly we define $\overline{h}_n$, $\overline{\beta}_n$, $\overline{m}_n$ and $\overline{m'}_n$. These prolongations are also called Rothe's (piecewise linear and continuous, or piecewise constant) functions. Now, we can rewrite (DPi) and (DMPi) on the whole time frame as

$$(\partial_t u_n(t), \varphi) + (\nabla \overline{u}_n(t), \nabla \varphi) = \overline{h}_n(t)\,(f, \varphi) + (\alpha(\overline{u}_n(t - \tau)), \varphi) + \left(\overline{\beta}_n(t), \varphi\right) \tag{DP}$$

and

$$\overline{m'}_n(t) - \int_\Gamma \Delta \overline{u}_n(t - \tau) = \overline{h}_n(t) \int_\Gamma f + \int_\Gamma \alpha(\overline{u}_n(t - \tau)) + \int_\Gamma \overline{\beta}_n(t). \tag{DMP}$$

Now, we are in a position to prove the existence of a weak solution to (P) and (MP).

Theorem 3.1 (*Existence of a Solution*)**.** *Let the assumptions of Lemma 2.1 be fulfilled. Then there exists a solution (u, h) to the (P), (MP) obeying $u \in C\left([0, T], L^2(\Omega)\right) \cap L^\infty\left((0, T), H^1(\Omega)\right)$ with $\partial_t u, \Delta u \in L^2\left((0, T), L^2(\Omega)\right)$, $\nabla \Delta u \in L^2((0, T), \mathbf{L}_2(\Omega))$, $h \in L^2(0, T)$.*

Proof. Theorem 2.1 says that $\|u_j\| + \|\nabla u_j\| \le C$ and $\sum_{i=1}^{n} \|\delta u_i\|^2 \tau \le C$, which means that for all $n > 0$ it holds $\|\overline{u}_n(t)\|_{H^1(\Omega)} \le C$ for all $t \in [0, T]$ and $\int_0^T \|\partial_t u_n\|^2 \,\mathrm{d}\xi \le C$. Using [23, Lemma 1.3.13] there exists $u \in C\left([0, T], L^2(\Omega)\right) \cap L^\infty\left((0, T), H^1(\Omega)\right)$ which is time-differentiable a.e. in $[0, T]$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$ (denoted by the same symbol again) such that

$$\begin{cases} u_n \to u, & \text{in } C\left([0, T], L^2(\Omega)\right) \quad \text{(a)}\\ u_n(t) \rightharpoonup u(t), & \text{in } H^1(\Omega),\ \forall t \in [0, T] \quad \text{(b)}\\ \overline{u}_n(t) \rightharpoonup u(t), & \text{in } H^1(\Omega),\ \forall t \in [0, T] \quad \text{(c)}\\ \partial_t u_n \rightharpoonup \partial_t u, & \text{in } L^2\left((0, T), L^2(\Omega)\right) \quad \text{(d)}. \end{cases} \tag{9}$$

Moreover $u : [0, T] \to L^2(\Omega)$ is Hölder continuous, i.e. $\|u(t) - u(t')\| \le C\sqrt{|t - t'|}$, which follows from

$$\|u_n(t) - u_n(t')\| = \left\|\int_t^{t'} \partial_t u_n \mathrm{d}\xi\right\| \le C\sqrt{|t - t'|}\sqrt{\int_0^T \|\partial_t u_n\|^2 \,\mathrm{d}\xi} \le C\sqrt{|t - t'|} \tag{10}$$

when passing to the limit for $n \to \infty$. Similarly we deduce

$$\|u_n(t) - \overline{u}_n(t)\| \le C\sqrt{\tau}\sqrt{\int_t^{t+\tau} \|\partial_t u_n(s)\|^2 \,\mathrm{d}s} \le C\sqrt{\tau}$$

and

$$\|\overline{u}_n(t) - \overline{u}_n(t-\tau)\| = \tau \,\|\partial_t u_n\| = \left|\int_{t-\tau}^{t} \|\partial_t u_n\|\right| \le C\sqrt{\tau}.$$

We can easily see that

$$\lim_{n\to\infty} \alpha(\overline{u}_n(t-\tau)) = \lim_{n\to\infty} \alpha(\overline{u}_n(t)) = \lim_{n\to\infty} \alpha(u_n(t)) = \alpha(u(t)) \quad \text{in } C\left([0,T], L^2(\Omega)\right) \tag{11}$$

as α is Lipschitz continuous and $u_n \to u$ in $C\left([0,T], L^2(\Omega)\right)$. Obviously $\lim_{n\to\infty} \overline{\beta}_n(t) = \beta(t)$ in $C\left([0,T], L^2(\Omega)\right)$. The reflexivity of $L^2(0,T)$ and the bound $\sum_{i=1}^n h_i^2 \tau \le C$ gives $\overline{h}_n \rightharpoonup h$ in $L^2\left((0,T), L^2(\Omega)\right)$. In the light of these considerations we can integrate (DP) in time, pass to the limit for $\tau \to 0$ and differentiate with respect to the time variable to arrive at (P).

Now, we would like to pass to the limit in (DMP). First we need some preparation work. Using Nečas's inequality (7), the fact that $\|\overline{u}_n\|_{H^1(\Omega)} \le C$ (Theorem 2.1) and $u \in L^\infty\left((0,T), H^1(\Omega)\right)$ we obtain

$$\begin{aligned}\|\alpha(\overline{u}_n(t-\tau)) - \alpha(u(t))\|_\Gamma^2 &\le \varepsilon \,\|\nabla(\alpha(\overline{u}_n(t-\tau)) - \alpha(u(t)))\|^2 + C_\varepsilon \,\|\alpha(\overline{u}_n(t-\tau)) - \alpha(u(t))\|^2 \\ &\le \varepsilon + C_\varepsilon \,\|\alpha(\overline{u}_n(t-\tau)) - \alpha(u(t))\|^2\end{aligned}$$

as α is Lipschitz continuous. Passing to the limit and applying (11) we have

$$\lim_{n\to+\infty} \|\alpha(\overline{u}_n(t-\tau)) - \alpha(u(t))\|_\Gamma^2 \le \varepsilon \Longrightarrow \alpha(\overline{u}_n(t-\tau)) \to \alpha(u(t)), \quad \text{in } C\left([0,T], L^2(\Gamma)\right). \tag{12}$$

Theorem 3.1 helps us to get

$$\int_0^T \|\Delta\overline{u}_n(t) - \Delta\overline{u}_n(t-\tau)\|^2 = \sum_{i=1}^n \|\Delta u_i - \Delta u_{i-1}\|^2 \,\tau = \mathcal{O}(\tau)$$

and analogously

$$\int_0^T \|\Delta\overline{u}_n(t) - \Delta u_n(t)\|^2 \le C\int_0^T \|\Delta\overline{u}_n(t) - \Delta\overline{u}_n(t-\tau)\|^2 = \mathcal{O}(\tau).$$

Using (9)(b) we see that for $\varphi \in H^1(\Omega)$

$$(\Delta u_n(t), \varphi) = -(\nabla u_n(t), \nabla\varphi) \to -(\nabla u(t), \nabla\varphi) = (\Delta u(t), \varphi).$$

According to $\sum_{i=1}^n \|\nabla\Delta u_i\|^2 \,\tau \le C$ we see that $\nabla\Delta\overline{u}_n \rightharpoonup \boldsymbol{z}$ in $L^2((0,T), \mathbf{L}_2(\Omega))$. Taking any $\boldsymbol{\varphi} \in \left(C_0^\infty(\Omega)\right)^N$ we deduce that

$$\int_0^T (\nabla\Delta\overline{u}_n, \boldsymbol{\varphi}) = -\int_0^T (\Delta u_n, \nabla\cdot\boldsymbol{\varphi}) \to -\int_0^T (\Delta u, \nabla\cdot\boldsymbol{\varphi}) = \int_0^T (\nabla\Delta u, \boldsymbol{\varphi}),$$

from which we deduce that $\boldsymbol{z} = \nabla\Delta u$.

Let us consider the following temporary Neumann problem $\Phi - \Delta\Phi = F$ in Ω along with $\nabla\Phi\cdot\boldsymbol{\nu} = 1$ on Γ. If $F \in \left(H^1(\Omega)\right)^*$ then by Lax–Milgram lemma we have $\Phi \in H^1(\Omega)$. If $F \in H^1(\Omega)$ we conclude that $\Delta\Phi \in H^1(\Omega)$ by the Hahn–Banach theorem. We see that

$$\begin{array}{llll}\displaystyle\int_0^t (\Delta\overline{u}_n, 1)_\Gamma & \displaystyle= \int_0^t (\Delta\overline{u}_n, \nabla\Phi\cdot\boldsymbol{\nu})_\Gamma & \displaystyle= \int_0^t (\Delta\overline{u}_n, \Delta\Phi) + \int_0^t (\nabla\Delta\overline{u}_n, \nabla\Phi) & \displaystyle= -\int_0^t (\nabla\overline{u}_n, \nabla\Delta\Phi) + \int_0^t (\nabla\Delta\overline{u}_n, \nabla\Phi) \\ & & & \qquad\qquad\qquad\qquad\downarrow \\ \displaystyle\int_0^t (\Delta u, 1)_\Gamma & \displaystyle= \int_0^t (\Delta u, \nabla\Phi\cdot\boldsymbol{\nu})_\Gamma & \displaystyle= \int_0^t (\Delta u, \Delta\Phi) + \int_0^t (\nabla\Delta u, \nabla\Phi) & \displaystyle= -\int_0^t (\nabla u, \nabla\Delta\Phi) + \int_0^t (\nabla\Delta u, \nabla\Phi).\end{array}$$

Now, we integrate (DMP) in time. Using $m' \in C([0,T])$, $\beta \in C\left([0,T], H^1(\Omega)\right)$ and the considerations above we may pass the limit for $n \to \infty$. Finally we differentiate the result with respect to the time and we arrive at (MP). □

Theorem 3.2 (*Uniqueness of a Solution*)**.** *There exists at most one solution to (P), (MP) obeying $u_0 \in \boldsymbol{V}$, $u \in C\left([0,T], L^2(\Omega)\right) \cap L^\infty\left((0,T), H^1(\Omega)\right)$ with $\partial_t u, \Delta u \in L^2\left((0,T), L^2(\Omega)\right)$, $\nabla\Delta u \in L^2((0,T), \mathbf{L}_2(\Omega))$, $h \in L^2(0,T)$ if one of the following assumptions is valid:*

(i) *α is linear*
(ii) *$\alpha \in C^2$ is nonlinear and $|\nabla u| \le C$ a.e. in $\Omega \times (0,T)$.*

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Proof. Let us have two solutions (u_1, h_1) and (u_2, h_2). Set $u := u_1 - u_2$ and $h := h_1 - h_2$. Subtracting the corresponding variational formulations (P) for particular solutions from each other and putting $\varphi = u$ we readily obtain

$$\|u(t)\|^2 + \int_0^t \|\nabla u\|^2 \,\mathrm{d}s \le C \int_0^t h^2 \mathrm{d}s \quad t \in [0, T] \tag{13}$$

similarly as in Lemma 2.2. Analogously as in Lemma 2.3 we deduce that

$$\|\nabla u(t)\|^2 + \int_0^t \|\partial_t u\|^2 \,\mathrm{d}s \le C \int_0^t h^2 \mathrm{d}s \quad t \in [0, T]. \tag{14}$$

If α is linear, then matching Lemma 2.4 we find that

$$\|\Delta u(t)\|^2 + \int_0^t \|\nabla \Delta u\|^2 \,\mathrm{d}s \le C \int_0^t h^2 \mathrm{d}s \quad t \in [0, T]. \tag{15}$$

If α is nonlinear, then we can follow Lemma 2.4, but the following term must be estimated in a different way

$$\begin{aligned} |\nabla \alpha(u_1) - \nabla \alpha(u_2)| &= \left|\alpha'(u_1)\nabla u_1 - \alpha'(u_2)\nabla u_2\right| \\ &= \left|\alpha'(u_1)(\nabla u_1 - \nabla u_2) + [\alpha'(u_1) - \alpha'(u_2)]\nabla u_2\right| \\ &\le C\,(|u| + |\nabla u|) \end{aligned}$$

using $\alpha \in C^2$ and $|\nabla u| \le C$ a.e. in $\Omega \times (0, T)$. Also in this case we reveal (15). By (7) we get

$$\left|\int_\Gamma \Delta u\right|^2 \le C\,\|\Delta u\|_\Gamma^2 \le \varepsilon\,\|\nabla \Delta u\|^2 + C_\varepsilon\,\|\Delta u\|^2\,,$$

and

$$\left|\int_\Gamma \alpha(u_1) - \alpha(u_2)\right|^2 \le C\,\|u\|_\Gamma^2 \le C\left(\|u\|^2 + \|\nabla u\|^2\right).$$

Subtracting the corresponding variational formulations (MP) for particular solutions from each other we easily see that

$$\int_0^t h^2 \mathrm{d}s \le \varepsilon \int_0^t \|\nabla \Delta u\|^2 \,\mathrm{d}s + C_\varepsilon \int_0^t \left(\|u\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2\right) \mathrm{d}s. \tag{16}$$

Relations (13)–(16) imply for a fixed small $\varepsilon > 0$

$$\|u(t)\|^2 + \|\nabla u(t)\|^2 + \|\Delta u(t)\|^2 + \int_0^t \|\nabla \Delta u\|^2 \,\mathrm{d}s \le C \int_0^t \left(\|u\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2\right) \mathrm{d}s.$$

Grönwall's argument ensures that $u = 0$. □

Conclusion

An ISP for a semilinear parabolic equation with a solely time-dependent unknown source is considered. The existence and uniqueness of a weak solution for the ISP are proved. The missing source term is recovered from an integral-type measurement over the boundary. A numerical algorithm based on Rothe's method is established and the convergence of approximations towards the exact solution is demonstrated.

We would like to point out that this technique can also be easily extended to a non-homogeneous Neumann BC, $f = f(x, t)$ and to a measurement (2) taken just over a measurable part of the boundary.

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