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Uniform isochronous cubic and quartic centers: Revisited

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ABSTRACT

In this paper we completed the classification of the phase portraits in the Poincaré disc of uniform isochronous cubic and quartic centers previously studied by several authors. There are three and fourteen different topological phase portraits for the uniform isochronous cubic and quartic centers respectively.

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1. Introduction and statement of the main results

The interest in the isochronous centers started in the XVII century with the works of C. Huygens, see [1]. The isochronicity phenomena appear in many physical problems, see for instance [2].

We say that $p \in \mathbb{R}^2$ is a *center* if it is a singular point of a planar differential system such that there is a neighborhood U of p where all the orbits of $U\backslash\{p\}$ are periodic. For every $q \in U\backslash\{p\}$ let $T(q)$ denote the period of the periodic orbit through q. When $T(q)$ is constant for all $q \in U\backslash\{p\}$ we say that p is an *isochronous center*. The fact that p is isochronous does not imply that the angular velocity of the vector $\overrightarrow{pq}$ is the same for all periodic orbits in $U\backslash\{p\}$. When such velocity is constant we say that p is a *uniform isochronous center* or a *rigid center*.

The uniform isochronous planar centers are characterized in the next result.

Proposition 1. *Assume that a planar polynomial differential system of degree n has a center at the origin of coordinates. Then this center is uniform isochronous if and only if by doing a linear change of variables and a scaling of time it can be written as*

$$\dot{x} = -y + x f(x, y), \qquad \dot{y} = x + y f(x, y), \tag{1}$$

with $f(x, y)$ a polynomial in x and y of degree $n-1$, $f(0, 0) = 0$.

Proposition 1 is well-known, a proof of it can be found in [3].

The next result due to Collins [4] in 1997, also obtained by Devlin, Lloyd and Pearson [5] in 1998, and by Gasull, Prohens and Torregrosa [6] in 2005 characterizes the uniform isochronous centers of cubic polynomial systems.

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Theorem 2. *A cubic polynomial differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\dot{x} = -y + xf(x, y), \qquad \dot{y} = x + yf(x, y), \tag{2}$$

where $f(x, y) = a_1x + a_2y + a_3x^2 + a_4xy - a_3y^2$, *and satisfies* $a_1^2a_3 - a_2^2a_3 + a_1a_2a_4 = 0$.

We note that the proof of Theorem 2 is obtained doing an affine transformation and a rescaling of the time.
Theorem 2 can be improved as follows.

Corollary 3. *A cubic polynomial differential system has a uniform isochronous center at the origin if and only if it can be written as*

$$\dot{x} = -y + xf(x, y), \qquad \dot{y} = x + yf(x, y), \tag{3}$$

where $f(x, y) = a_1x + a_2y + a_4xy$, *and satisfies* $a_1a_2 = 0$ *and* $a_4 \neq 0$.

Corollary 3 is proved in Section 2. We note that (3) has one parameter less than in (2) and eases the polynomial condition that the parameters must hold.

Using Theorem 2 Collins obtains two normal forms for all the uniform isochronous cubic centers, one with one parameter and the other a given system. We prefer to work with the unique normal form given in Corollary 3.

In the next theorem we present the first integrals of the uniform isochronous cubic centers described by systems (3). More complicated first integrals were obtained in [7] using the normal form (2) and were given only in polar coordinates. The new normal form (3) allows to provide easier expressions of these first integrals in Cartesian coordinates.

Theorem 4. *The first integrals H of system* (3) *are described in what follows.*

Case 1: $\mathbf{a_1^2 + a_2^2 \neq 0}$.

Subcase 1.1: $\mathbf{4a_4 > a_1^2}$ *and* $\mathbf{a_2 = 0}$.

$$H = e^{-2\arctan\left[\frac{2a_4y+a_1}{S}\right]}\left[\frac{x^2+y^2}{a_4y^2+a_1y+1}\right]^{S/a_1},$$

where $S = \sqrt{4a_4 - a_1^2}$.

Subcase 1.2: $\mathbf{4a_4 < a_1^2}$ *and* $\mathbf{a_2 = 0}$.

$$H = \frac{(x^2+y^2)^{S/a_1}(S + a_1 + 2a_4y)^{1-S/a_1}}{(S - a_1 - 2a_4y)^{1+S/a_1}},$$

where $S = \sqrt{a_1^2 - 4a_4}$.

Subcase 1.3: $\mathbf{4a_4 < -a_2^2}$ *and* $\mathbf{a_1 = 0}$.

$$H = e^{-2\arctan\left[\frac{2a_4x+a_2}{S}\right]}\left[\frac{x^2+y^2}{a_4x^2+a_2x-1}\right]^{S/a_2},$$

where $S = \sqrt{-4a_4 - a_2^2}$.

Subcase 1.4: $\mathbf{4a_4 > -a_2^2}$ *and* $\mathbf{a_1 = 0}$.

$$H = \frac{(x^2+y^2)^{S/a_2}(S + a_2 + 2a_4x)^{1-S/a_2}}{(S - a_2 - 2a_4x)^{1+S/a_2}},$$

where $S = \sqrt{4a_4 + a_2^2}$.

Subcase 1.5: $\mathbf{4a_4 = a_1^2}$ *and* $\mathbf{a_2 = 0}$.

$$H = \frac{(x^2+y^2)e^{\frac{4}{2+a_1y}}}{(2+a_1y)^2}.$$

Subcase 1.6: $\mathbf{4a_4 = -a_2^2}$ *and* $\mathbf{a_1 = 0}$.

$$H = \frac{(x^2+y^2)e^{\frac{4}{2-a_2x}}}{(2-a_2x)^2}.$$

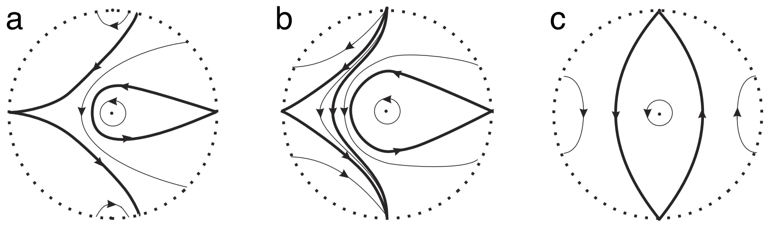

Fig. 1. Phase portraits of uniform isochronous cubic centers.

Case 2: $\mathbf{a_1 = a_2 = 0}$.

$$H = \frac{x^2 + y^2}{1 - a_4 x^2}.$$

Theorem 4 is proved in Section 2.

Collins [4] found that differential systems with uniform isochronous cubic centers may have three topologically different phase portraits, that is the phase portraits denoted by (a), (b) and (c) in Fig. 1. These results are correct but some steps in their proofs are not and other steps are not completely justified, see Section 2. Also in [7] the authors omitted one separatrix in one of the phase portraits that they took from [4]. In short the three correct phase portraits of uniform isochronous cubic centers are given in Fig. 1, for more details see Section 2.

More precisely, a phase portrait of (3) is topologically equivalent to the phase portrait (a) of Fig. 1 if one of the following conditions holds

- $a_4 < -a_2^2/4 < 0$ and $a_1 = 0$;
- $0 < a_1^2/4 < a_4$ and $a_2 = 0$;

the phase portrait (b) if one of the following conditions holds

- $-a_2^2/4 \leq a_4 < 0$ and $a_1 = 0$;
- $0 < a_4 \leq a_1^2/4$ and $a_2 = 0$;

the phase portrait (c) if one of the following conditions holds

- $a_1 = 0$, $a_2 \neq 0$, and $a_4 > 0$;
- $a_1 \neq 0$, $a_2 = 0$, and $a_4 < 0$;
- $a_1 = a_2 = 0$.

In what follows we study the phase portraits of uniform isochronous quartic centers. The first studies on some of these phase portraits are due to Algaba et al. [8]. The phase portraits of uniform isochronous quartic centers whose nonlinear part is not homogeneous were studied in [3], and the ones whose nonlinear part is homogeneous in [9]. As we shall see in Section 3, the study done in [3] has some mistakes.

We denote by $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. The following result classifies all the phase portraits of the uniform isochronous quartic centers.

Theorem 5. *Consider a quartic polynomial vector field X and assume that X has a uniform isochronous center at the origin. Then the phase portrait of X is topologically equivalent to the phase portraits (a), (m) or (n) of Fig. 2 when the nonlinear part of the system is homogeneous. When this nonlinear part is not homogeneous then the phase portrait of X is topologically equivalent to one of the 12 phase portraits (from (a) to (l)) of Fig. 2 according to the following conditions:*

(a) *of Fig. 2 if either $C_1C_3 > 0$, or $C_3 = 0$ and $B_2 < 0$, or $C_1 = 0$ and $C_3 \neq 0$, or $r_3 = r_2 = r_1$, $\forall r_1, r_2, r_3 \in \mathbb{R}^*$, or $r_1 \neq 0$ and $r_{2,3} = a \pm bi$, $\forall r_1, b \in \mathbb{R}^*$, $a \in \mathbb{R}$;*
(b) *of Fig. 2 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1, r_2, r_3 > 0$, or $r_1, r_2, r_3 < 0$ or $r_1r_2 > 0$, $r_3 = r_2$, or $r_2 = r_1$, $r_1r_3 > 0$;*
(c) *of Fig. 2 if $C_1 = 0$, $C_3 \neq 0$ and if either $r_1 < 0$, $r_2, r_3 > 0$, or $r_1, r_2 < 0$, $r_3 > 0$, or $r_1 < 0$, $r_2 > 0$, $r_3 = r_2$, or $r_2 = r_1$, $r_1 < 0$, $r_3 > 0$;*
(d) *of Fig. 2 if $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 > -A_1B_2$;*
(e) *of Fig. 2 if either $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 = -A_1B_2$, or $B_2 = C_3 = 0$;*
(f) *of Fig. 2 if $C_3 = 0$, $C_1 \neq 0$, $B_2 > 0$, $C_1 < -A_1B_2$;*
(g) *or (h) or (i) of Fig. 2 if $C_1C_3 < 0$, $B_2 = 0$;*
(h) *or (k) or (l) of Fig. 2 if $C_1C_3 < 0$, $B_2 \neq 0$;*

where in the cases with $C_1 = 0$, we have that r_1, r_2, r_3 are the roots of the polynomial $-C_3 - B_2x - A_1x^2 - x^3$ and we assume that $r_1 \leq r_2 \leq r_3$ when these roots are real.

Theorem 5 is proved in Section 3.

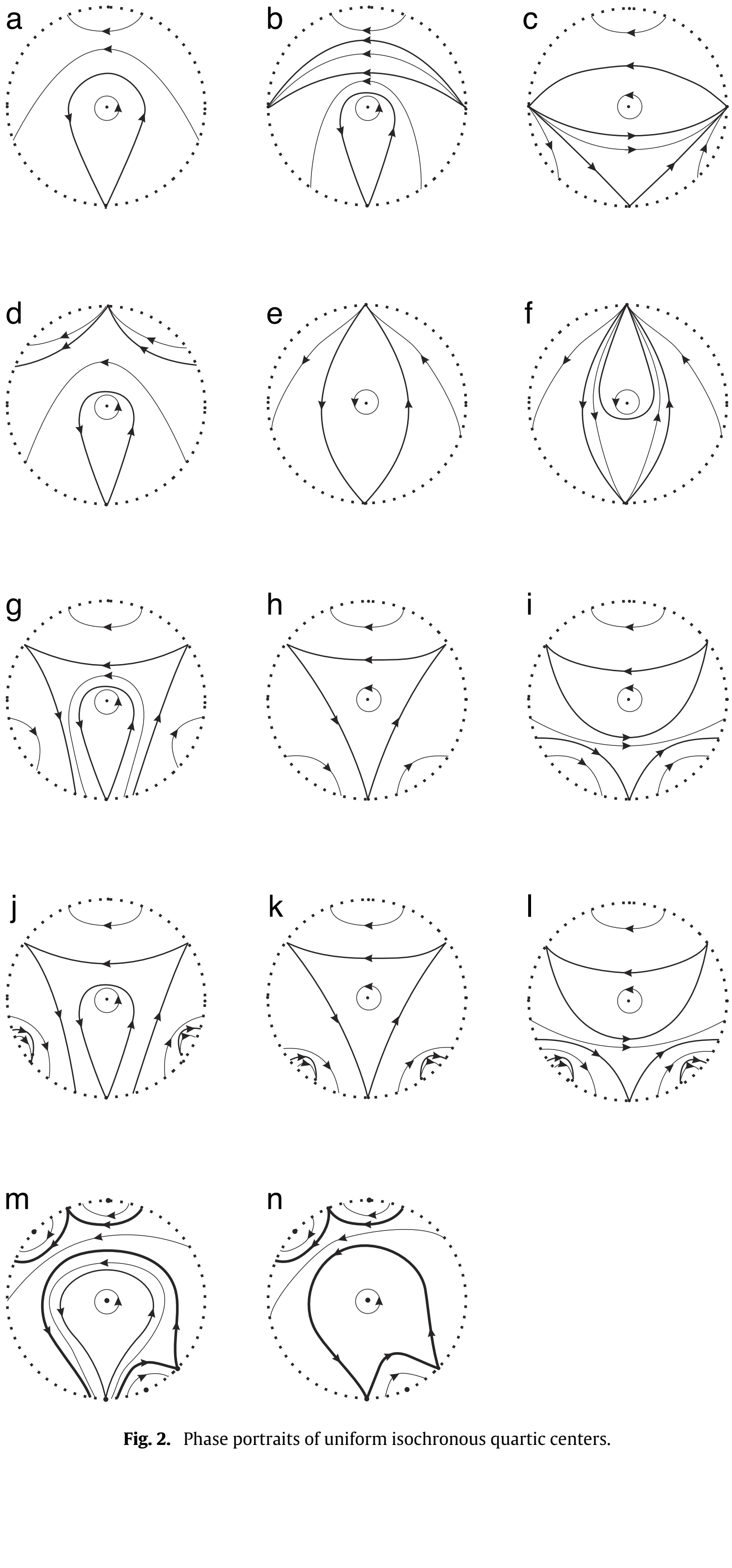

Fig. 2. Phase portraits of uniform isochronous quartic centers.

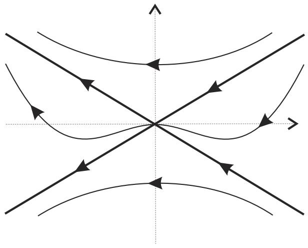

Fig. 3. Correct local phase portrait corresponding to Figure 20 of [3].

2. Uniform isochronous cubic centers

In this section we prove all our results on uniform isochronous cubic centers.

Proof of Corollary 3. The corollary follows doing the change of variables to system (2) given by the rotation

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},$$

with $\cot\alpha = -a_4/(2a_3)$, and renaming the parameters. □

Proof of Theorem 4. It is easy to check that $(\partial H/\partial x)\dot{x} + (\partial H/\partial y)\dot{y} = 0$ for the functions H given in the statement of Theorem 4, and hence they are first integrals of this system under the corresponding conditions. □

Since planar polynomial differential systems of degree >1 with uniform isochronous centers have the infinity filled with singular points (this is an immediate consequence of their normal form (1)), one must study the singular points that remain on the line at infinity after removing it which we call *the singular points of the reduced infinity*. Then for the uniform isochronous cubic centers it happens that the reduced infinity always has two singular points, a saddle and an anti-saddle which may be either a focus, or a center, or a topological node. In the case focus or center we obtain phase portrait (a) or (c) respectively. It is not possible to have a saddle–node as Collins claimed in Figure 2-d of [4], equivalent to our Fig. 1(b). In fact our Fig. 1(b) is also equivalent to Collins' Figure 2-c. This Figure 2-c is what one obtains when the node at infinity is generic (and produces then two invariant straight lines in the equations used by Collins), but the transition from a node to a focus cannot occur by means of a saddle–node, but in this case it occurs by means of one-directional node, and the two previous invariant straight lines become a single one. We note that this transition from node to focus can also occur in other systems by means of a star node. Note also that the normal form (3) does not exhibit invariant straight lines but this does not affect the topological phase portraits. The existence of the invariant straight lines facilitates the study of the phase portraits.

It is also worth of remark that the transition from the phase portraits (b)–(c) cannot be done inside the cubic systems, because when $a_4 = 0$ the system becomes quadratic.

3. Uniform isochronous quartic centers

In this section we prove Theorem 5 on uniform isochronous quartic centers. Since this theorem was already proved in [3] for the non-homogeneous nonlinear part and in [9] for the homogeneous nonlinear part, we will only detail the cases which were not correct in [3]. All steps in [9] are correct.

The first mistake is that the pairs $(a) \equiv (f)$, $(b) \equiv (d)$ and $(c) \equiv (e)$ of phase portraits of Figure 1 of [3] are topologically equivalent. These three pairs correspond to our phase portraits (a), (b) and (c) respectively. Moreover we have improved phase portraits (b) and (c) by reducing the width of one orbit which resembled a separatrix, and we have added two orbits in phase portrait (c) because the orbits of two canonical regions were not defined.

The second mistake is that the phase portraits (g) and (h) of Figure 1 of [3] (denoted here with (d) and (e) respectively) are not correct because the orbits in two canonical regions are not well drawn. The problem arises from an incorrect blow-up of a singularity. Precisely, the incorrect blow-up is the one corresponding to Figure 20 of [3]. The correct Figure 20 is our Fig. 3. This produces two parabolic sectors that do not appear in (g) and (h) of Figure 1 of [3].

The third and last mistake is that a phase portrait is missing in Figure 1 of [3]. Concretely our phase portrait (f) was omitted. In fact, phase portraits (d) and (f) bifurcate from (e) according to the sign of $C_1 + A_1B_2$.

We have finally renamed and reordered the phase portraits from (i) to (n) of Figure 1 of [3] to our (g) to (l). And we have finally added the results from [9] in this new complete theorem for uniform isochronous quartic centers.

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