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On compact representations for the solutions of linear difference equations with variable coefficients

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ABSTRACT

A comprehensive treatment on compact representations for the solutions of linear difference equations with variable coefficients, of both n th and unbounded order, is presented. The equivalence between their celebrated combinatorial and determinantal representations is considered. A corresponding representation is proposed using determined nested sums of their variable coefficients. It makes explicit all the sum of products involved in the previous representations of such solutions. Some basic applications are also illustrated.

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1. Introduction

Compact representations for the solutions of linear difference equations with variable coefficients, LDE for short, of both finite and unbounded order are of interest in many branches of science and engineering; see e.g. [1]. Some approaches for representing the solutions of LDE have been introduced in the literature. Among these, most noteworthy have been the determinantal representations [2,3], and the combinatorial one [4]. The determinantal representation uses hessenbergians, determinants of Hessenberg submatrices (see e.g. [5,6]) of a single solution matrix. The combinatorial representation is based on determined combinations of sums of products of their variable coefficients.

The nested sums have resulted to be useful for obtaining explicit representations of complex combinatorial formulas, e.g. with binomial, Gaussian binomial, or Stirling-like coefficients [7]. These nested structures have been applied on the expansion of transcendental functions and multiscale multiloop integrals [8], on orthogonal polynomials, linear non-autonomous area-preserving maps, representations for the inverses of tridiagonal matrices, and also on continued fractions; see e.g. [9,10] and the references therein. Relative to LDE, the nested sums are suitable for representations of the solutions of parameterized LDE [11], and of the second-order LDE [9].

Our purpose is twofold. First, it is natural to consider the equivalence of the hessenbergian representation for the solutions of LDE [2,3], with respect to the combinatorial one [4]. Furthermore, it is also of use to establish simpler representations of such solutions. The suitability of the nested sums regarding more explicit representation for the solutions of LDE will be pointed up.

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The outline is as follows. A comprehensive treatment on compact representations for the solutions of LDE of both n th and unbounded order is discussed in Section 2. The equivalence of the hessenbergian representation respecting the combinatorial one is checked. A more explicit representation for the solutions of LDE of unbounded order based on nested sums is proposed in Section 3. Although it presents a more involved problem, a representation for the n th order LDE with nested sums is also introduced. We illustrate in Section 4 with basic examples of their potential applications. Thus compact representations based on nested sums for hessenbergians, inverses of triangular matrices, multinomial distribution, and the Roger–Szegő polynomials, are managed.

2. Representations for the solutions of LDE and their equivalence

The equivalence between the hessenbergian [2,3] and the combinatorial representation [4] is checked. With this aim, we begin focusing on the notation and results about the hessenbergian representation given in [2].

2.1. LDE of unbounded order

Following [2], a LDE of unbounded order can be formulated as

$$\sum_{i=1}^k p(k, i)y_i = f(k), \quad (1)$$

the coefficients $p(k, i)$, and the nonhomogeneous terms $f(k)$, are known functions. Here the coefficients $p(k, k)$ satisfying $p(k, k) \neq 0$, for every $k \in \mathbb{Z}^+$. Since y_n depends only on y_1, y_2, \dots, y_{n-1} , the first n equations allow us to attain y_n . Indeed, given the (infinite) lower unreduced Hessenberg matrix

$$\mathbf{R} = \begin{bmatrix} f(1) & p(1, 1) & 0 & 0 & \cdots \\ f(2) & p(2, 1) & p(2, 2) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (2)$$

a representation of the solutions of (1) using hessenbergians [5,6] is

$$y_n = \frac{(-1)^{n-1}}{\prod_{i=1}^n p(i, i)} \det \mathbf{R}_n. \quad (3)$$

The finite matrix \mathbf{R}_n ,

$$\mathbf{R}_n = \begin{bmatrix} f(1) & p(1, 1) & 0 & \cdots & 0 \\ f(2) & p(2, 1) & p(2, 2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ f(n-1) & p(n-1, 1) & p(n-1, 2) & \cdots & p(n-1, n-1) \\ f(n) & p(n, 1) & p(n, 2) & \cdots & p(n, n-1) \end{bmatrix}, \quad (4)$$

is the n th section of the matrix \mathbf{R} . Using elementary properties of the determinants, and defining the ratios $x_i = \frac{f(i)}{p(i, i)}$, $b_{ki} = -\frac{p(k, i)}{p(k, k)}$, formula (3) yields

$$y_n = \det \mathbf{R}_n^* = |\mathbf{R}_n^*|, \quad (5)$$

with the matrix

$$\mathbf{R}_n^* = \begin{bmatrix} x_1 & -1 & 0 & \cdots & 0 \\ x_2 & b_{21} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ x_{n-1} & b_{n-1,1} & b_{n-1,2} & \cdots & -1 \\ x_n & b_{n1} & b_{n2} & \cdots & b_{n,n-1} \end{bmatrix}. \quad (6)$$

Furthermore, expanding the hessenbergian (5) by the first column of the matrix \mathbf{R}_n^* ,

$$y_n = \sum_{i=1}^n C_{i,1}^{(n)} x_i, \quad (7)$$

where the $C_{i,1}^{(n)} = (-1)^{i+1} \det(\mathbf{R}_n^*)_{i,1}$ are the cofactors of the first column of \mathbf{R}_n^* . This form is similar to Eq. (2) given in [4]. The cofactors $C_{i,1}^{(n)}$ are also hessenbergians, $C_{i,1}^{(n)} = \det \mathbf{R}_n^*(i+1:n, i+1:n)$,

$$C_{i,1}^{(n)} = \begin{vmatrix} b_{i+1,i} & -1 & 0 & \cdots & 0 \\ b_{i+2,i} & b_{i+2,i+1} & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ b_{n-1,i} & b_{n-1,i+1} & b_{n-1,i+2} & \cdots & -1 \\ b_{n,i} & b_{n,i+1} & b_{n,i+2} & \cdots & b_{n,n-1} \end{vmatrix}. \quad (8)$$

Notice $C_{n,1}^{(n)} = 1$. Comparing (7) and (8) with the combinatorial form given in [4, Eqs. (2)–(3)], the equivalence between two representations for the solutions of the unbounded LDE (1) comes out.

Proposition 2.1. *The hessenbergians (8) have the compact representations*

$$C_{i,1}^{(n)} = \begin{cases} b_{n,i} + \sum_{j=2}^{n-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ l_1, l_2, \dots, l_j \geq 1 \\ l_1 + l_2 + \dots + l_j = n-i}} b_{n, n-l_1} \left[\prod_{m=2}^j b_{n - \sum_{k=1}^{m-1} l_k, n - \sum_{k=1}^m l_k} \right], & \text{if } 1 \leq i < n-1; \\ b_{n, n-1}, & \text{if } i = n-1; \\ 1, & \text{if } i = n. \end{cases} \quad (9)$$

Proof. The values for $C_{n,1}^{(n)}$ and $C_{n-1,1}^{(n)}$ are obtained trivially. We use (complete) mathematical induction for proving the remaining cofactors. Notice $C_{i,1}^{(i)} = 1$, $C_{i,1}^{(i+1)} = b_{i+1,i}$, and $C_{i,1}^{(i+2)} = b_{i+2,i} + b_{i+2,i+1}b_{i+1,i}$, satisfying (9) for $n = i$, $n = i+1$, and $n = i+2$, respectively.

Assume that the cofactors $C_{i,1}^{(n-r)}$ satisfy (9) when $1 \leq r < n-i$, and $r \in \mathbb{Z}^+$. Thus

$$\begin{aligned} C_{i,1}^{(n-r)} &= b_{n-r,i} + \sum_{j=2}^{n-r-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ l_1, l_2, \dots, l_j \geq 1 \\ l_1 + l_2 + \dots + l_j = n-r-i}} b_{n-r, n-r-l_1} \left[\prod_{m=2}^j b_{n-r - \sum_{k=1}^{m-1} l_k, n-r - \sum_{k=1}^m l_k} \right] \\ &= \sum_{j=1}^{n-r-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ l_1, l_2, \dots, l_j \geq 1 \\ l_1 + l_2 + \dots + l_j = n-r-i}} b_{n-r, n-r-l_1} \left[\prod_{m=2}^j b_{n-r - \sum_{k=1}^{m-1} l_k, n-r - \sum_{k=1}^m l_k} \right]. \end{aligned} \quad (10)$$

We have taken in (10) the usual conventions on sums and products. Expanding $C_{i,1}^{(n)}$ from (8) by the last row and taking into account the previous assumptions on the hessenbergians $C_{i,1}^{(n-r)}$ involved in such an expansion,

$$\begin{aligned} C_{i,1}^{(n)} &= b_{n,i} + \sum_{r=1}^{n-i-1} b_{n, n-r} C_{i,1}^{(n-r)} \\ &= b_{n,i} + \sum_{r=1}^{n-i-1} b_{n, n-r} \sum_{j=1}^{n-r-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ l_1, l_2, \dots, l_j \geq 1 \\ l_1 + l_2 + \dots + l_j = n-r-i}} b_{n-r, n-r-l_1} \left[\prod_{m=2}^j b_{n-r - \sum_{k=1}^{m-1} l_k, n-r - \sum_{k=1}^m l_k} \right]. \end{aligned}$$

We conclude the proof relabeling the dummy indexes $l_i \rightarrow l_{i+1}$, $i = 1, 2, \dots, j$, and $r \rightarrow l_1$.

$$\begin{aligned} C_{i,1}^{(n)} &= b_{n,i} + \sum_{l_1=1}^{n-i-1} \sum_{j=1}^{n-l_1-i} \sum_{\substack{(l_2, l_3, \dots, l_{j+1}) \\ l_2, l_3, \dots, l_{j+1} \geq 1 \\ l_2 + l_3 + \dots + l_{j+1} = n-l_1-i}} b_{n, n-l_1} \left[\prod_{m=2}^{j+1} b_{n-l_1 - \sum_{k=2}^{m-1} l_k, n-l_1 - \sum_{k=2}^m l_k} \right] \\ &= b_{n,i} + \sum_{j=2}^{n-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ l_1, l_2, \dots, l_j \geq 1 \\ l_1 + l_2 + \dots + l_j = n-i}} b_{n, n-l_1} \left[\prod_{m=2}^j b_{n - \sum_{k=1}^{m-1} l_k, n - \sum_{k=1}^m l_k} \right]. \quad \square \end{aligned}$$

Remark 2.2. The previous combinatorial formula was introduced in [4] for the solutions of unbounded LDE. When it is applied on the hessenbergians (8), gives a more explicit representation than Leibnitz's formula. However, in order to expand the formula (9) for obtaining all the sums of products involved, a Diophantine equation on the dummy natural indexes l_j should be solved, whose complexity increases with $n - i$. For simpler formulas of hessenbergians using nested sums, see Section 4.1.

2.2. LDE of order n

A general n th order LDE was considered in [2] under the form

$$\sum_{i=0}^n q(k, i+k)y_{i+k} = g(k), \quad (11)$$

with initial conditions $y_j = c_j, j = 1, 2, \dots, n$. The $g(k)$ are known functions. It is an initial value problem, discrete analogous of Cauchy's problem in differential equations. A determinantal solution was also provided in [2],

$$y_{m+n} = \frac{(-1)^{m-1}}{\prod_{i=1}^m q(i, i+n)} \det \mathbf{S}_m, \quad (12)$$

with the $m \times m$ matrix \mathbf{S}_m having the entries, $1 \leq i, j \leq m$,

$$[\mathbf{S}_m]_{i,j} = \begin{cases} g(i) - \sum_{r=0}^{n-i} q(i, i+r)c_{i+r}, & \text{if } j = 1, 1 \leq i \leq n; \\ g(i), & \text{if } j = 1, n < i; \\ q(i, n+j-1), & \text{if } 1 < j, j-1 \leq i \leq n+j-1; \\ 0, & \text{otherwise.} \end{cases}$$

Defining $x_i = \frac{[\mathbf{S}_m]_{i,1}}{q(i, n+i)}$, and $b_{i,j} = \frac{[\mathbf{S}_m]_{i,j+1}}{q(i, n+i)}$, the solution (12) takes the form,

$$y_{m+n} = \det \mathbf{S}_m^*, \quad (13)$$

with the $m \times m$ lower Hessenberg matrix \mathbf{S}_m^* as the matrix \mathbf{R}_m^* given in (6), but now $b_{ij} = 0$ for $i > n+j$. That is, if $m > n+1$, the submatrix $\mathbf{S}_m^*(2 : m, 2 : m)$ is also a Hessenberg banded matrix of bandwidth $n+1$.

Example 2.3. As an illustration, the solution (13), with $m+n = 8$ and $n = 3$, for the 3-th order LDE as given in (11), is considered. The set of solutions of a general n th order LDE has structure of affine space over the vectorial space of solutions of the corresponding homogeneous LDE. This algebraic structure is showed by taking, for $n = 3$, the solution in the usual form $y_{m+3} = y_{m+3}^{(h)} + y_{m+3}^{(p)}$. Here $y_{m+3}^{(h)} = \lambda_1 y_1(m+3) + \lambda_2 y_2(m+3) + \lambda_3 y_3(m+3)$, is the general solution of the homogeneous part. The λ_i and $y_i, i = 1, 2, 3$, are arbitrary complex coefficients and the canonical homogeneous solutions, respectively. The remaining term $y_{m+3}^{(p)}$ is a particular solution of the general nonhomogeneous 3-th order LDE.

In the initial value problem the initial conditions are fixed, $y_i^{(h)} = c_i$, and $i = 1, 2, 3$. Hence, the general solution of the homogeneous part is

$$y_{5+3}^{(h)} = c_1 \begin{vmatrix} -\frac{q(1,1)}{q(1,4)} & -1 & 0 & 0 & 0 \\ 0 & b_{21} & -1 & 0 & 0 \\ 0 & b_{31} & b_{32} & -1 & 0 \\ 0 & b_{41} & b_{42} & b_{43} & -1 \\ 0 & 0 & b_{52} & b_{53} & b_{5,4} \end{vmatrix} + c_2 \begin{vmatrix} -\frac{q(1,2)}{q(1,4)} & -1 & 0 & 0 & 0 \\ -\frac{q(2,2)}{q(2,5)} & b_{21} & -1 & 0 & 0 \\ 0 & b_{31} & b_{32} & -1 & 0 \\ 0 & b_{41} & b_{42} & b_{43} & -1 \\ 0 & 0 & b_{52} & b_{53} & b_{5,4} \end{vmatrix} \\ + c_3 \begin{vmatrix} -\frac{q(1,3)}{q(1,4)} & -1 & 0 & 0 & 0 \\ -\frac{q(2,3)}{q(2,5)} & b_{21} & -1 & 0 & 0 \\ -\frac{q(3,3)}{q(3,6)} & b_{31} & b_{32} & -1 & 0 \\ 0 & b_{41} & b_{42} & b_{43} & -1 \\ 0 & 0 & b_{52} & b_{53} & b_{5,4} \end{vmatrix}.$$

Null initial conditions are chosen for the particular solution $y_{m+3}^{(p)}$,

$$y_{5+3}^{(p)} = \begin{bmatrix} \frac{g(1)}{q(1,4)} & -1 & 0 & 0 & 0 \\ \frac{g(2)}{q(2,5)} & b_{21} & -1 & 0 & 0 \\ \frac{g(3)}{q(3,6)} & b_{31} & b_{32} & -1 & 0 \\ x_4 & b_{41} & b_{42} & b_{43} & -1 \\ x_5 & 0 & b_{52} & b_{53} & b_{5,4} \end{bmatrix}.$$

Therefore in agreement with (13),

$$y_8 = y_{5+3}^{(h)} + y_{5+3}^{(p)} = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 \\ x_2 & b_{21} & -1 & 0 & 0 \\ x_3 & b_{31} & b_{32} & -1 & 0 \\ x_4 & b_{41} & b_{42} & b_{43} & -1 \\ x_5 & 0 & b_{52} & b_{53} & b_{5,4} \end{bmatrix}.$$

A combinatorial representation for the hessenbergians (13) is straightforward using Proposition 2.1.

Proposition 2.4. The hessenbergians (13) have the compact representations

$$y_{m+n} = \sum_{i=1}^m C_{i,1}^{*(m)} x_i, \quad \text{where}$$

$$C_{i,1}^{*(m)} = \begin{cases} \sum_{j=1}^{m-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ 1 \leq l_1, l_2, \dots, l_j \leq n \\ l_1 + l_2 + \dots + l_j = m-i}} b_{m, m-l_1} \left[\prod_{r=2}^j b_{m - \sum_{k=1}^{r-1} l_k, m - \sum_{k=1}^r l_k} \right], & \text{if } 1 \leq i \leq m-1; \\ 1, & \text{if } i = m. \end{cases} \quad (14)$$

Proof. Expanding the hessenbergians (13) by their first columns, the formula for y_{m+n} using the cofactors $C_{i,1}^{*(m)}$ is trivial. The cofactors $C_{i,1}^{*(m)}$ are as given in (9), but now as a general rule, the lower Hessenberg matrices involved are also banded matrices. The null entries outer the bandwidth should be avoided in the sum of products. Therefore, the conditions on the indexes should be modified. Note that the entries involved in such a combinatorial representation have the form $b_{k+l_r, k}$, $k \in \mathbb{Z}^+$. The nonzero entries satisfy the conditions $k + l_r - k \leq n$. Therefore, $l_r \leq n$, $r = 1, 2, \dots, j$, and the new condition for the indexes is simply $1 \leq l_1, l_2, \dots, l_j \leq n$. \square

The n th order LDE was also considered in [4], where a slight different formulation was introduced. A combinatorial representation for the solution was provided in [4], Proposition 2. Its equivalence with respect to the representation (14) can be obtained introducing adequate changes of indexes.

3. A representation for the solutions of LDE with nested sums

3.1. LDE of unbounded order

We consider the nested sums representing a sum of products; see e.g. [7]. That is, for $i < n = k_0, k_j, j \in \mathbb{Z}^+$, we handle the nested sums

$$\sum_{k_1=i}^{k_0-1} \sum_{k_2=i}^{k_1-1} \cdots \sum_{k_j=i}^{k_{j-1}-1} f(k_0, k_1, \dots, k_j), \quad (15)$$

where the functions $f(k_0, k_1, \dots, k_j) = \prod_{m=1}^j g(k_{m-1}, k_m)$, and $g(k_{m-1}, k_m)$ are known functions on the indexes. The index j gives the depth (level) of each nested structure, i.e. the number of sums involved starting by the left. Such a nested structure of level j , for $j = 1, 2, \dots, n-i$, results in a sum of j -tuples, products with j elements.

We now relate the nested sums (15) with the solutions of LDE of unbounded order via formula (9), by choosing $g(k_{m-1}, k_m) = b_{k_{m-1}, k_m}$, the matrix entries from (8), to obtain $f(k_0, k_1, \dots, k_j) = \prod_{m=1}^j b_{k_{m-1}, k_m}$. As an advantage of using nested sums for compact representations of the solutions of LDE, all the sums of products involved are given explicitly.

Theorem 3.1. A representation with nested sums for the hessenbergians (8) is given by

$$C_{i,1}^{(n)} = \begin{cases} b_{n,i} + \sum_{j=2}^{n-i} \sum_{k_1=i+j-1}^{k_0-1} \sum_{k_2=i+j-2}^{k_1-1} \cdots \sum_{k_{j-1}=i+1}^{k_{j-2}-1} \prod_{m=1}^{j-1} b_{k_{m-1},k_m} b_{k_{j-1},i}, & \text{if } i \leq n-1; \\ 1, & \text{if } i = n. \end{cases} \quad (16)$$

Proof. We establish the result from the combinatorial representation of $C_{i,1}^{(n)}$ (9), by introducing determined nested sums (15) to solve the Diophantine equations on the indexes that appear in (9). Indeed, for $i \leq n-1$,

$$\begin{aligned} C_{i,1}^{(n)} &= \sum_{j=1}^{n-i} \sum_{\substack{(l_1, l_2, \dots, l_j) \\ l_1, l_2, \dots, l_j \geq 1 \\ l_1 + l_2 + \dots + l_j = n-i}} b_{n, n-l_1} \left[\prod_{m=2}^j b_{n-\sum_{k=1}^{m-1} l_k, n-\sum_{k=1}^m l_k} \right] \\ &= \sum_{l_1=1}^{n-i} b_{n, n-l_1} \sum_{j=2}^{n-i-l_1} \sum_{\substack{(l_2, \dots, l_j) \\ l_2, \dots, l_j \geq 1 \\ l_2 + \dots + l_j = n-l_1-i}} b_{n-l_1, n-l_1-l_2} \left[\prod_{m=3}^j b_{n-\sum_{k=1}^{m-1} l_k, n-\sum_{k=1}^m l_k} \right] \\ &= \sum_{l_1=1}^{n-i} b_{n, n-l_1} C_{i,1}^{(n-l_1)}. \end{aligned}$$

Using again the same procedure, we obtain

$$C_{i,1}^{(n)} = \sum_{l_1=1}^{n-i} b_{n, n-l_1} \sum_{l_2=1}^{n-i-l_1} b_{n-l_1, n-l_1-l_2} C_{i,1}^{(n-l_1-l_2)}.$$

After $n-i$ iterations, it yields

$$C_{i,1}^{(n)} = \sum_{l_1=1}^{n-i} b_{n, n-l_1} \sum_{l_2=1}^{n-i-l_1} b_{n-l_1, n-l_1-l_2} \cdots \sum_{l_{n-i}=1}^{n-i-\sum_{m=1}^{n-i-1} l_m} b_{n-\sum_{m=1}^{n-i-1} l_m, n-\sum_{m=1}^n l_m}.$$

The following change of dummy indexes simplifies notably the formula, $k_1 = n - l_1$, $k_m = k_{m-1} - l_m$, and $m = 2, \dots, j$,

$$C_{i,1}^{(n)} = \sum_{k_1=n-1}^i b_{n, k_1} \sum_{k_2=k_1-1}^i b_{k_1, k_2} \cdots \sum_{k_j=k_{j-1}-1}^i b_{k_{j-1}, k_j}.$$

Taking $n = k_0$ by convenience in the notation, after a trivial rearranging of indexes, we obtain for $i = 1, \dots, n-1$,

$$C_{i,1}^{(n)} = \sum_{k_1=i}^{k_0-1} b_{k_0, k_1} \sum_{k_2=i}^{k_1-1} b_{k_1, k_2} \cdots \sum_{k_j=i}^{k_{j-1}-1} b_{k_{j-1}, k_j}.$$

Using the new indexes k_j , the condition $l_1 + l_2 + \dots + l_j = n-i$ is simply $k_j = i$. Therefore, the preceding formula can be simplified by classifying in terms of the j -level of the nested structures involved in the nested sum to accomplish the simpler representation,

$$\begin{aligned} C_{i,1}^{(n)} &= b_{n,i} + \sum_{j=2}^{n-i} \sum_{k_1=i+j-1}^{k_0-1} b_{k_0, k_1} \sum_{k_2=i+j-2}^{k_1-1} b_{k_1, k_2} \cdots \sum_{k_{j-1}=i+1}^{k_{j-2}-1} b_{k_{j-2}, k_{j-1}} b_{k_{j-1}, i} \\ &= b_{n,i} + \sum_{j=2}^{n-i} \sum_{k_1=i+j-1}^{k_0-1} \sum_{k_2=i+j-2}^{k_1-1} \cdots \sum_{k_{j-1}=i+1}^{k_{j-2}-1} \prod_{m=1}^{j-1} b_{k_{m-1}, k_m} b_{k_{j-1}, i}. \quad \square \end{aligned}$$

Corollary 3.2. A compact representation with nested sums for the solutions of LDE of unbounded order (1) is given by

$$y_n = x_n + \sum_{i=1}^{n-1} b_{n,i} x_i + \sum_{i=1}^{n-2} \sum_{j=2}^{n-i} \sum_{k_1=i+j-1}^{k_0-1} \sum_{k_2=i+j-2}^{k_1-1} \cdots \sum_{k_{j-1}=i+1}^{k_{j-2}-1} \prod_{m=1}^{j-1} b_{k_{m-1}, k_m} b_{k_{j-1}, i} x_i.$$

As an advantage of the representation with nested sums respecting those from [2–4], all the sums of products involved can be handled easily with the usual packages of symbolic computation, e.g. *Maple*®.

3.2. LDE of order n

In a similar way as Proposition 2.4, a representation using nested sums (15), for the solutions y_{m+n} ($m > n + 1$) of the n th order LDE (11), can be derived from Theorem 3.1. In order to avoid the computation of unnecessary zero entries, finer conditions on the limits of the nested sums are compulsory. Notice that for $m \leq n + 1$, the formula (16) is also applicable.

Theorem 3.3. A representation with nested sums for the hessenbergians (13) is given by

$$y_{m+n} = \sum_{i=1}^m C_{i,1}^{*(m)} x_i, \quad \text{where, for } m = k_0 > n + 1,$$

$$C_{i,1}^{*(m)} = \begin{cases} \sum_{j=\alpha}^{m-i} \sum_{k_1=F_{k_0}}^{f_{k_0}} \sum_{k_2=F_{k_1}}^{f_{k_1}} \cdots \sum_{k_{j-1}=F_{k_{j-2}}}^{f_{k_{j-2}}} \prod_{r=1}^{j-1} b_{k_{r-1},k_r} b_{k_{j-1},i}, & \text{if } i \leq m-1; \\ 1, & \text{if } i = m, \end{cases} \quad (17)$$

with $\alpha \in \mathbb{Z}^+$ satisfying $(\alpha - 1)n < m - i \leq \alpha n$. The lower limits are $F_{k_{r-1}} = \max\{i + j - r, k_{r-1} - n\}$. The upper limits are $f_{k_0} = \min\{i + (j - 1)n, k_0 - 1\}$, for $j = \alpha$; $f_{k_0} = m - 1 = k_0 - 1$, for $j > \alpha$; and

$$f_{k_{r-1}} = \begin{cases} \min\{i + (j - r)n, k_{r-1} - 1\}, & \text{if } i + (j - r)n < m - 1; \\ k_{r-1} - 1, & \text{if } i + (j - r)n \geq m - 1, \end{cases}$$

for $r = 2, \dots, j - 1$.

Proof. First, for $(\alpha - 1)n < m - i \leq \alpha n$, the shorter length in the j -tuples that contribute is $j = \alpha$. All the nested structures of greater level, $\alpha + 1, \alpha + 2, \dots, m - i$, also contribute in the nested sum. Besides, the banded structure of bandwidth $n + 1$ on the hessenbergians of type (8), associated to $C_{i,1}^{*(m)}$ from (14), should be incorporate to the limits of summation of the nested sum (16).

For a determined nested structure of level j in representation (16), the condition on the lower limit on the sum of index k_r is $k_r = i + j - r$, where $r = 1, \dots, j - 1$. Here, the upper limit is $k_{r-1} - 1$. Therefore taking into account the bandwidth of the Hessenberg matrix, we incorporate in the representation (17) a new condition $k_{r-1} - n$ in the lower limit of such a sum. Thus, to avoid the null entries outer the bandwidth, the lower limit of summation is $k_r = F_{k_{r-1}} = \max\{i + j - r, k_{r-1} - n\}$.

The first upper limit f_{k_0} is related with the last row m of the lower Hessenberg matrix \mathbf{S}_m^* from (6) and (13). For $j = \alpha = 2$, we consider the possibility that some entries of the last row do not contribute in the α -tuples of the sum of products by defining $f_{k_0} = \min\{i + n, k_0 - 1\}$. For a level $j > 2$, $f_{k_{j-2}} = \min\{i + n, k_{j-2} - 1\}$, $f_{k_{j-3}} = \min\{i + 2n, k_{j-3} - 1\}$, and $f_{k_{r-1}} = \min\{i + (j - r)n, k_{r-1} - 1\}$; for $2 \leq r \leq j - 1$. Since the condition $m - i \leq (j - 1)n$ is accomplished for $j > \alpha$, we can take $f_{k_0} = m - 1 = k_0 - 1$. We take for $j = \alpha$, $f_{k_0} = \min\{i + (j - 1)n, k_0 - 1\}$. \square

4. Some applications

The preceding results regarding compact representations for the solutions of LDE and nested sums are illustrated with some basic applications.

4.1. Generality on hessenbergians

We begin with hessenbergians $|\mathbf{H}_n|$, determinants of $n \times n$ Hessenberg matrices [5,6]. We take without loss of generality lower unreduced Hessenberg matrices of the form $\mathbf{H}_n = \{h_{i,j}\}_{1 \leq i,j \leq n}$, with $h_{i,j} = 0$ when $j - i > 1$ and $h_{i,i+1} \neq 0$, $i = 1, 2, \dots, n - 1$. If some superdiagonal entries are null, $h_{i,i+1} = 0$, we take simply the product of the determinants of the adequate submatrices. From the general representation of a hessenbergian, we derive the representation with entries of value -1 on the superdiagonal by dividing the i th row of \mathbf{H}_n by $-h_{i,i+1}$,

$$|\mathbf{H}_n| = (-1)^{n-1} \prod_{i=1}^{n-1} h_{i,i+1} |\mathbf{B}_n|, \quad (18)$$

where \mathbf{B}_n is a lower Hessenberg matrix with entries $b_{ij} = \frac{-h_{ij}}{h_{i,i+1}}$, $1 \leq i \leq n - 1$, and $b_{nj} = h_{nj}$. It is well known that such a hessenbergian can be computed in $O(n^2)$ time.

A compact representation for hessenbergians (18) using nested sums follows as a consequence of Theorem 3.1. We only must adjust the indexes to the nonzero entries of the Hessenberg matrix \mathbf{B}_n .

Corollary 4.1. A compact representation for the hessenbergians (18) is

$$|\mathbf{H}_n| = (-1)^{n-1} \prod_{i=1}^{n-1} h_{i,i+1} \left(b_{n,1} + \sum_{j=2}^n \sum_{k_1=j}^n \sum_{k_2=j-1}^{k_1-1} \cdots \sum_{k_{j-1}=2}^{k_{j-2}-1} b_{n,k_1} \prod_{m=2}^{j-1} b_{k_{m-1}-1, k_m} b_{k_{j-1}-1, 1} \right). \quad (19)$$

Another consequence of Theorem 3.3 is the representation for the determinants of $m \times m$ banded Hessenberg matrices of bandwidth $n+1 \leq m$.

Corollary 4.2. A representation with nested sum for the hessenbergians (18), of $m \times m$ banded Hessenberg matrices \mathbf{H}_m of bandwidth $n+1 \leq m$, is

$$|\mathbf{H}_m| = (-1)^{m-1} \prod_{i=1}^{m-1} h_{i,i+1} \sum_{j=\alpha}^m \sum_{k_1=F_{k_0}}^{f_{k_0}} \sum_{k_2=F_{k_1}}^{f_{k_1}} \cdots \sum_{k_{j-1}=F_{k_{j-2}}}^{f_{k_{j-2}}} b_{m,k_1} \prod_{r=2}^{j-1} b_{k_{r-1}-1, k_r} b_{k_{j-1}-1, 1},$$

with α the shorter length of the j -tuples, satisfying $(\alpha-1)n < m \leq \alpha n$. The lower limits $F_{k_{r-1}}$, and the upper limits $f_{k_{r-1}}$ ($r = 1, \dots, j-1$) are as given in Theorem 3.3 for $C_{1,1}^{*(m+1)}$. That is, $F_{k_0} = \max\{j, m+1-n\}$,

$$f_{k_0} = \begin{cases} \min\{1+(j-1)n, m\}, & \text{if } j = \alpha; \\ m, & \text{if } j > \alpha, \end{cases}$$

for $r = 1$. $F_{k_{r-1}} = \max\{j - (r-1), k_{r-1} - n\}$,

$$f_{k_{r-1}} = \begin{cases} \min\{1+(j-r)n, k_{r-1}-1\}, & \text{if } 1+(j-r)n < m; \\ k_{r-1}-1, & \text{if } 1+(j-r)n \geq m, \end{cases}$$

for $r = 2, \dots, j-1$.

Leibnitz's formula for hessenbergians

The determinant of an $n \times n$ matrix \mathbf{H}_n is represented currently in compact form using Leibnitz's formula

$$|\mathbf{H}_n| = \sum_{\rho \in S_n} \text{sgn}(\rho) \prod_{i=1}^n h_{i, \rho_i}, \quad (20)$$

where S_n represents the permutation group. The sum is over all $n!$ permutations of the set $\{1, \dots, n\}$, $\text{sgn}(\rho) = \pm 1$ according to whether the permutation ρ is even or odd. The new indexes ρ_i give a new ordering of the set $\{1, \dots, n\}$. Computation of $|\mathbf{H}_n|$ requires about $n!n$ operations, which rapidly becomes infeasible when n increases. In practice, for a general square matrix, $|\mathbf{H}_n|$ is evaluated in $O(n^3)$ time using the *PLU* factorization with pivoting strategies. If in addition matrix \mathbf{H}_n is (lower) Hessenberg, instead of summing over the complete permutation group S_n , one has to sum over the set $S_n^* \subset S_n$, $S_n^* = \{\rho \in S_n : \rho_i \leq i+1\}$, $i < n$, with 2^{n-1} terms, and the *PLU* method supplies $|\mathbf{H}_n|$ in $O(n^2)$ time. Leibnitz's formula for hessenbergians can be also handled using (19) and symbolic computation.

Corollary 4.3. Leibnitz's formula for the hessenbergian of an $n \times n$ (lower) Hessenberg matrix \mathbf{H}_n has the following representation with nested sums

$$|\mathbf{H}_n| = \sum_{\rho \in S_n^*} \text{sgn}(\rho) \prod_{i=1}^n h_{i, \rho_i} \\ = (-1)^{n-1} \prod_{i=1}^{n-1} h_{i,i+1} \left(b_{n,1} + \sum_{j=2}^n \sum_{k_1=j}^n \sum_{k_2=j-1}^{k_1-1} \cdots \sum_{k_{j-1}=2}^{k_{j-2}-1} b_{n,k_1} \prod_{m=2}^{j-1} b_{k_{m-1}-1, k_m} b_{k_{j-1}-1, 1} \right).$$

Compact formulas for the entries of the inverses of triangular matrices

Given an $n \times n$ nonsingular lower triangular matrix $\mathbf{L} = \{l_{ij}\}_{1 \leq i, j \leq n}$, a well-known application of hessenbergians is Cramer's formula for triangular systems $\mathbf{L}\vec{a} = \vec{x}$; see e.g. page 14 from [5]. We denote the vectors in a non-standard form by convenience in the following. The component solutions are $a_i = \frac{1}{l_{ii}} \det \mathbf{R}_i^*$, with the matrix \mathbf{R}_i^* as given in (6), and the entries

$$b_{jk} = -\frac{l_{jk}}{l_{jj}}.$$

Taking $\vec{x} = \vec{e}_i$, and $i = 1, 2, \dots, n$, we obtain trivially a closed form for the entries of the inverse matrix,

$$(\mathbf{L}^{-1})_{ij} = \begin{cases} \frac{1}{l_{ii}} C_{j,1}^{(i)}, & \text{if } i \geq j; \\ 0, & \text{if } i < j. \end{cases}$$

The hessenbergian $C_{j,1}^{(i)} = \det \mathbf{R}_i^*(j+1 : i, j+1 : i)$ is as given in (8), with $b_{jk} = -\frac{l_{jk}}{l_{jj}}$. These are nothing else but compact formulas of the usual forward substitution procedure, column version, for inverting lower triangular matrices.

The representations of the entries $(\mathbf{L}^{-1})_{ij}$ with nested sums are immediate from Theorem 3.1. The stimulus–response relationships among the entries of a triangular matrix and the entries of its inverse are of practical interest. It can be fully controlled using nested sums and symbolic computation.

Triangular Toeplitz matrices. Although some literature for inverting triangular Toeplitz matrices with entries $l_{ij} = l_{i+k,j+k} = t_{i-j}$, is available; see e.g. [12] and the references therein, the representation of their inverse entries is trivial using the preceding result. Since the inverse is also triangular Toeplitz, we only need to evaluate the first column, $(\mathbf{L}^{-1})_{ij} = \begin{cases} \frac{1}{t_0} C_{1,1}^{(i-j+1)}, & \text{if } i \geq j; \\ 0, & \text{if } i < j. \end{cases}$

The hessenbergians $C_{1,1}^{(i-j+1)} = \det \mathbf{R}_1^*(2 : i-j+1, 2 : i-j+1)$ are as given in (8), with $b_{jk} = b_{j-k} = -\frac{t_{j-k}}{t_0}$, $k \leq j$, and $b_{jk} = 0$, for $k > j$. Another immediate consequence of Theorem 3.1 is the following.

Corollary 4.4. A representation with nested sums for the entries of the inverses of (lower) triangular Toeplitz matrices is

$$(\mathbf{L}^{-1})_{ij} = \begin{cases} \frac{1}{t_0} \left(b_{k_0-1} + \sum_{r=2}^{k_0-1} \sum_{k_1=r}^{k_0-1} \sum_{k_2=r-1}^{k_1-1} \cdots \sum_{k_{r-1}=2}^{k_{r-2}-1} \prod_{m=1}^{r-1} b_{k_{m-1}-k_m} b_{k_{r-1}-1} \right), & \text{if } i > j; \\ \frac{1}{t_0}, & \text{if } i = j; \\ 0, & \text{if } i < j, \end{cases}$$

where $k_0 = i - j + 1$.

Thus the entries of the inverse can be represented in terms of entries from the original matrix \mathbf{L} using symbolic computation. Similarly, applying Theorem 3.3 for the inversion of band triangular Toeplitz matrices.

Corollary 4.5. A representation with nested sums for the entries of the inverses of (lower) $m \times m$ band triangular Toeplitz matrices of bandwidth n is, for $i - j > n$,

$$(\mathbf{L}^{-1})_{ij} = \frac{1}{t_0} \sum_{r=\alpha}^{i-j} \sum_{k_1=F_{k_0}}^{f_{k_0}} \sum_{k_2=F_{k_1}}^{f_{k_1}} \cdots \sum_{k_{r-1}=F_{k_{r-2}}}^{f_{k_{r-2}}} \prod_{s=1}^{r-1} b_{k_{s-1}-k_s} b_{k_{r-1}-1},$$

where $k_0 = i - j + 1$. The lower limits $F_{k_{s-1}}$, the upper limits $f_{k_{s-1}}$, $s = 1, \dots, r - 1$, and α , are as given in Theorem 3.3 for $C_{1,1}^{*(i-j+1)}$. For $i - j \leq n$, the formula of $(\mathbf{L}^{-1})_{ij}$ is obtained from Corollary 4.4.

Example 4.6. Corollary 4.5 is illustrated using Maple® for computing the entries $(\mathbf{L}^{-1})_{ij}$, $m \geq 12$, $n = 5$, and $i - j = 11$, in terms of entries of \mathbf{L} ,

$$\begin{aligned} t_0(\mathbf{L}^{-1})_{ij} = & 3b_1b_5^2 + 3b_3b_4^2 + 3b_3^2b_5 + 6b_2b_4b_5 + 12b_1b_2b_4^2 + 12b_1b_3^2b_4 + 24b_1b_2b_3b_5 \\ & + 12b_1^2b_4b_5 + 4b_2b_3^2 + 12b_2^2b_3b_4 + 4b_2^3b_5 + 30b_1b_2^2b_3^2 + 20b_1b_2^3b_4 \\ & + 60b_1^2b_2b_3b_4 + 30b_1^2b_2^2b_5 + 10b_1^2b_3^2 + 20b_1^3b_3b_5 + 10b_1^3b_4^2 + 5b_2^4b_3 \\ & + 6b_1b_2^5 + 60b_1^2b_2^3b_3 + 60b_1^3b_2b_3^2 + 60b_1^3b_2^2b_4 + 30b_1^4b_2b_5 + 30b_1^4b_3b_4 \\ & + 35b_1^3b_2^4 + 105b_1^4b_2^2b_3 + 42b_1^5b_2b_4 + 21b_1^5b_3^2 + 7b_1^6b_5 + 56b_1^5b_2^3 + 8b_1^7b_4 \\ & + 56b_1^6b_2b_3 + 9b_1^8b_3 + 36b_1^7b_2^2 + 10b_1^9b_2 + b_1^{11}. \end{aligned}$$

Each nested sum of level j , $j = 3, 4, \dots, 11$, can be computed independently. For example, the related 7-tuples in terms of entries of \mathbf{L} are

$$\sum_{k_1=7}^{11} \sum_{k_2=6}^{k_1-1} \sum_{k_3=5}^{k_2-1} \sum_{k_4=4}^{k_3-1} \sum_{k_5=3}^{k_4-1} \sum_{k_6=2}^{k_5-1} \prod_{s=1}^6 b_{k_{s-1}-k_s} b_{k_6-1} = 35b_1^3b_4^2 + 105b_1^4b_2^2b_3 + 42b_1^5b_2b_4 + 21b_1^5b_3^2 + 7b_1^6b_5.$$

This procedure for managing stimulus–response relationships among the entries of a triangular matrix and those of its inverse, should be extended to another nonsingular matrix through its *PLU* factorization.

4.2. The multinomial distribution

The multinomial expansion, for $m, n \in \mathbb{Z}^+$,

$$\left(\sum_{r=1}^n p_r \right)^m = \sum_{\substack{0 \leq l_1, l_2, \dots, l_n \leq m \\ l_1 + l_2 + \dots + l_n = m}} \frac{m!}{l_1! l_2! \dots l_n!} p_1^{l_1} p_2^{l_2} \dots p_n^{l_n}, \quad (21)$$

generalizes the binomial one. If in addition $\sum_{r=1}^n p_r = 1$, the expansion (21) yields the multinomial distribution. The term p_r is the probability that an event E_r occurs in a trial, among n mutually exclusive events E_1, E_2, \dots, E_n . Suppose now that m independent trials are performed. The probability that, for $r = 1$ to n , the events E_r occur l_r times, with $0 \leq l_r \leq m$ and $l_1 + l_2 + \dots + l_n = m$, is $\frac{m!}{l_1! l_2! \dots l_n!} p_1^{l_1} p_2^{l_2} \dots p_n^{l_n}$. The representation of the multinomial expansion with nested sums is simpler than its well-known counterpart (21).

Lemma 4.7 (Representation of the Multinomial Expansion with Nested Sums).

$$\begin{aligned} \left(\sum_{r=1}^n p_r \right)^m &= \sum_{\substack{0 \leq l_1, l_2, \dots, l_n \leq m \\ l_1 + l_2 + \dots + l_n = m}} \frac{m!}{l_1! l_2! \dots l_n!} p_1^{l_1} p_2^{l_2} \dots p_n^{l_n} \\ &= \sum_{k_1=1}^n \sum_{k_2=1}^n \dots \sum_{k_m=1}^n \prod_{j=1}^m p_{k_j}. \end{aligned} \quad (22)$$

Proof. By mathematical induction on $m \in \mathbb{Z}^+$. \square

Example 4.8. For $m = 4, n = 3$, and $\sum_{r=1}^3 p_r = 1$, we use the multinomial distribution (22) and Maple® to attain the entire space of events,

$$\begin{aligned} \sum_{k_1=1}^3 \sum_{k_2=1}^3 \sum_{k_3=1}^3 \sum_{k_4=1}^4 \prod_{j=1}^4 p_{k_j} &= p_1^4 + 4p_1^3 p_2 + 4p_1^3 p_3 + 6p_1^2 p_2^2 + 12p_1^2 p_2 p_3 + 6p_1^2 p_3^2 \\ &\quad + 4p_1 p_2^3 + 12p_1 p_2^2 p_3 + 12p_1 p_2 p_3^2 + 4p_1 p_3^3 + p_2^4 + 4p_2^3 p_3 + 6p_2^2 p_3^2 + 4p_2 p_3^3 + p_3^4 = 1. \end{aligned}$$

For example, the probability for the events $l_1 = 1, l_2 = 3, l_3 = 0$, is $4p_1 p_2^3$.

4.3. Roger–Szegő's polynomials

Representations with nested sums of the orthogonal polynomials on the real line [9], and on the unit circle [10], have been introduced recently. We illustrate here another application of the results from Section 3 with Roger–Szegő's polynomials, also known as the q -deformed Hermite polynomials,

$$H_m(x; q) = \sum_{r=0}^m \begin{bmatrix} m \\ r \end{bmatrix}_q x^r,$$

where $x = \frac{z}{-\sqrt{q}}$, and

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{(q)_m}{(q)_r (q)_{m-r}} = \frac{(1-q)(1-q^2) \dots (1-q^m)}{(1-q) \dots (1-q^r) (1-q) \dots (1-q^{m-r})}$$

are the q -binomial (Gaussian binomial) coefficients; see e.g. [7] for a representation of Gaussian binomial coefficients with nested sums. These polynomials are of use in applied mathematics, mathematical physics, and quantum mechanics [13]. Roger–Szegő's polynomials are q -orthogonal on the unit circle, $z = e^{i\varphi}$, with respect to the (weight) Jacobi function $\vartheta_3(q; \varphi)$. The three-term recurrence relation satisfied by such polynomials,

$$H_{m+1}(x; q) = (x+1)H_m(x; q) + (q^m - 1)xH_{m-1}(x; q), \quad (m \geq 0)$$

with customary initial conditions $H_{-1}(x; q) = 0, H_0(x; q) = 1$, allows us to obtain the determinantal representation [14],

$$H_m(x; q) = \begin{vmatrix} x+1 & -1 & 0 & \dots & \dots & 0 \\ (q-1)x & x+1 & -1 & \ddots & \dots & 0 \\ 0 & (q^2-1)x & x+1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & x+1 & -1 \\ 0 & 0 & \dots & \dots & (q^{m-1}-1)x & x+1 \end{vmatrix}_{m \times m}. \quad (23)$$

Therefore, we can use [Corollary 4.2](#), with $n = 2$, on the hessenbergian (23) to obtain a compact representation of $H_m(x; q)$ with nested sums.

Another representation of Roger–Szegő's polynomials with nested sums can be obtained easily from the identity [7],

$$\left[\begin{matrix} r+p \\ r \end{matrix} \right]_q = \sum_{k_p=0}^r \sum_{k_{p-1}=0}^{k_p} \cdots \sum_{k_1=0}^{k_2} q^{k_1+k_2+\cdots+k_p},$$

that relates Gaussian binomial coefficients with nested sums. After introducing elementary changes of indexes, with $k_0 = r$,

$$\begin{aligned} H_m(x; q) &= \sum_{r=0}^{m-1} \left[\sum_{k_1=0}^r \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-r}=0}^{k_{m-r-1}} q^{k_1+k_2+\cdots+k_{m-r}} \right] x^r + x^m \\ &= \sum_{k_0=0}^{m-1} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{m-k_0}=0}^{k_{m-k_0-1}} x^{k_0} q^{k_1+k_2+\cdots+k_{m-k_0}} + x^m. \end{aligned} \quad (24)$$

Example 4.9. For $m = 4$, using the representation (24) for computing $H_4(x; q) = 1 + x(1 + q + q^2 + q^3) + x^2(1 + q + 2q^2 + q^3 + q^4) + x^3(1 + q + q^2 + q^3) + x^4$. The same result can be computed from [Corollary 4.2](#) and (23),

$$\begin{aligned} H_4(x; q) &= \sum_{j=2}^4 \sum_{k_1=f_{k_0}}^{f_{k_0}} \sum_{k_2=f_{k_1}}^{f_{k_1}} \cdots \sum_{k_{j-1}=f_{k_{j-2}}}^{f_{k_{j-2}}} b_{4,k_1} \prod_{r=2}^{j-1} b_{k_{r-1}-1,k_r} b_{k_{j-1}-1,1} \\ &= b_{43}b_{21} + b_{43}b_{22}b_{11} + b_{44}b_{32}b_{11} + b_{44}b_{33}b_{21} + b_{44}b_{33}b_{22}b_{11} \\ &= (q^3 - 1)x(q - 1)x + ((q^3 - 1) + (q^2 - 1) + (q - 1))x(x + 1)^2 + (x + 1)^4. \end{aligned}$$

It is straightforward to check that the two polynomials are identical.

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