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F. Goodarzi, M. Amini, G.R. Mohtashami Borzadaran

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Some results on upper bounds for the variance of functions of the residual life random variables

F. Goodarzi, M. Amini* and G. R. Mohtashami Borzadaran
Department of Statistics, Ordered and Spatial Data Center of Excellence
Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract

As a measure of maximum dispersion from the mean, upper bounds on variance have applications in all areas of theoretical and applied mathematical sciences. In this paper, we obtain an upper bound for the variance of a function of the residual life random variable X_t. Since one of the most important types of system structures is the parallel structure, we give an upper bound for the variance of a function of this system consisting of n identical and independent components, under the condition that, at time t, $n-r+1$, $r=1,\ldots,n$ of its components are still working. Here we characterize the Pareto distribution through Cauchy's functional equation for mean residual life. It is shown that the underlying distribution function F can be recovered from the proposed mean and variance residual life function of the system for $r=1$. Moreover, we see that the variance residual lifetime of the components of the system is not necessarily a decreasing function of r and increasing of n for $r=1$, unlike their mean residual lifetime. As an application, the variance of $X_{F^{-1}(p_0)}$ for all $p_0 \in [0,1)$ is investigated and also a real data analysis is presented.

Keywords: Characterization, Mean residual life function, Variance residual life function, Parallel systems, Variance bound.
Mathematical Subject Classification: Primary 60E15; Secondary 62N05.

1 Introduction

The mean residual life (MRL) function has been widely used in reliability. For example, it is used to design burn-in programs, plan spare provision, and formulate warranty policies. Let X be a non-negative random variable having absolutely continuous distribution function $F(t)$ and survival function $\overline{F}(t) = 1 - F(t)$ and probability density function $f(t)$. Then the hazard rate function of X is defined as $r(t) = -\frac{d}{dt}\log\overline{F}(t) = \frac{f(t)}{\overline{F}(t)}$.

An useful reliability measure of X is mean residual life, which is defined as expectation of the residual life random variable $X_t = (X - t \mid X > t)$, given by

$$m(t) = \frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(x)dx. \tag{1}$$

*Corresponding author.
E-mail address: m-amini@um.ac.ir

The hazard rate and MRL functions are related by

$$r(t) = \frac{1 + m'(t)}{m(t)}. \tag{2}$$

It is well known that $r(t)$ determines the distribution function uniquely and hence $m(t)$ also characterizes the distribution. In addition $\overline{F}(t)$ and $r(t)$ are connected by

$$\overline{F}(t) = \exp\left\{-\int_0^t r(x)dx\right\}. \tag{3}$$

The survival function of $X - t$ given that $X > t$, is

$$\overline{F}_t(x) = P\{X - t > x \mid X > t\} = \frac{\overline{F}(t+x)}{\overline{F}(t)}, \tag{4}$$

and thus the probability density function of X_t, is given by

$$f_t(x) = \begin{cases} \frac{f(t+x)}{\overline{F}(t)} & 0 < x < \infty, \\ 0 & otherwise. \end{cases} \tag{5}$$

Hall and Wellner (1981) and Bhattacharjee (1982) have characterized the class of mean residual life functions. It has been shown by Gupta (1975, 1981) that MRL function determines the distribution uniquely. In particular, it is well known that a constant MRL characterizes the exponential distribution, as it was shown by Nanda (2010). For a comprehensive review, see Guess and Proschan (1988).

When discussing the variance of the residual lifetime X_t, it will be assumed that $E(X^2) < \infty$. The variance residual life (VRL) function is defined as

$$\sigma_F^2(t) = \mathrm{Var}(X - t|X > t) = \frac{2}{\overline{F}(t)} \int_t^\infty \int_x^\infty \overline{F}(y)\, dy\, dx - m^2(t). \tag{6}$$

F is said to have increasing failure rate, IFR if $\overline{F}_t(x)$ is decreasing in t. If F is absolutely continuous with density f, then F is in the IFR class if $r(t)$ is increasing in t.

Karlin (1982) has studied the monotonic behaviour of $\sigma_F^2(t)$ when the density is log-convex (log-concave). Defining the residual coefficient as $\gamma_F(t) = \sigma_F(t)/m(t)$, Gupta (1987) has characterized the monotonic behaviour of $\sigma_F^2(t)$ in terms of $\gamma_F(t)$.

We refer to Gupta and Kirmani (1998), Gupta et al. (1987) and Gupta (2006) for more details about the MRL and the VRL functions. We also refer to Navarro and Ruiz (2004) for the discussion on characterization of probability distributions based on the relationship between hazard rate and conditional moments

In the study of reliability of technical systems and subsystems, parallel systems play an important role. A parallel system $\mathcal{S}_n$, consisting of n components, is a system which functions if and only if at least one of its n components functions. Let $X_i,\ i = 1, 2, \ldots, n$ be the time up to the failure of i^{th} component, such that $X_1, X_2, \ldots, X_n$ are independent, identically distributed (i.i.d.) random variables with continuous distribution function F and probability density function f. Let $X_{i:n}$ $(i = 1, 2, \ldots, n)$ be the lifetime of the component having i^{th} smallest lifetime among n independent and identical components. Then $X_{k:n}$ represents the lifetime of an $(n-k+1)$-out-of-n system. It should be noted that, a $(n-k+1)$-out-of-n system is a system consisting of n components (usually the same) and functions if and only if at least $n-k+1$ out of n components are operating ($k \leq n$). Here we consider a parallel system (a system which fails when the component with lifetime $X_{n:n}$ fails). Suppose that the system fails at or before time t.

In a parallel system, the survival probability of the system corresponding to a mission of duration of x is $\overline{S}(x) = P\{X_{n:n} > x\}$ and the life function is $S(x) = 1 - \overline{S}(x)$.

The conditional probability of survival of system in the interval $(t, t+x)$, with no failed component at time t (the probability of the system having non failure elements at time t function at time $t+x$) is

$$\overline{S}(x \mid t) = P\{X_{n:n} > t + x \mid X_{1:n} > t\}.$$

The conditional probability of the system's failing in the interval $(t, t+x]$, with no failing components at time t is

$$S(x \mid t) = P\{X_{n:n} \leq t + x \mid X_{1:n} > t\} = \left[1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\right]^n \text{ if } \overline{F}(t) > 0, \tag{7}$$

and therefore the density probability function of $\Psi_n^1(X;t) = (X_{n:n} - t | X_{1:n} > t)$ is

$$f_n^1(x \mid t) = \begin{cases} n\left[1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\right]^{n-1} \frac{f(t+x)}{\overline{F}(t)} & 0 < x < \infty, \\ 0 & otherwise. \end{cases} \tag{8}$$

The conditional expectation of residual life length of the system $\mathcal{S}_n$ having parallel structure

$$m_{(n)}^1(t) = E(X_{n:n} - t | X_{1:n} > t),$$

given $X_{1:n} > t$ (all elements of $\mathcal{S}_n$ function at time t) is called the mean residual life function of parallel system.

Lastly, let us consider $\Psi_n^r(X;t) = (X_{n:n} - t | X_{r:n} > t)$ defined in Asadi and Bairamov (2005) with the conditional probability of system's failing in the interval $(t, t+x]$, under the condition that, $n-r+1$, $r = 1, 2, \ldots, n$, components of the system are still working. Explicitly it is given by

$$P\{X_{n:n} < x + t | X_{r:n} > t\} = \frac{\sum_{i=0}^{r-1} \binom{n}{i} F^i(t)[\overline{F}(t) - \overline{F}(t+x)]^{n-i}}{\sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \overline{F}^{n-i}(t)}, \tag{9}$$

and the probability density function of $\Psi_n^r(X;t)$ is as follows:

$$f_n^r(x|t) = \begin{cases} \frac{1}{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t)} \sum_{i=0}^{r-1} (n-i) \binom{n}{i} \phi^i(t) \left[1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\right]^{n-i-1} \frac{f(t+x)}{\overline{F}(t)} & x > 0, \\ 0 & otherwise, \end{cases} \tag{10}$$

where $\phi(t) = \dfrac{F(t)}{\overline{F}(t)}$.

Asadi and Bairamov (2005) showed that

$$m^r_{(n)}(t) = E(X_{n:n} - t | X_{r:n} > t)$$
$$= \frac{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} m_j(t)}{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t)} \qquad r = 1, 2, \dots n, \tag{11}$$

where $m_j(t) = \dfrac{\int_t^\infty \overline{F}^j(x)dx}{\overline{F}^j(t)}$. They showed that if the components of the system have an increasing failure rate then $m^r_{(n)}(t)$ is a decreasing function of t. It was also shown that $m^r_{(n)}(t)$ for fixed n is a decreasing function of r, $r = 1, \dots, n$ and moreover, $m^1_n(t)$ is an increasing function of n.

The following remark is also their observation.

Remark 1 To define the MRL $m^r_{(n)}(t)$ and obtain (11), one does not actually need to restrict the support of F on $(0, \infty)$. In general, $m^r_{(n)}(t)$ can be defined for the distribution functions with left extremity $-\infty \le a$ and right extremity $b \le \infty$, respectively.

As a measure of maximum dispersion from the mean, upper bounds on variance have applications in all areas of theoretical and applied mathematical sciences. Since the pricing and valuation of actuarial and financial risks often depends on variance, appropriate bounds are of considerable practical interest (Hürlimann 2012). Furthermore, in some situations we need to show that the variance is finite. In this paper, we obtain upper bounds for the variance of a function of random variables discussed in the literature. Upper bounds for the variance of a function $g(X)$ of a normal random variable X in terms of derivative g' are known as the inequality of Chernoff (see Chernoff 1981). Upper and lower variance bounds of $g(X)$ for an arbitrary random variable X were considered in Cacoullos (1982), Cacoullos and Papathanasiou (1985) and Prakasa Rao and Sreehari (1997). Both upper and lower variance bounds may be obtained by Cauchy-Schwarz inequality. Now, in order to find the desired bounds, we use the following lemma.

Lemma 1 *(Cacoullos and Papathanasiou 1985) Let X be a continuous random variable with density $f(x)$ and $E(X) = \mu$. Let g and g' be real-valued functions on $\mathbb{R}$ such that g is an indefinite integral of g' and $Var[g(X)] < \infty$. Then*

$$Var[g(X)] \le \int_{-\infty}^{\infty} [g'(x)]^2 \left\{ \int_{-\infty}^{x} (\mu - t) f(t) dt \right\} dx = \int_{-\infty}^{\infty} [g'(x)]^2 \left\{ \int_{x}^{\infty} (t - \mu) f(t) dt \right\} dx, \tag{12}$$

where equality holds if and only if g is linear.

The rest of the paper is organized as follows. In Section 2, we derive an upper bound for the variance of a function of the residual life random variable. Also the Pareto distribution characterized through the additive functional equation of mean residual life function m. In continued, we give an upper bound for the variance of a function of random variable $\Psi^r_n(X; t)$, for n identical and independent components. We show that the underlying distribution function F can be recovered from the proposed mean and variance of $\Psi^1_n(X; t)$ and $\Psi^1_{n-1}(X; t)$. It is also shown that the variance residual lifetime of the components of the system is not necessarily a decreasing function of r and increasing of n for $r = 1$. Application in a study is very important. Most of researchers are interested in finding the use of their work in practical issues, for example Chau and Wu (2010), Taormina and Chau (2015), Wang et al. (2015), Wu et al. (2009) and etc. So, the behaviour of the variance of $X_{F^{-1}(p_0)}$ for all $p_0 \in (0,1)$ and analysis of real data are provided in section 3.

2 Main Results

In this section, we obtain upper bounds for the variance of a function of random variables X_t and $\Psi_n^r(X;t), r = 1,\ldots,n$. In the following proposition, we find the bound for X_t.

Proposition 1 *Let X be a non-negative random variable with density function $f(x)$ and survival function $\overline{F}(x) = 1 - F(x)$. If g is an absolutely continuous function with derivative g' then*

$$Var[g(X_t)] \leq E\left[\frac{1}{r(X_t+t)}\Big(m(X_t+t) - m(t) + X_t\Big){g'}^2(X_t)\right], \tag{13}$$

where equality holds if and only if g is a linear function.

Proof: Using Lemma 1, and since $E[X_t] = m(t)$, we can write that

$$\begin{aligned}\int_x^\infty (y - m(t))\frac{f(t+y)}{\overline{F}(t)}dy &= \frac{1}{\overline{F}(t)}\left\{\int_x^\infty yf(t+y)dy - m(t)\int_x^\infty f(t+y)dy\right\}\\ &= \frac{1}{\overline{F}(t)}\left\{\int_{x+t}^\infty (y-t)f(y)dy - m(t)\int_{x+t}^\infty f(y)dy\right\}\\ &= \frac{\overline{F}(t+x)}{\overline{F}(t)}\Big\{m(t+x) - m(t) + x\Big\},\end{aligned}$$

and thereby

$$\begin{aligned}\mathrm{Var}[g(X_t)] &\leq \int_0^\infty {g'}^2(x)\frac{\overline{F}(t+x)}{\overline{F}(t)}\Big\{m(t+x) - m(t) + x\Big\}dx\\ &= \int_0^\infty {g'}^2(x)\frac{1}{r(t+x)}\Big\{m(t+x) - m(t) + x\Big\}\frac{f(x+t)}{\overline{F}(t)}dx,\end{aligned}$$

and this completes the proof.

The equality is obvious in view of Lemma 1.

Remark 2 In Proposition 1, if X is a non-negative random variable and

(a) F is IFR then

$$\mathrm{Var}[g(X_t)] \leq E\left[\frac{X}{r(X)}{g'}^2(X)\right]. \tag{14}$$

Especially, if $g(x) = x^k$ where $k > 0$ is an integer, since $\int_0^\infty x^{2k-1}\overline{F}(x)dx = \frac{1}{2k}E(X^{2k})$ then

$$\mathrm{Var}({X_t}^k) \le \frac{k}{2}E(X^{2k}).$$

The upper bound of (13) is equal to (14) if and only if X have exponential distribution.

(b) $g(x) = -\log f_t(x)$, then

$$\mathrm{Var}[-\log f_t(X_t)] \le E\left[\frac{\eta^2\,(t+X_t)}{r\,(t+X_t)}\Big(m(t+X_t) - m(t) + X_t\Big)\right], \tag{15}$$

where $\eta(x) = -\frac{f'(x)}{f(x)}$ is eta function and log denotes the natural logarithm.

It should be noted that according to, the variance entropy (varentropy) of a random variable X is defined as

$$\mathrm{Var}(-\log f(X)) = \int_{\mathbb{R}} f(x)(\log f(x))^2 dx - \left(\int_{\mathbb{R}} f(x)\log f(x)dx\right)^2,$$

so we call $\mathrm{Var}[-\log f_t(X_t)]$ as variance residual entropy.
Wang (2014) proved that given a random vector $\mathbf{X}$ in $\mathbb{R}^n$ with log-concave density f,

$$\mathrm{Var}(-\log f(\mathbf{X})) \le n,$$

which is a particular case of inequality (15), on the space $\mathbb{R}^+$.

Notice that, the right hand side of equation (13) motivate us to characterize a distribution for which $m(X_t + t) = m(X_t) + m(t)$ and therefore the upper bound becomes somewhat easier.

Proposition 2 *Let F a continuous distribution function with left extremity $a > 0$ then F is Pareto distribution with a positive scale parameter and shape parameter larger than 1 if and only if $m(t+s) = m(t) + m(s)$.*

Proof: If F have Pareto distribution with the scale and shape parameters, we can show that $m(t) = ct$ with $c > 0$ constant. Thus the result is obtained.

For the proof of the sufficient condition, since, $m(t) = ct$ is the only solution of functional equation $m(t+s) = m(t) + m(s)$, on using (2) and (3):

$$\begin{aligned}\overline{F}(x) &= \exp\left\{-\int_a^x \frac{c+1}{ct}dt\right\} \\ &= \left(\frac{a}{x}\right)^{(c+1)/c}.\end{aligned}$$

Now, since $c > 0$ with considering $k = \dfrac{c+1}{c}$, we have Pareto distribution with density function $f(x) = \dfrac{ka^k}{x^{k+1}}$, $x > a$, for $a > 0$ and $k > 1$.

At present, we attain an upper bound for the variance of a function of $\Psi_n^1(X;t)$. We first give a lemma that will be used in the proof of Theorem 1.

Lemma 2 *Let $m^1_{(n)}(t)$ be the mean residual life function of the parallel system $\mathcal{S}_n$, consisting of n identical and mutually independent components with continuous life distribution function F and density function f. Then the following identity holds*

$$\int_0^\infty x \left\{ \frac{\overline{F}(t+x)}{\overline{F}(t)} \right\}^{n-1} \frac{f(t+x)}{\overline{F}(t)}\, dx = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} m^1_{(j)}(t). \tag{16}$$

Proof: The proof is easily obtained by induction on n and hence omitted.

Theorem 1 *If $\Psi^1_n(X;t)$ show that, the system having non failure elements at time t, function at time $t+x$ and g is an absolutely continuous function with derivative g' then*

$$\begin{aligned} Var[g(\Psi^1_n(X;t))] \le \int_0^\infty g'^2(x) &\left[\sum_{i=1}^{n} \binom{n}{i} \left(m^1_{(i)}(t+x) - m^1_{(n)}(t) + x \right) \left\{ \frac{\overline{F}(t+x)}{\overline{F}(t)} \right\}^i \right. \\ &\left. \times \left\{ 1 - \frac{\overline{F}(t+x)}{\overline{F}(t)} \right\}^{n-i} \right] dx, \end{aligned} \tag{17}$$

where equality holds if and only if g is linear.

Proof: Using (12), we have

$$\begin{aligned} &\int_x^\infty (y - m^1_{(n)}(t))\, n \left\{ 1 - \frac{\overline{F}(t+y)}{\overline{F}(t)} \right\}^{n-1} \frac{f(t+y)}{\overline{F}(t)}\, dy \\ &= n \int_0^\infty (u + x - m^1_{(n)}(t)) \left\{ 1 - \frac{\overline{F}(t+x+u)}{\overline{F}(t)} \right\}^{n-1} \frac{f(t+x+u)}{\overline{F}(t)}\, du \\ &= n \int_0^\infty u \left\{ 1 - \frac{\overline{F}(t+x+u)}{\overline{F}(t)} \right\}^{n-1} \frac{f(t+x+u)}{\overline{F}(t)}\, du \\ &+ (x - m^1_{(n)}(t)) \left(1 - \left\{ 1 - \frac{\overline{F}(t+x)}{\overline{F}(t)} \right\}^n \right), \end{aligned} \tag{18}$$

now by Lemma 2 we get

$$n\int_0^\infty u\left\{1-\frac{\overline{F}(t+x+u)}{\overline{F}(t)}\right\}^{n-1}\frac{f(t+x+u)}{\overline{F}(t)}du$$

$$=n\int_0^\infty u\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left\{\frac{\overline{F}(t+x+u)}{\overline{F}(t)}\right\}^i\frac{f(t+x+u)}{\overline{F}(t)}du$$

$$=n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{i+1}\int_0^\infty u\left\{\frac{\overline{F}(t+x+u)}{\overline{F}(t+x)}\right\}^i\frac{f(t+x+u)}{\overline{F}(t+x)}du$$

$$=\sum_{i=0}^{n-1}(-1)^{i+1}\binom{n}{i+1}\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{i+1}\sum_{j=1}^{i+1}(-1)^j\binom{i+1}{j}m^1_{(j)}(t+x)$$

$$=\sum_{j=1}^{n}(-1)^j m^1_{(j)}(t+x)\sum_{i=j}^{n}(-1)^i\binom{n}{i}\binom{i}{j}\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^i$$

$$=n\,m^1_{(1)}(t+x)\,\frac{\overline{F}(t+x)}{\overline{F}(t)}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^i$$

$$+\binom{n}{2}m^1_{(2)}(t+x)\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^2\sum_{i=2}^{n}(-1)^i\binom{n-2}{i-2}\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{i-2}$$

$$+\cdots+\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^n m^1_{(n)}(t+x)$$

$$=\sum_{i=1}^{n}\binom{n}{i}m^1_{(i)}(t+x)\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^i\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-i}, \tag{19}$$

and further using (12) and substituting the right hand side (19) in (18), we have

$$\mathrm{Var}[g(\Psi^1_n(X;t))]\le\int_0^\infty g'^2(x)\quad\left[\sum_{i=1}^{n}\binom{n}{i}m^1_{(i)}(t+x)\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^i\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-i}\right.$$
$$\left.+(x-m^1_{(n)}(t))\left(1-\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^n\right)\right], \tag{20}$$

and therefore simplicity (17) is obtained.

By Lemma 1, the equality holds if and only if g is a linear function.

Example 1 Let F be the exponential distribution function $F(x)=1-\exp(-\lambda x)$ $x\ge 0$, $\lambda>0$. Then using the well known representation

$$X_{n:n}\overset{\mathrm{d}}{=}\frac{X_1}{n}+\frac{X_2}{n-1}+\cdots+X_n, \tag{21}$$

(where $\stackrel{\mathrm{d}}{=}$ denotes the equality in distribution), Bairamov et. al (2002) have shown that for the exponential distribution

$$m^1_{(n)}(t) = E(X_{n:n} - t \mid X_{1:n} > t) = E(X_{n:n}) = \frac{1}{\lambda}\Big(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1\Big).$$

Therefore, using (20), an upper bound for $\mathrm{Var}[g(\Psi^1_n(X,t))]$ is given as follows

$$\int_0^\infty g'^2(x)\left\{\Big(x - m^1_{(n)}(t)\Big)\Big(1-(1-e^{-\lambda x})^n\Big) + \sum_{i=1}^n \binom{n}{i} m^1_{(i)}(t+x)e^{-i\lambda x}(1-e^{-\lambda x})^{n-i}\right\}dx$$

$$= \int_0^\infty g'^2(x)\left\{\Big(x - m^1_{(n)}(t)\Big)\sum_{i=1}^n (-1)^{i+1}\binom{n}{i}e^{-i\lambda x}\right.$$

$$\left. + \sum_{i=1}^n \binom{n}{i}\frac{1}{\lambda}\Big(1+\frac{1}{2}+\cdots+\frac{1}{i}\Big)e^{-i\lambda x}(1-e^{-\lambda x})^{n-i}\right\}dx$$

$$= \sum_{i=1}^n (-1)^{i+1}\binom{n}{i}\int_0^\infty g'^2(x)(x - m^1_{(n)}(t))e^{-i\lambda x}dx$$

$$+ \sum_{i=1}^n \binom{n}{i}\frac{1}{\lambda}\Big(1+\frac{1}{2}+\cdots+\frac{1}{i}\Big)\int_0^\infty g'^2(x)e^{-i\lambda x}(1-e^{-\lambda x})^{n-i}dx. \tag{22}$$

If $g(x) = x$, then

$$\mathrm{Var}[\Psi^1_n(X;t)] = \sum_{i=1}^n (-1)^{i+1}\binom{n}{i}\frac{1}{(i\lambda)^2} - \frac{1}{\lambda^2}\Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)\sum_{i=1}^n (-1)^{i+1}\binom{n}{i}\frac{1}{i}$$

$$+ \frac{1}{\lambda^2}\sum_{i=1}^n \binom{n}{i}\Big(1+\frac{1}{2}+\cdots+\frac{1}{i}\Big)\frac{\Gamma(i)\Gamma(n-i+1)}{\Gamma(n+1)}, \tag{23}$$

now, as is shown in (Gradshteyn and Ryzhik 2007), since

$$\sum_{i=1}^n (-1)^{i+1}\binom{n}{i}\frac{1}{i} = \sum_{m=1}^n \frac{1}{m},$$

we have

$$\mathrm{Var}[\Psi^1_n(X;t)] = \frac{1}{\lambda^2}\left\{\sum_{i=1}^n (-1)^{i+1}\binom{n}{i}\frac{1}{i^2} - \Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)^2 + \sum_{i=1}^n \Big(1+\frac{1}{2}+\cdots+\frac{1}{i}\Big)\frac{1}{i}\right\}.$$

On the other hand, since

$$E[\Psi^1_n(X;t)]^2 = 2\int_0^\infty \sum_{i=1}^n \binom{n}{i} m^1_{(i)}(t+x)\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^i\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-i}dx$$

$$= 2\int_0^\infty x\Big(1 - \Big\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\Big\}^n\Big)dx, \tag{24}$$

we have the following identity

$$\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{1}{i^2}=\sum_{i=1}^{n}\Big(1+\frac{1}{2}+\cdots+\frac{1}{i}\Big)\frac{1}{i},$$

thus

$$Var[\Psi_n^1(X;t)]=\frac{1}{\lambda^2}\left\{2\sum_{i=1}^{n}\Big(1+\frac{1}{2}+\cdots+\frac{1}{i}\Big)\frac{1}{i}-\Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)^2\right\}.$$

Example 2 For $x\geq a>0$ and $k>0$, let X have the Pareto distribution, with density function and distribution function $f(x)=\dfrac{ka^k}{x^{k+1}}$ and $F(x)=1-(a/x)^k$ respectively. It can be readily shown that $S(x\mid t)=\Big[1-\Big(\dfrac{t}{t+x}\Big)^k\Big]^n$ and consequently for $k>1$

$$\begin{aligned}
m^1_{(n)}(t)&=\int_0^\infty\Big(1-\Big[1-\Big(\frac{t}{t+x}\Big)^k\Big]^n\Big)\,dx\\
&=\int_0^\infty\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\Big(\frac{t}{t+x}\Big)^{ik}dx\\
&=\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{t}{ik-1}=c_{n,k}t,
\end{aligned}\tag{25}$$

here $c_{n,k}$ is a constant depending on n and k. So the upper bound for $\mathrm{Var}[g(\Psi_n^1(X,t))]$ given by

$$\begin{aligned}
\mathrm{Var}[g(\Psi_n^1(X;t))]\leq&\int_0^\infty g'^2(x)\left\{\sum_{i=1}^{n}\binom{n}{i}c_{i,k}(t+x)\Big(\frac{t}{t+x}\Big)^{ik}\Big[1-\Big(\frac{t}{t+x}\Big)^k\Big]^{n-i}\right.\\
&\left.+(x-c_{n,k}t)\Big(1-\Big[1-\Big(\frac{t}{t+x}\Big)^k\Big]^n\Big)\right\}dx\\
=&\int_0^\infty g'^2(x)\left\{\sum_{i=1}^{n}c_{i,k}\binom{n}{i}\frac{t^{ik}}{(t+x)^{ik-1}}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}\Big(\frac{t}{t+x}\Big)^{jk}\right.\\
&\left.+\,(x-c_{n,k}t)\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\Big(\frac{t}{t+x}\Big)^{ik}\right\}dx,
\end{aligned}\tag{26}$$

in this case, we must choose the value of k such that the above integral is convergent. For example, if $g(x)=x$, then for $k>2$ this integral converges.

Example 3 Let F be the Rayleigh distribution function $F(x)=1-e^{-x^2/2},\ x>0$. It is well known that the distribution is IFR and do not have an explicit expression for

$$m_n^1(t)=\int_t^\infty\Big\{1-\Big(1-e^{-(x^2-t^2)/2}\Big)^n\Big\}\,dx,\tag{27}$$

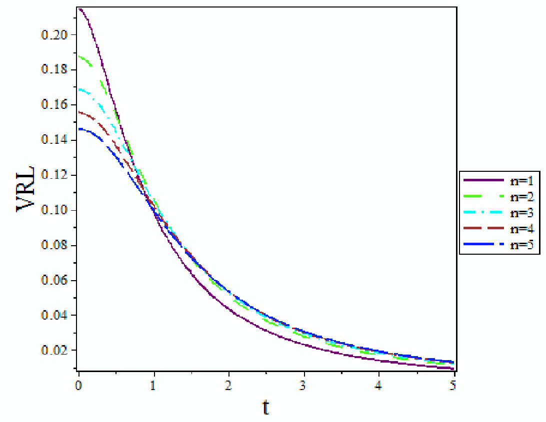

Figure 1: The variance of $\Psi_n^1(X;t)$ of a parallel system with n independent Rayleigh components for n=1, ..., 5.

and

$$\mathrm{Var}[\Psi_n^1(X;t)] = 2\int_t^{\infty}(x-t)\left\{1-\left(1-e^{-(x^2-t^2)/2}\right)^n\right\}dx - (m_n^1(t))^2. \tag{28}$$

Figure 1 shows $\mathrm{Var}[\Psi_n^1(X;t)]$ for $n = 1, 2, 3, 4, 5$. Unlike $m_n^1(t)$, it shows that $\mathrm{Var}[\Psi_n^1(X;t)]$ is not an increasing function of n for any $t > 0$.

The next theorem characterize the distribution by the knowledge of $\mathrm{Var}[\Psi_n^1(X;t)]$ and $m_{(n)}^1(t)$ for two consecutive integers n.

Theorem 2 *Let the components of the system have a common absolutely continuous strictly increasing distribution function F, then*

$$\overline{F}(x) = \exp\left\{-\frac{1}{n}\int_0^x \frac{\frac{d}{dt}Var[\Psi_n^1(X;t)]}{Var[\Psi_n^1(X;t)] - Var[\Psi_{n-1}^1(X;t)] - (m_{(n)}^1(t) - m_{(n-1)}^1(t))^2}dt\right\}, \tag{29}$$

with $m_{(0)}^1(t) = Var[\Psi_0^1(X;t)] = 0$ for $n = 1$.

Proof: From Theorem 1 Bairamov et. al (2002), we have

$$\frac{d}{dt}m_{(n)}^1(t) = nr(t)\Big(m_{(n)}^1(t) - m_{(n-1)}^1(t)\Big) - 1. \tag{30}$$

On the other hand

$$\begin{aligned} E[\Psi_{n-1}^1(X;t)]^2 &= 2\int_0^{\infty} x\left\{1-\left(1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right)^n\right\}dx \\ &= 2\int_t^{\infty} x\,\{1-(1-\theta_t(x))^n\}dx - 2t\int_t^{\infty}\{1-(1-\theta_t(x))^n\}dx, \end{aligned} \tag{31}$$

where $\theta_t(x) = \frac{\overline{F}(x)}{\overline{F}(t)}$.

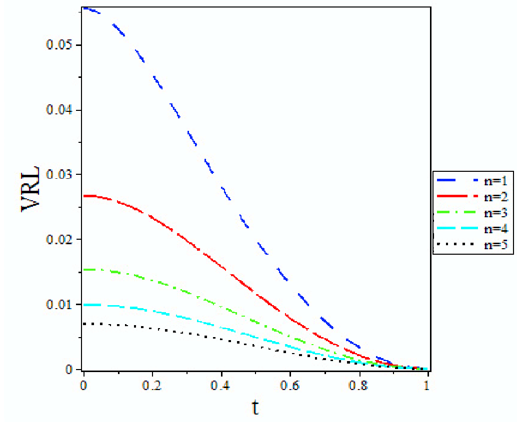

Figure 2: The variance of $\Psi_n^1(X;t)$ of a parallel system with n independent Beta components for $n = 1,\dots,5$.

Differentiating (31) with respect to t leads to

$$\begin{aligned}\frac{d}{dt}E[\Psi_{n-1}^1(X;t)]^2 &= -2t + 2\int_t^\infty x\frac{d}{dt}\{1-(1-\theta_t(x))^n\}dx \\ &\quad -2\int_t^\infty \{1-(1-\theta_t(x))^n\}dx - 2t\frac{d}{dt}\int_t^\infty \{1-(1-\theta_t(x))^n\}dx \\ &= -2t + 2nr(t)\int_t^\infty x(1-\theta_t(x))^{n-1}\theta_t(x)dx - 2m_{(n)}^1(t) \\ &\quad -2t\left\{nr(t)\left(m_{(n)}^1(t) - m_{(n-1)}^1(t)\right) - 1\right\} \\ &= nr(t)\Big\{E[\Psi_n^1(X;t)]^2 - E[\Psi_{n-1}^1(X;t)]^2\Big\} - 2m_{(n)}^1(t). \qquad (32)\end{aligned}$$

Then, using (30) and (32), one can obtain

$$\frac{d}{dt}\mathrm{Var}[\Psi_n^1(X;t)] = nr(t)\left\{\mathrm{Var}[\Psi_n^1(X;t)] - \mathrm{Var}[\Psi_{n-1}^1(X;t)] - (m_{(n)}^1(t) - m_{(n-1)}^1(t))^2\right\}. \qquad (33)$$

After some derivation from (33), we have

$$\frac{d}{dt}(\ln \overline{F}(t)) = -\frac{1}{n}\frac{\frac{d}{dt}\mathrm{Var}[\Psi_n^1(X;t)]}{\left\{\mathrm{Var}[\Psi_n^1(X;t)] - \mathrm{Var}[\Psi_{n-1}^1(X;t)] - (m_{(n)}^1(t) - m_{(n-1)}^1(t))^2\right\}}. \qquad (34)$$

Finally, integrating (34) over $[0,x]$ the proof is complete.

Remark 3 In view of (33), if $\mathrm{Var}[\Psi_n^1(X;t)]$ is decreasing in n, then $\mathrm{Var}[\Psi_n^1(X;t)]$ is a decreasing function in t. For example, if $X_1, \dots, X_n$ are i.i.d random variables with density function $f(x) = 2(1-x)$, $0 < x < 1$, then $\mathrm{Var}[\Psi_n^1(X;t)]$ is decreasing in n. Figure 2 depicts this fact.

Theorem 3 *If $\Psi_n^r(X;t)$ shows that, the system having $n-r+1$ surviving component at time t function*

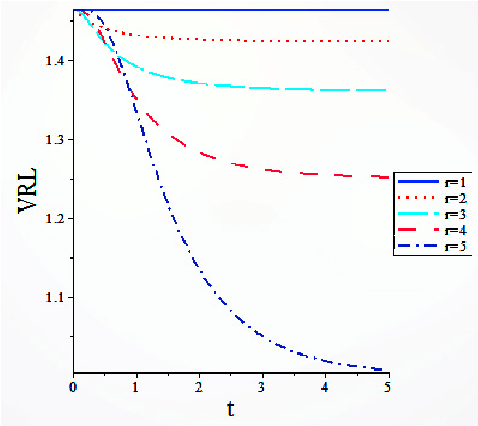

Figure 3: The variance of $\Psi_n^r(X;t)$ of a parallel system with $n=5$ independent exponential components with parameter 1.

at time $t+x$ and g is an absolutely continuous function with derivative g' then

$$Var[g(\Psi_n^r(X;t))] \le \int_0^\infty g'^2(x)\frac{1}{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)}\left\{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)\sum_{j=1}^{n-i}\binom{n-i}{j}m^1_{(j)}(t+x)\right.$$
$$\times\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^j\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-j-i}$$
$$\left.+\,(x-m^r_{(n)}(t))\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)\left[1-\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-i}\right]\right\}dx. \quad (35)$$

The equality holds if and only if g is a linear function of $\Psi_n^r(X;t)$.

Proof: In order to prove inequality (35), we resort to an argument similar to the proof of Theorem 1 and Equation (10), and hence one can obtain the given equality in the following:

$$\int_x^\infty (y-m^r_{(n)}(t))\,\frac{1}{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)}\sum_{i=0}^{r-1}(n-i)\binom{n}{i}\phi^i(t)\left\{1-\frac{\overline{F}(t+y)}{\overline{F}(t)}\right\}^{n-i-1}\frac{f(t+y)}{\overline{F}(t)}\,dy$$
$$=\frac{1}{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)}\left\{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)\sum_{j=1}^{n-i}\binom{n-i}{j}m^1_{(j)}(t+x)\left\{\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^j\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-j-i}\right.$$
$$\left.+(x-m^r_{(n)}(t))\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)\left[1-\left\{1-\frac{\overline{F}(t+x)}{\overline{F}(t)}\right\}^{n-i}\right]\right\}, \quad (36)$$

and consequently the desired result is obtained. By Lemma 1, the equality holds if and only if g is linear.

Example 4 As in Example 1, let $X_1,\ldots,X_n$ denote the independent random variables having exponential distribution with $\lambda=1$. We can show that the upper bound for the variance of $g(\Psi_n^r(X;t))$

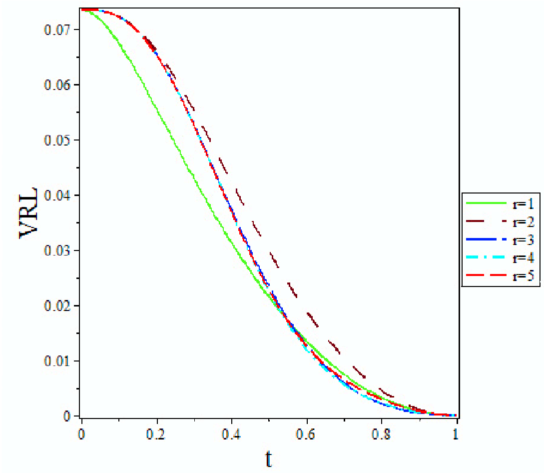

Figure 4: The variance of $\Psi_n^r(X;t)$ of a parallel system with $n=5$ independent Beta components with parameters $(2,1)$.

is given by

$$\mathrm{Var}[g(\Psi_n^r(X;t))] \leq \frac{1}{\sum_{i=0}^{r-1}\binom{n}{i}(e^t-1)^i}\left\{\sum_{i=0}^{r-1}\binom{n}{i}(e^t-1)^i\sum_{j=1}^{n-i}\binom{n-i}{j}\left(1+\frac{1}{2}+\cdots+\frac{1}{j}\right)\right.$$

$$\times\int_0^\infty g'^2(x)e^{-jx}(1-e^{-x})^{n-j-i}\,dx$$

$$\left.+\sum_{i=0}^{r-1}\binom{n}{i}(e^t-1)^i\int_0^\infty g'^2(x)(x-m_{(n)}^r(t))\left[1-\{1-e^{-x}\}^{n-i}\right]\,dx\right\},$$

where, by using (11), $m_{(n)}^r(t)$ has the following form:

$$m_{(n)}^r(t)=\frac{\sum_{i=0}^{r-1}\binom{n}{i}(e^t-1)^i\sum_{j=1}^{n-i}\frac{1}{j}}{\sum_{i=0}^{r-1}\binom{n}{i}(e^t-1)^i}.$$

Example 5 Suppose that $X_1,\ldots,X_n$ are i.i.d. random variables with density function $f(x)=2x$, $0<x<1$. Then, the upper bound for the variance of $\Psi_n^r(X,t)$, $r=1,2,\ldots,n$, is given by

$$\mathrm{Var}[g(\Psi_n^r(X;t))] \leq \int_t^1\frac{g'^2(x)}{\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{t^2}{1-t^2}\right)^i}\left\{\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{t^2}{1-t^2}\right)^i\sum_{j=1}^{n-i}\binom{n-i}{j}m_{(j)}^1(y)\left(\frac{1-y^2}{1-t^2}\right)^j\right.$$

$$\left.\times\left(\frac{y^2-t^2}{1-t^2}\right)^{n-j-i}+\left(y-t-m_{(n)}^r(t)\right)\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{t^2}{1-t^2}\right)^i\left[1-\left(\frac{y^2-t^2}{1-t^2}\right)^{n-i}\right]\right\}dy,$$

where

$$m_{(n)}^r(t)=\frac{\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{t^2}{1-t^2}\right)^i\sum_{j=1}^{n-i}(-1)^{j+1}\binom{n-i}{j}\left\{\int_t^1(1-y^2)^j dy/(1-t^2)^j\right\}}{\sum_{i=0}^{r-1}\binom{n}{i}\left(\frac{t^2}{1-t^2}\right)^i}.$$

In Figure 3 and 4, Var$[\Psi_n^r(X,t)]$ is a decreasing function of t and unlike $m_{(n)}^r(t)$, for fixed n, it is not a decreasing function of r, $r = 1, 2, \ldots, n$. It should be noted that distributions given in examples 4 and 5 are IFR and as Gupta (2006) indicates, since $m_{(n)}^r(t)$ is a decreasing function of t , we can guess the behaviour of Var$[\Psi_n^r(X,t)]$ with respect to t.

3 An application

The concept of variability is a basic one in statistics, probability, and many other related areas. The simplest way of comparing the variability of two distributions is by comparison of the standard deviations; however the comparison of numerical measures is not always informative. In the past two decades, several more refined transforms and stochastic orders, which measure and compare variabilities of random variables based on their entire distribution functions, have been introduced. Shaked and Shanthikumar (1998) presented discussion on excess wealth transform, as a measure of spread. It was also independently proposed and studied by Fernández-Ponce et al. (1998) and they called it as right spread transform. More specifically, for a random variable X the quantile function or the inverse distribution function, is defined by $F^{-1}(p) = \inf\{x : F(x) \geq p\}$ for $p \in (0,1)$ and $F^{-1}(0)$ and $F^{-1}(1)$ are defined as the left and right extremes of the support, respectively. Note that the excess wealth function (or right spread function) is defined as

$$
\begin{aligned}
W(p;F) &= E[(X - F^{-1}(p))^+] \\
&= \int_{F^{-1}(p)}^{\infty} \overline{F}(x)dx = \int_p^1 (F^{-1}(q) - F^{-1}(p))dq, \qquad (37)
\end{aligned}
$$

where $(Z)^+ = \max\{Z, 0\}$. Also, the right spread function of random variable X can be called by the notation $S_X^+(p)$. The excess wealth function can be considered as a measure of spread to the right of every quantile $F^{-1}(x)$. This function is also related to its mean residual life function by the relationship

$$E[(X - F^{-1}(p))^+] = (1-p)m(F^{-1}(p)).$$

Fernández-Ponce et al. (1998) showed that for a continuous random variable X with distribution function F which is strictly increasing on its support, then

$$Var[X] = \int_0^1 \left[\frac{W(p;F)}{1-p}\right]^2 dp.$$

Furthermore Kochar and Xu (2013) proved for any $p_0 \in [0,1)$,

$$Var[X|X > F^{-1}(p_0)] = \frac{1}{1-p_0}\int_{p_0}^1 \left[\frac{W(p;F)}{1-p}\right]^2 dp. \qquad (38)$$

The following examples show the behaviour of $Var[X|X > F^{-1}(p_0)]$.

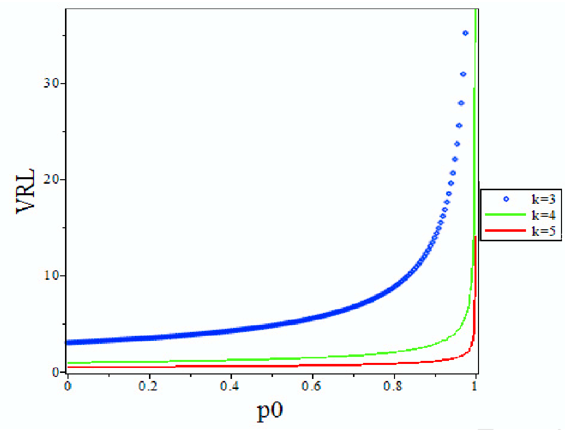

Figure 5: The variance of $X_{F^{-1}(p_0)}$ of Pareto distribution for $a = 2$ and different values of k.

Example 6 Let X be a random variable with uniform distribution on (α, β). The excess wealth function is given by

$$W(p; F) = \frac{(\beta - \alpha)(1 - p)^2}{2},$$

on using (37) and

$$Var[X|X > F^{-1}(p_0)] = \frac{(\beta - \alpha)^2(1 - p_0)^2}{12},$$

by using (38).

Example 7 Let X have Pareto distribution given in Example 2, then by using (37) and (38), we have respectively

$$W(p; F) = \frac{a(1 - p)^{-\frac{1}{k}+1}}{k - 1}, \text{ for } k > 1,$$

and

$$Var[X|X > F^{-1}(p_0)] = \frac{ka^2}{(k - 2)(k - 1)^2(1 - p_0)^{2/k}}, \text{ for } k > 2.$$

Trivially $\frac{d}{dp_0}(Var[X|X > F^{-1}(p_0)) = \frac{2a^2(1-p_0)^{1-2/k}}{(k-2)(k-1)^2}$ and thus for fixed values of a and k, conditional variance is increasing in $p_0 \in [0, 1)$. Also, since for fixed values of a and p_0,

$$\frac{d}{dk}(Var[X|X > F^{-1}(p_0)]) = \frac{(k - 2)(k - 1)\ln(1 - p_0) + k(1 + k - k^2)}{(k - 1)^3(k - 2)^2 k} < 0, \text{ for } k > 2,$$

therefore the variance of $X_{F^{-1}(p_0)}$ with respect to k is decreasing.

In Figure 5 conditional variance curve is shown with different values of k and $a = 2$. It is evident that, if the value of k is increased then the variance is decreased.

Now, to illustrate with real data sets, it is necessary to introduce the non-parametric estimator of excess wealth function. A nonparametric estimator for $E[(X - F^{-1}(p))^+]$ given a random sample $X_1, X_2, \ldots, X_n$, was obtained by Kochar and Xu (2013). They showed the empirical version of this transform for $i = 1, \ldots, n - 1$ as follows

$$W_i = W(\frac{i}{n}; F_n) = \sum_{j=i+1}^{n} \frac{n - j + 1}{n}(X_{j:n} - X_{j-1:n}),$$

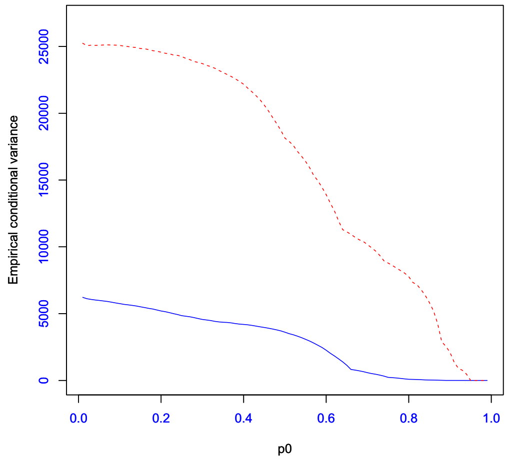

Figure 6: Empirical conditional variance for the laboratory environment (continuous line) and the germ free environment (dashed line).

where F_n is the empirical distribution function. It is observed that

$$W_0 = \overline{X}, \quad W_{i+1} = W_i - \frac{n-i}{n}(X_{i+1:n} - X_{i:n}), \quad 0 \leq i \leq n-1. \tag{39}$$

The real data set arose in a study of survival times in the presence of pollutants (see Hoel 1972, Page 483). The data set consists of two groups of survival times of RFM strain male mice. We consider the data set corresponding to the thymic lymphoma death. The first group lived in a conventional laboratory environment while the second group was in a germ free environment. The reported data are the time to death in days.

Belzunce et al. (2016), by drawing the plot of nonparametric estimators of the excess wealth, showed that the survival times in the germ free environment are more dispersed than that of the laboratory environment. Here we draw the plot of nonparametric estimators of conditional variance for two groups. As we can see in Figure 6, for all $p_0 \in (0,1)$ variance of $X_{F^{-1}(p_0)}$ for the survival times in the germ free environment larger or equal than that in the laboratory environment and thus the number of days until death for the group of RFM strain male mice lived in a germ free environment has a greater dispersion.

4 Conclusion

In this article, we obtained an upper bound for a function of the residual life random variable X_t and $\Psi_n^r(X;t)$ in parallel systems for n identical and independent components. Moreover, we studied properties VRL in parallel systems and characterized the distribution by the knowledge of mean and variance $\Psi_n^1(X;t)$ for two consecutive integers n. We also obtained an application of our achievements via real data in the practical example with our interpretation of the results. An interesting extension of the concept of VRL at the system level for more complex systems, can be considered for the case where the system has a $(n-k+1)$-out-of-n structure, $k = 1, 2, \ldots, n$ (see Asadi and Bayramoglu 2006) as well as the MOSE type $(n-k+1)$-out-of-n: G system (see Bayramoglu and Ozkut 2016).

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