Accepted Manuscript

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PII: S0377-0427(18)30371-6
DOI: https://doi.org/10.1016/j.cam.2018.06.019
Reference: CAM 11749

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 31 August 2017
Revised date : 10 May 2018

CK transformations, symmetries, exact solutions and conservation laws of the generalized variable-coefficient KdV types of equations†

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Abstract *In this paper, the improved CK reduction transformation is performed on the generalized variable-coefficient KdV (vc-KdV) type of equation, then these variable-coefficient equations are transformed into its constant-coefficient counterparts under some conditions. Moreover, the complete Lie group classification is presented, all of the point symmetries of the equations are obtained. Furthermore, the exact solutions and conservation laws of the equations are investigated.*

Keywords *CK transformation, Lie group classification, Exact solution, Conservation law, Variable-coefficient equation*

Mathematics Subject Classification (2000) 37K10, 35C05

PACS numbers 02.30.Jr, 02.30.Ik

1 Introduction

It is well known that the CK direct transformation is a powerful method for dealing with similarity reductions and exact solutions to nonlinear partial differential equations (NLPDEs) [1-4]. In this paper, by the improved CK transformation method, we consider the generalized

†This work was supported by the National Natural Science Foundation of China under grant Nos. 11171041 and 11505090, the Natural Science Foundation of Shandong Province under grant No. ZR2018MA025, and the high-level personnel foundation of Liaocheng University under grant No. 31805.

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variable-coefficient KdV type of equation as follows:

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_x + l(t)u = 0, \tag{1.1}$$

where $u = u(x,t)$ denotes the unknown function of space variable x and time t, the coefficients $f = f(t)$, $g = g(t)$, $h = h(t)$ and $l = l(t)$ are all arbitrary analytic functions, $fg \neq 0$ is assumed throughout this paper.

In particular, if $l(t) = 0$, then Eq. (1.1) is reduced to the following equation

$$u_t + f(t)uu_x + g(t)u_{xxx} + h(t)u_x = 0. \tag{1.2}$$

If $h(t) = l(t) = 0$, then Eq. (1.1) is reduced to the following vc-KdV equation

$$u_t + f(t)uu_x + g(t)u_{xxx} = 0. \tag{1.3}$$

These variable-coefficient nonlinear evolution equations (vc-NLEEs) play a significant role in mathematical physics, integrable system and physical applications [4-7]. However, the vc-NLEEs differ greatly from its constant-coefficient counterparts, and they are more involved for investigating exact solutions and other properties of these generalized variable-coefficient equations. For dealing with the exact solutions and other properties of vc-NLEEs, a lot of methods were developed such as the Painlevé test [6-10], Lie symmetry analysis [10-15] and the various trial function methods based on the homogeneous balance principle (HBP) [16,17], and so on. Ghany and Mohammed [18,19] studied some complicated vc-NLEEs such as the generalized stochastic Hirota-Satsuma coupled KdV equation and fractional KdV-Brugers-Kuramoto equation by the white noise and homotopy analysis method along with the Hermit transform. But such methods are more involved and the solutions are not presented explicitly. Clarkson and Kruskal [1] proposed the CK direct method for dealing with similarity reductions of the nonlinear partial differential equations, Lou et al considered the similarity reductions and exact solutions to some NLPDEs by using this method [2-4]. In the present paper, we shall develop the improved CK transformation method for transforming the vc-NLEEs into its constant-coefficient counterparts, so the exact explicit solutions to the vc-NLEEs are obtained accordingly.

The remainder of this paper is organized as follows. In Section 2, through the improved CK transformation method, the vc-NLEEs are transformed into its constant-coefficient counterparts in the same form of the former. In Section 3, the complete Lie group classification of the generalized equation is performed, all of the point symmetries of the equations are obtained. In Section 4, the exact explicit solutions to the generalized vc-NLEEs are investigated through the similarity reductions and CK transformations. In Section 5, the explicit conservation laws of the equations are given in terms of the nonlinear self-adjoint method. Finally, the conclusion and some remarks will be given in Section 6.

2 CK transformations for the vc-NLEEs

In this section, we employ the direct reduction method for investigating the relationship between variable-coefficient equations (1.1)-(1.3) and its corresponding constant-coefficient counterparts.

Firstly, we assume that Eq. (1.1) can be transformed into the following constant-coefficient equation

$$u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_x + \delta u = 0, \tag{2.1}$$

by the CK transformation as follows:

$$u \equiv u(x,t) = A(x,t) + B(x,t)U(X,T), \tag{2.2}$$

here $X = X(x,t)$, $T = T(x,t)$ and $A = A(x,t)$, $B = B(x,t)$ are functions of x and t to be determined by requiring that $U = U(X,T)$ satisfies the same KdV type of equation as $u = u(x,t)$ with the transformation $\{u, x, t\} \to \{U, X, T\}$. That is, requiring that $\{U, X, T\}$ satisfy Eq. (2.1), i.e.,

$$U_T + \alpha U U_X + \beta U_{XXX} + \gamma U_X + \delta U = 0, \tag{2.3}$$

where α, β, γ and δ are all constants, and $\alpha\beta \neq 0$ in Eqs. (2.1) and (2.3).

Then, substituting (2.2) into Eq. (1.1), through the CK reduction method, we get the following results:

Case I In general, $l(t) \neq 0$. In this case, we have

$$A = p(t)x - e^{-\int (fp+l)dt}\Big[\int hpe^{\int (fp+l)dt}dt - c_2\Big], \quad B = \frac{\alpha g r^2}{\beta f},$$
$$T = \frac{1}{\beta}\int g r^3 dt + c_4, \quad X = r(t)x + \frac{\gamma}{\beta}\int g r^3 dt - \int f g r dt + c_5, \tag{2.4}$$

where $p = p(t) = \frac{l}{e^{(t-c_1)l} \pm f}$, $r = r(t) = c_3 e^{-\int f p dt}$, c_i $(i = 1, 2, ..., 5)$ are arbitrary constants.

Thus, if we obtain the exact solution to Eq. (2.1), then the exact solution to Eq. (1.1) can be given through the transformation as follows

$$u = p(t)x - e^{-\int (fp+l)dt}\Big[\int hpe^{\int (fp+l)dt}dt - c_2\Big]$$

$$+\frac{\alpha g r^2}{\beta f}U\Big(r(t)x + \frac{\gamma}{\beta}\int gr^3 dt - \int fgr dt + c_5, \frac{1}{\beta}\int gr^3 dt + c_4\Big), \tag{2.5}$$

under the condition $\alpha\beta f g' - \alpha\beta f' g + \beta f g l - \alpha\beta f^2 g p - \alpha\delta f g^2 r^3 = 0$, here $p = p(t)$ and $r = r(t)$ are given by (2.4). In other words, under this condition, Eq. (1.1) can be transformed into the constant-coefficient equation (2.3) through the transformation (2.2).

Case II If $l(t) = 0$, then Eq. (1.1) becomes Eq. (1.2), and $\delta = 0$ in Eqs. (2.1) and (2.3), correspondingly. In this case, we have

$$A = \frac{1}{F(t)+c_1}(x + c_2 - H(t)), \quad B = \frac{c_3^2\alpha g}{\beta(F+c_1)^2 f},$$

$$X = \frac{c_3(x+c_2)}{F+c_1} + \frac{c_3^3\gamma}{\beta}\int \frac{g}{(F+c_1)^3}dt - \frac{c_3 H}{F+c_1} + c_4, \quad T = \frac{c_3^3}{\beta}\int \frac{g}{(F+c_1)^3}dt + c_5, \tag{2.6}$$

where $F = F(t) = \int f(t)dt$, $H = H(t) = \int h(t)dt$, c_i $(i = 1, 2, ..., 5)$ are arbitrary constants.

Thus, if we obtain the exact solution to Eq. (2.1) with $\delta = 0$, then the exact solution to Eq. (1.2) can be given through the transformation as follows

$$u = \frac{1}{F(t)+c_1}(x + c_2 - H(t))$$

$$+\frac{c_3^2\alpha g}{\beta(F+c_1)^2 f}U\Big(\frac{c_3(x+c_2)}{F+c_1} + \frac{c_3^3\gamma}{\beta}\int \frac{g}{(F+c_1)^3}dt - \frac{c_3 H}{F+c_1} + c_4, \frac{c_3^3}{\beta}\int \frac{g}{(F+c_1)^3}dt + c_5\Big), \tag{2.7}$$

under the condition $fg' - f'g - \frac{f^2 g}{F+c_1} = 0$, here $F = F(t)$ is given by (2.6). In other words, under this condition, Eq. (1.2) can be transformed into the constant-coefficient equation (2.3) with $\delta = 0$, through the transformation (2.2).

In particular, if $h(t) = l(t) = 0$, then Eq. (1.1) reduces to Eq. (1.3), and $\gamma = \delta = 0$ in Eqs. (2.1) and (2.3) at the same time. In this case, we have

$$A = \frac{1}{F(t)+c_1}(x + c_2), \quad B = \frac{c_3^2\alpha g}{\beta(F+c_1)^2 f},$$

$$X = \frac{c_3}{F+c_1}(x+c_2) + c_4, \quad T = \frac{c_3^3}{\beta}\int \frac{g}{(F+c_1)^3}dt + c_5, \tag{2.8}$$

where $F = F(t) = \int f(t)dt$, c_i $(i = 1, 2, ..., 5)$ are arbitrary constants.

Thus, if we obtain the exact solution to Eq. (2.1) with $\gamma = \delta = 0$, then the exact solution to Eq. (1.3) can be given through the transformation as follows

$$u = \frac{1}{F(t)+c_1}(x+c_2)$$
$$+\frac{c_3^2\alpha g}{\beta(F+c_1)^2 f}U\Big(\frac{c_3(x+c_2)}{F+c_1}+c_4, \frac{c_3^3}{\beta}\int\frac{g}{(F+c_1)^3}dt+c_5\Big), \tag{2.9}$$

under the condition $fg' - f'g - \frac{f^2 g}{F+c_1} = 0$, here $F = F(t)$ is given by (2.6). In other words, under this condition, Eq. (1.3) can be transformed into the constant-coefficient equation (2.3) with $\gamma = \delta = 0$, through the transformation (2.2).

Summering the above discussion, we have

Theorem 2.1 If $U = U(X, T)$ is a solution to Eq. (2.3), then $u = A + BU(X, T)$ is a solution to Eq. (1.1), where A, B, X and T are given by (2.4) under the condition (2.5). In particular, if $l = 0$, then $u = A + BU(X, T)$ is a solution to Eq. (1.2), where A, B, X and T are given by (2.6) under the condition (2.7). If $h = l = 0$, then $u = A + BU(X, T)$ is a solution to Eq. (1.3), where A, B, X and T are given by (2.8) under the condition (2.9). □

Therefore, if the exact solutions to the corresponding cc-NLEEs are obtained, then the exact solutions to the vc-NLEEs are presented through the CK transformations immediately. So in what follows, we consider the symmetry reductions and exact solutions to the former only.

Remark 2.1 From our previous discussion, we can see that the coefficient function $l = l(t)$ rather than $h = h(t)$ affects the CK transformation of the vc-NLEEs greatly, in addition to the coefficients functions $fg \neq 0$.

3 Complete Lie group classification of Eq. (2.1)

In this section, we give the symmetries of Eq. (2.1).

Recall that the geometric vector field of a NLEE is as follows:

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{3.1}$$

where $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ are coefficient functions of the vector fields to be determined.

If the vector field (3.1) generates a symmetry of Eq. (2.1), then V must satisfy the Lie's symmetry condition

$$\mathrm{pr}^{(3)}V(\Delta)|_{\Delta=0}=0, \tag{3.2}$$

where $\Delta = u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_x + \delta u$. Thus, by the Lie symmetry analysis method, we get the complete Lie group classification of Eq. (2.1) as follows

Case I In general, if $\delta \neq 0$, then the vector field of Eq. (2.1) is

$$V_1=\partial_x, \quad V_2=\partial_t, \quad V_3=\alpha e^{-\delta t}\partial_x-\delta e^{-\delta t}\partial_u, \tag{3.3}$$

where $\alpha\delta \neq 0$ are arbitrary constants.

Case II In particular, if $\delta = 0$, then the vector field of Eq. (2.1) is

$$V_1=\partial_x, \quad V_2=\partial_t, \quad V_3=\alpha x\partial_x+\partial_u, \quad V_4=\alpha x\partial_x+3\alpha t\partial_t-(2\alpha u+2\gamma)\partial_u, \tag{3.4}$$

where $\alpha \neq 0$ and γ are arbitrary constants.

Remark 3.1 We note that (3.3) and (3.4) is the complete Lie group classification of Eq. (2.1) actually. If $\gamma = \delta = 0$, then we get the vector field of the general KdV equation

$$u_t+\alpha u u_x+\beta u_{xxx}=0 \tag{3.5}$$

as follows

$$V_1=\partial_x, \quad V_2=\partial_t, \quad V_3=\alpha x\partial_x+\partial_u, \quad V_4=x\partial_x+3t\partial_t-2u\partial_u, \tag{3.6}$$

where $\alpha\beta \neq 0$ are arbitrary constants.

4 Exact solutions to the vc-NLEEs

In Section 2, we give the transformation between the vc-NLEEs and cc-NLEEs. Thus, if the exact solutions to the cc-NLEEs are obtained, then the exact solutions to the corresponding vc-NLEEs are given through the CK transformations immediately. In Section 3, we present all of the point symmetries of the cc-NLEEs, now we investigate the exact solutions to these vc-NLEEs by the symmetry reductions and CK transformations. In what follows, we consider the general case $\delta \neq 0$. In this case, the vector field of Eq. (2.1) is (3.3).

(i) For $V_1 = \partial_x$, we have

$$u=\phi(\xi), \tag{4.1}$$

where $\xi = t$. Substituting (4.1) into Eq. (2.1), we have

$$\phi' + \delta\phi = 0, \tag{4.2}$$

where $\phi' = d\phi/d\xi$. Solving this equation, we get $\phi(\xi) = ce^{-\delta\xi}$. Thus, we obtain the exact solution to Eq. (1.1) as follows

$$u(x,t) = A + cBe^{-\delta T}, \tag{4.3}$$

where $A = A(x,t)$, $B = B(x,t)$ and $T = T(x,t)$ are given by (2.4), under the condition (2.5), c is an arbitrary constant.

(ii) For $V_3 = \alpha e^{-\delta t}\partial_x - \delta e^{-\delta t}\partial_u$, we have

$$u = \phi(\xi) - \frac{\delta}{\alpha}x, \tag{4.4}$$

where $\xi = t$. Substituting (4.4) into Eq. (2.1), we have

$$\phi' - \frac{\gamma\delta}{\alpha} = 0, \tag{4.5}$$

where $\phi' = d\phi/d\xi$. So we have $\phi(\xi) = \frac{\gamma\delta}{\alpha}\xi + c$. Thus, we obtain the exact solution to Eq. (1.1) as follows

$$u(x,t) = A + cB - \frac{\delta}{\alpha}B(X - \gamma T), \tag{4.6}$$

where $A = A(x,t)$, $B = B(x,t)$ and $T = T(x,t)$ are given by (2.4), under the condition (2.5), c is an arbitrary constant.

(iii) For $V = v\partial_x + \partial_t$ (v is a constant), we have

$$u = \phi(\xi), \tag{4.7}$$

where $\xi = x - vt$. Substituting (4.4) into Eq. (2.1), we have

$$\alpha\phi\phi' + \beta\phi''' + (\gamma - v)\phi' + \delta\phi = 0, \tag{4.8}$$

where $\phi' = d\phi/d\xi$.

(iv) For $V = v\partial_t + \alpha e^{-\delta t}\partial_x - \delta e^{-\delta t}\partial_u$ ($v \neq 0$ is a constant), we have

$$u = \frac{1}{v}e^{-\delta t} + \phi(\xi), \tag{4.9}$$

where $\xi = x + \frac{\alpha}{v\delta}e^{-\delta t}$. Substituting (4.9) into Eq. (2.1), we have

$$\alpha\phi\phi' + \beta\phi''' + \gamma\phi' + \delta\phi = 0, \tag{4.10}$$

where $\phi' = d\phi/d\xi$. We note that Eqs. (4.8) and (4.10) are the same type of nonlinear higher-order ODEs, they cannot be solved by the classical integration method in the general case $\delta \neq 0$.

Now, we seek the exact power series solutions to such equations. First, we suppose a solution to Eq. (4.8) in a power series form as follows

$$\phi(\xi) = \sum_{n=0}^{\infty} c_n \xi^n, \tag{4.11}$$

where the coefficients c_n $(n = 0, 1, 2, ...)$ are constants to be determined.

Substituting (4.11) into (4.8), and comparing coefficients, we obtain

$$c_{n+3} = \frac{-1}{\beta(n+1)(n+2)(n+3)}\Big[\alpha \sum_{k=0}^{n}(n+1-k)c_k c_{n+1-k} + (\gamma - v)(n+1)c_{n+1} + \delta c_n\Big], \tag{4.12}$$

for all $n = 0, 1, 2, ...$

On the other hand, we note that Eq. (4.8) has a constant solution $\phi = \phi(\xi) = 0$ (for $\delta \neq 0$), so we have $c_0 = 0$. Thus, for arbitrarily chosen constants c_1 and c_2, from (4.12), we have

$$c_3 = -\frac{1}{6\beta}(\gamma - v)c_1, \quad c_4 = -\frac{1}{24\beta}[\alpha c_1^2 + 2(\gamma - v)c_2 + \delta c_1],$$

and

$$c_5 = -\frac{1}{60\beta}[3\alpha c_1 c_2 + 3(\gamma - v)c_3 + \delta c_2],$$

and so on.

Hence, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (4.12) in a unique manner. This implies that for Eq. (4.8), there exists a power series solution (4.11) with the coefficients given by (4.12).

Now, we show the convergence of the power series solution (4.11) to Eq. (4.8). In fact, in view of (4.12), we have

$$|c_{n+3}| \leq M\Big(\sum_{k=0}^{n} |c_k||c_{n+1-k}| + |c_{n+1}| + |c_n|\Big), \quad n = 0, 1, 2, ..., \tag{4.13}$$

where $M = \max\{|\frac{\alpha}{\beta}|, |\frac{\gamma - v}{\beta}|, |\frac{\delta}{\beta}|\}$.

If we define a power series $P = P(\xi) = \sum_{n=0}^{\infty} p_n \xi^n$ by $p_0 = |c_0| = 0$, $p_1 = |c_1|$, $p_2 = |c_2|$ and

$$p_{n+3} = M\Big[\sum_{k=0}^{n} p_k p_{n+1-k} + p_{n+1} + p_n\Big], \quad n = 0, 1, 2, ...,$$

then by the induction method, it is easily seen that

$$|c_n| \leq p_n, \quad n = 0, 1, 2, ...$$

In other words, $P(\xi) = \sum_{n=0}^{\infty} p_n \xi^n$ is a majorant series of (4.11).

Now, we prove that $P = P(\xi)$ has a positive radius of convergence. Indeed, noting that by the formal power series calculation, we have

$$\begin{aligned} P = P(\xi) &= p_1\xi + p_2\xi^2 + \sum_{n=0}^{\infty} p_{n+3}\xi^{n+3} \\ &= p_1\xi + p_2\xi^2 + M\Big[\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} p_k p_{n+1-k}\Big)\xi^{n+3} + \sum_{n=0}^{\infty} p_{n+1}\xi^{n+3} + \sum_{n=0}^{\infty} p_n \xi^{n+3}\Big] \\ &= p_1\xi + p_2\xi^2 + M\xi^2[P^2(\xi) + \xi P(\xi) + P(\xi)]. \end{aligned}$$

Consider now the implicit functional equation

$$F = F(\xi, P) = P - p_1\xi - p_2\xi^2 - M\xi^2(P^2 + \xi P + P) = 0.$$

Since F is analytic on a disk with the origin as the center and $F(0,0) = 0$, $F'_P(0,0) = 1 \neq 0$, by the implicit function theorem [20,21], we see that $P = P(\xi)$ is analytic in a neighborhood of the origin and with a positive radius. This completes the proof.

Therefore, the general solution in power series form of Eq. (4.8) can be written as follows:

$$\begin{aligned} \phi(\xi) &= c_1\xi + c_2\xi^2 + \sum_{n=0}^{\infty} c_{n+3}\xi^{n+3} = c_1\xi + c_2\xi^2 - \frac{1}{6\beta}(\gamma - v)c_1\xi^3 \\ &- \sum_{n=1}^{\infty} \frac{1}{\beta(n+1)(n+2)(n+3)}\Big[\alpha\sum_{k=0}^{n}(n+1-k)c_k c_{n+1-k} + (\gamma - v)(n+1)c_{n+1} + \delta c_n\Big]\Big]\xi^{n+3}. \end{aligned} \tag{4.14}$$

So, we obtain the corresponding exact traveling wave solution to vc-NLEE (1.1) in power series form as follows

$$\begin{aligned} u(x,t) &= A + B\Big[c_1(X - vT) + c_2(X - vT)^2 - \frac{1}{6\beta}(\gamma - v)c_1(X - vT)^3 \\ &- \sum_{n=1}^{\infty} \frac{1}{\beta(n+1)(n+2)(n+3)}\Big(\alpha\sum_{k=0}^{n}(n+1-k)c_k c_{n+1-k} + (\gamma - v)(n+1)c_{n+1} + \delta c_n\Big)(X - vT)^{n+3}\Big], \end{aligned} \tag{4.15}$$

where c_1 and c_2 are arbitrary constants, c_{n+3} $(n = 0, 1, 2, ...)$ are given by (4.12) successively, $A = A(x,t)$, $B = B(x,t)$, $X = X(x,t)$ and $T = T(x,t)$ are given by (2.4), under the condition (2.5), v is the wave speed.

Second, we suppose that Eq. (4.10) has a solution in a power series of the form (4.11) also. Then substituting (4.11) into (4.10), and comparing coefficients, we obtain

$$c_{n+3} = \frac{-1}{\beta(n+1)(n+2)(n+3)}\Big[\alpha\sum_{k=0}^{n}(n+1-k)c_k c_{n+1-k} + (n+1)\gamma c_{n+1} + \delta c_n\Big], \tag{4.16}$$

for all $n = 0, 1, 2, ...$

Similar to the above, we obtain the exact solution to vc-NLEE (1.1) in power series form as follows

$$u(x,t) = A + B\Big[\frac{1}{v}e^{-\delta T} + c_1(X + \frac{\alpha}{\delta v}e^{-\delta T}) + c_2(X + \frac{\alpha}{\delta v}e^{-\delta T})^2 - \frac{\gamma c_1}{6\beta}(X + \frac{\alpha}{\delta v}e^{-\delta T})^3$$
$$-\sum_{n=1}^{\infty}\frac{1}{\beta(n+1)(n+2)(n+3)}\Big(\alpha\sum_{k=0}^{n}(n+1-k)c_k c_{n+1-k} + (n+1)\gamma c_{n+1} + \delta c_n\Big)(X + \frac{\alpha}{\delta v}e^{-\delta T})^{n+3}\Big], \tag{4.17}$$

where c_1 and c_2 are arbitrary constants, c_{n+3} $(n = 0, 1, 2, ...)$ are given by (4.16) successively, $A = A(x,t)$, $B = B(x,t)$, $X = X(x,t)$ and $T = T(x,t)$ are given by (2.4), under the condition (2.5), $v \neq 0$ is an arbitrary constant.

5 Conservation laws of Eq. (2.1)

In the preceding section, we give all of the point symmetries of Eq. (2.1), the symmetries of the other equations are presented successively. In this section, we give the explicit conservation laws in terms of the point symmetries. Since this equation is not a self-adjoint in general, so we employ the the nonlinear self-adjoint concept [22], it is the generalization of the self-adjoint method.

First, for Eq. (2.1), if we write that

$$F = u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_x + \delta u = 0, \tag{5.1}$$

then the adjoint equation to this equation is as follows

$$F^* = v_t + \alpha u v_x + \beta v_{xxx} + \gamma v_x - \delta v = 0, \tag{5.2}$$

where $v = v(x,t)$ is a new dependent variable of x and t.

Clearly, if $\delta \neq 0$, then Eq. (2.1) is not self-adjoint. In particular, if $\delta = 0$, then it is self-adjoint.

Second, for Eq. (2.1), the formal Lagrangian is

$$L = v(u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_x + \delta u), \tag{5.3}$$

where $v = v(x,t)$ is defined by (5.2). Moreover, we consider the conservation laws of Eq. (2.1) in following cases, respectively.

(I) In general, $\delta \neq 0$. In this case, Eq. (2.1) is not self-adjoint. If we get $v = \varphi(x,t,u) \neq 0$, such that

$$F^*|_{v=\varphi} = \lambda F, \tag{5.4}$$

then Eq. (2.1) is said to be nonlinearly self-adjoint. Furthermore, through the similar procedure to symmetry analysis, we get $\varphi = ue^{2L}(c_3 - c_1 \int f e^{-L} dt) + e^L(c_1 x - c_1 H + c_2)$, where $H = H(t) = \int h(t)dt$, $L = L(t) = \int l(t)dt$, c_i $(i = 1,2,3)$ are constants.

Based on the generators $V = (k_1 e^{-\delta t} + k_3)\partial_x + k_2 \partial_t - \frac{\delta}{\alpha} k_1 e^{-\delta t}\partial_u$ (k_1, k_2 and k_3 are arbitrary constants), the conservation laws of Eq. (2.1) is as follows

$$D_t(C^1) + D_x(C^2)|_{(2.1)} = 0, \tag{5.5}$$

with the following components of the conserved vector $C = (C^1, C^2)$:

$$C^1 = \varphi(k_2 F + W),$$

$$\begin{aligned} C^2 = {} & \varphi[(k_1 e^{-\delta t} + k_3)F - \beta(k_1 e^{-\delta t} + k_3)u_{xxx} - \beta k_2 u_{xxt}] + W\Big[\alpha\varphi u + \gamma\varphi \\ & + \beta\Big(c_3 - c_1 \int f e^{-L} dt\Big)e^{2L} u_{xx}\Big] + \beta[(k_1 e^{-\delta t} + k_3)u_{xx} - k_2 u_{xt}]\Big[c_1 e^L + \Big(c_3 - c_1 \int f e^{-L} dt\Big)e^{2L} u_x\Big], \end{aligned} \tag{5.6}$$

where $\varphi = ue^{2L}(c_3 - c_1 \int f e^{-L} dt) + e^L(c_1 x - c_1 H + c_2)$, $F = u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_x + \delta u$, $W = -\frac{\delta}{\alpha} k_1 e^{-\delta t} - (k_1 e^{-\delta t} + k_3)u_x - k_2 u_t$, $H = H(t) = \int h(t)dt$, $L = L(t) = \int l(t)dt$, c_i and k_i $(i = 1,2,3)$ are arbitrary constants.

(II) In particular, $\delta = 0$. In this case, Eq. (2.1) is self-adjoint. So we have $v = u$.

Based on the generators $V = (c_1 x + c_3 t + c_4)\partial_x + (3c_1 t + c_2)\partial_t - (2c_1 u + \frac{2\gamma}{\alpha} c_1 x + \frac{1}{\alpha} c_3)\partial_u$ (c_1, c_2 and c_3 are arbitrary constants), the conservation laws of Eq. (2.1) is (5.5) with the following components of the conserved vector $C = (C^1, C^2)$:

$$C^1 = \alpha(3c_1 t + c_2)u^2 u_x + \beta(3c_1 t + c_2)u u_{xxx} - [c_1 x - (3\gamma c_1 - c_3)t + c_4]u u_x - 2c_1 u^2 - \frac{2\gamma c_1 - c_3}{\alpha} u,$$

$$C^2 = [c_1 x - (3\gamma c_1 - c_3)t - \gamma c_2 + c_4]uu_t + \beta(c_1 x + c_3 t + c_4)uu_{xxx} - 2\alpha c_1 u^3 - 2\beta c_1 uu_{xx}$$

$$-(4\gamma c_1 - c_3)u^2 - \frac{\gamma}{\alpha}(2\gamma c_1 - c_3)u - \frac{\beta}{\alpha}(2\gamma c_1 - c_3)u_{xx} + 2\beta(c_1 x + c_3 t + c_4)u_x u_{xx}$$

$$-\alpha(3c_1 t + c_2)u^2 u_t - \beta(3c_1 t + c_2)u_{xx}u_t + 3\beta c_1 u_x^2 + \beta(3c_1 t + c_2)u_x u_{xt} - 4\beta c_1 u_{xx}^2$$

$$-\beta(c_1 x + c_3 t + c_4)u_{xx}u_{xxx} - \beta(3c_1 t + c_2)u_{xx}u_{xxt}, \quad (5.7)$$

where c_i $(i = 1, ..., 4)$ are arbitrary constants.

6 Conclusion and remarks

In the current paper, the generalized variable-coefficient KdV types of equations are investigated by the improved CK reduction transformation method. Under some constraint conditions, the vc-PDEs are transformed into its constant-coefficient counterparts. Then, all of the point symmetries are obtained, the exact solutions to the vc-PDEs are presented through the symmetry reduction and CK transformation method. Moreover, the explicit conservation laws of the equations are investigated by the nonlinear self-adjoint method. However, how to consider the other properties of vc-PDEs through such CK transformation, it is an interesting and promising problem, we hope to investigate it in the future.

Remark 6.1 Generally speaking, the improved CK transformations in the current paper are equivalent transformations, they transform the vc-PDEs into its constant-coefficient counterparts under some conditions. Conversely, they can transform the exact solutions and other properties of the latter into the former. From the above discussion, we can see that compare to the other aforementioned methods, this CK transformation method is more effective and direct for tackling exact solutions to the vc-PDEs. Moreover, we think it will plays a more great role in studying other properties of vc-PDEs.

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