

Accepted Manuscript

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PII: S0377-0427(18)30159-6

DOI: <https://doi.org/10.1016/j.cam.2018.03.013>

Reference: CAM 11565

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 8 September 2017

Revised date: 1 February 2018

Please cite this article as: J.M. Gutiérrez, M.Á. Hernández-Verón, An acceleration of the continuous Newton's method, *Journal of Computational and Applied Mathematics* (2018), <https://doi.org/10.1016/j.cam.2018.03.013>

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AN ACCELERATION OF THE CONTINUOUS NEWTON'S METHOD

J. M. GUTIÉRREZ AND M. Á. HERNÁNDEZ-VERÓN

ABSTRACT. In this work we study some numerical properties of the continuous Newton's method, the continuous version of the classical Newton's method for solving nonlinear equations $p(z) = 0$. In fact, continuous Newton's method is an initial value problem whose solutions flow to a root of the equation. We show the influence of the multiplicity of the roots of the considered equation in the Jacobian matrix related to the problem. In addition we study some modifications of the continuous Newton's method that allow us to increase the velocity of the convergence of the solutions towards the roots of $p(z) = 0$.

1. INTRODUCTION

In this work we are concerned with some numerical properties of the initial value problem

$$(1) \quad z(0) = z_0, \quad z'(t) = -\frac{p(z(t))}{p'(z(t))},$$

where $z : \mathbb{R} \rightarrow \mathbb{C}$ and $p(z)$ is a given complex function. This problem is called continuous Newton's method and it was initially posed in the late 1980s (see [5], [7], [8]). Later, the method was popularized by Neuberger [6].

Note that the classical Newton's method for solving a nonlinear equation $p(z) = 0$,

$$(2) \quad z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)},$$

can be seen as an Euler step to the differential equation (1). Even more, a general Euler approximation of the differential equation (1) with step size $h > 0$ gives rise to the damped Newton's method

$$(3) \quad z_{n+1} = z_n - h \frac{p(z_n)}{p'(z_n)}.$$

This relationship between the continuous and discrete Newton's method has been analyzed by Jacobsen et al. ([4]). Actually, they have studied the basins of attraction of the roots for methods (3) and they have realized that their fractal structure shrinks away when $h \rightarrow 0$. This fact has been also pointed out by Epureanu and Greenside ([1]). In addition, in [4] other ODE solvers, like Runge-Kutta methods or Adams-Bashforth methods have been considered. The influence of the step size in the fractal dimension of the basin boundaries is the same than in the Euler's case. However, higher-order approximations for solving ODEs did not necessarily produce root-finding methods with a good numerical efficiency. In [2] the relationship between the step size and the order of convergence is particularly considered. For a constant step size, these iterative root-finding methods have a low order of convergence (linear) but a high computational cost. For a non-constant step size, a plethora of root-finding methods can be constructed. In particular, the famous Chebyshev-Halley family of methods can be constructed in this way.

Even more, we show in this paper that higher-order methods for solving nonlinear equations can be derived from continuous Newton's method. Actually, the well-known family of iterative methods discovered by Schröder (see [3], for instance) can be derived from the continuous Newton's method. Let us define $q(u)$ the function inverse of $p(z)$ and let z^* be a simple root of p ; thus

$$p(z) = u, \quad q(u) = z$$

and

$$p(z^*) = 0, \quad q(0) = z^*.$$

In order to look forward a one-point iterative method that converges with order $k \geq 1$ to z^* , we consider the Taylor's approximation of the inverse function q :

$$(4) \quad \begin{aligned} q(0) \approx & q(u(t)) - q'(u(t))u(t) + \frac{1}{2}q''(u(t))u(t)^2 + \dots \\ & + \dots + \frac{1}{j!}q^{(j)}(u(t))(-1)^j u(t)^j + \dots \end{aligned}$$

If we consider the symbolic operators

$$[\cdot]' = \frac{d}{dt}, \quad D = \frac{d}{du},$$

we have

$$q^{(j)}(u(t)) = D^{(j-1)} \left[\frac{1}{p'(z(t))} \right].$$

From equation (4) the following iterative methods of order $k \geq 2$ can be derived, $z_{n+1} = \varphi_k(z_n)$, where

$$\varphi_k(z) = z - \sum_{j=0}^{k-2} \frac{(-1)^j}{j!} D^{(j-1)} \left[\frac{1}{p'(z)} \right] p(z)^{j+1}$$

Note that for $k = 2$, Newton's method (2) is obtained whereas Chebyshev's method appears for $k = 3$:

$$\varphi_3(z) = z - \left(1 + \frac{1}{2} \frac{p(z)p''(z)}{p'(z)^2} \right) \frac{p(z)}{p'(z)}.$$

In general, φ_k gives raise to methods with order of convergence k (see [3] for details).

On the other hand, Neuberger ([6]) used the following improved version of continuous Newton's method for obtaining roots of a nonconstant complex polynomial p . For a given $z_0 \in \mathbb{C}$, the target is to get a continuous function $z : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$(5) \quad z(0) = z_0, \quad p(z)'(t) = -p(z(t)), \quad t \in \mathbb{R}.$$

Neuberger characterized the domains of attraction for continuous Newton's method in three theorems, that can be summarize in the following results.

Theorem 1. (Neuberger, 1999) Let $\mathcal{Q}$ be the set of functions $z : \mathbb{R} \rightarrow \mathbb{C}$ which solve the differential equation (5). Then, if $z \in \mathcal{Q}$, then $u = \lim_{t \rightarrow \infty} z(t)$ exists and $p(u) = 0$.

The range of a member of $\mathcal{Q}$ is called trajectory.

Theorem 2. (Neuberger, 1999) Every point of $\mathbb{C}$ is contained in some trajectory.

An incoming trajectory is a subset $z((-\infty, d])$ such that

$$p(y) \neq 0, \quad p'(y) = 0, \quad z(d) = y$$

for some $y \in \mathbb{C}$, $d \in \mathbb{R}$ and $z \in \mathcal{Q}$. An outgoing trajectory is $\omega([d, \infty))$. Let us denote $\mathcal{M}$ the members in $\mathbb{C}$ which belong to no incoming trajectory.

Theorem 3. (Neuberger, 1999) Every component (maximal connected subset) of $\mathcal{M}$ contains just one root of p . If $z(t)$ intersects a component of $\mathcal{M}$ then $u = \lim_{t \rightarrow \infty} z(t)$ is the root in such component.

In this paper we consider the modified continuous Newton's method

$$(6) \quad \omega(0) = \omega_0, \quad p(\omega)'(t) = -kp(\omega(t)), \quad k > 0.$$

Our aim is to extend the results given by Neuberger and to analyze the influence of the parameter k in the velocity of convergence of the flow to the solutions.

2. A MODIFIED CONTINUOUS NEWTON'S METHOD

In this section we show how the theoretical framework established by Neuberger for continuous Newton's method (5) naturally extend to its modified version (6). As in the previous section, we assume that p is a nonconstant complex polynomial.

Lemma 4. *Let ω be a solution of (6). Then, $p(\omega(t)) = p(\omega_0)e^{-kt}$.*

Proof. This first result follows immediately after integrating in the formula

$$\frac{p'(\omega(t))}{p(\omega(t))} \omega'(t) = -k,$$

with $\omega(0) = \omega_0$ □

As in the case of the continuous Newton's method, for each $k > 0$, let us introduce the set $\mathcal{Q}_k$ of functions $\omega : \mathbb{R} \rightarrow \mathbb{C}$ which solve the differential equation (6). We can deduce that the modified continuous Newton's method has the same properties than the classical Neuberger's problem (5). Actually, the change of variable $\omega(t) = z(kt)$ reduces (6) to (5). In this way, the extension of Theorems 1, 2 and 3 to (6) follows straightforward. Anyway, to show some of the ideas behind Neuberger's proof, we develop here the following result.

Theorem 5. *For a given $k > 0$, if $\omega \in \mathcal{Q}_k$, then $u = \lim_{t \rightarrow \infty} \omega(t)$ exists and $p(u) = 0$.*

Proof. Let ω be a solution of (6). The set $\Omega = \{\omega(t); t \geq 0\}$ is bounded. Actually, if $s \geq 0$, $\omega(s)$ is a root of the polynomial

$$p_{s, \omega_0}(y) = p(y) - e^{-ks}p(\omega_0), \quad y \in \mathbb{C}.$$

As a consequence of Rouché's theorem, the set of roots of $p_{s, \omega_0}(y)$ is bounded.

Now let us suppose that $\lim_{t \rightarrow \infty} \omega(t)$ does not exist. As Ω is bounded, for each sequence $\{t_k\}_{k \geq 0}$ there exists a point $v \in \mathbb{C}$ such that $\lim_{k \rightarrow \infty} \omega(t_k) = v$. By Lemma 4, v is a root of p . Consequently, the trajectory $z(t)$ is moving infinitely between the roots of p . Let us denote α_i , $i = 1, \dots, n$, these roots and $\mathcal{U} = \bigcup_{i=1}^n D(\alpha_i, r_i)$ a disjoint union of open disks, containing one root each of them. As z is a continuous function, it takes infinitely many values in $\Omega \setminus \mathcal{U}$. But $\Omega \setminus \mathcal{U}$ is a compact set, so there exists a sequence $\{s_k\}_{k \geq 0} \subseteq \Omega \setminus \mathcal{U}$ with $\lim_{k \rightarrow \infty} \omega(s_k) = v'$, with v' a root of p , that is a contradiction. So $\lim_{t \rightarrow \infty} \omega(t)$ exists. □

Now, we are interested in the study of the fixed points and equilibrium solutions of the modified continuous Newton's method (6).

Generally speaking, the fixed points (x^*, y^*) of a system of differential equations

$$(7) \quad \begin{cases} x'(t) &= f(x(t), y(t)), \\ y'(t) &= g(x(t), y(t)), \end{cases}$$

represent equilibrium solutions of the problem, since if $x(t) = x^*$ and $y(t) = y^*$ initially, then $x(t) = x^*$ and $y(t) = y^*$ for all time. We are interested in the study

of the equilibria of the modified continuous Newton's method (6). Actually, if we make $z = x + iy$ we obtain a system of differential equations (7) where

$$(8) \quad f(x, y) = \operatorname{Re} \left(-\frac{p(x + iy)}{p'(x + iy)} \right), \quad g(x, y) = \operatorname{Im} \left(-\frac{p(x + iy)}{p'(x + iy)} \right).$$

Our first result characterizes the equilibria of (7) and establishes their local dynamical behavior in terms of the corresponding Jacobian matrices.

Theorem 6. *Let $z^* = x^* + iy^*$ be a zero with multiplicity m of $p(z)$. Then (x^*, y^*) is an asymptotically stable fixed point of the system (7) with f and g given by (8). In addition, if we denote $F(x, y)$ to the vector field $F(x, y) = (f(x, y), g(x, y))$, we have that the Jacobian matrix of $F(x, y)$ at (x^*, y^*) is*

$$(9) \quad F'(x^*, y^*) = \begin{pmatrix} -k/m & 0 \\ 0 & -k/m \end{pmatrix}.$$

Proof. Let us write $p(z) = (z - z^*)^m q(z)$ with $q(z^*) \neq 0$ and $R(z) = -kp(z)/p'(z)$. Then

$$R(z) = (z - z^*)R_1(z), \quad R_1(z) = \frac{-kq(z)}{mq(z) + (z - z^*)q'(z)}.$$

Therefore $R'(z) = R_1(z) + (z - z^*)R_1'(z)$ and $R'(z^*) = -k/m$.

Now we consider the real and imaginary parts of $R(x + iy) = f(x, y) + ig(x, y)$, where f and g given by (8), and we take into account the Cauchy-Riemann equations to deduce the following relationships

$$-\frac{k}{m} = R'(x^* + iy^*) = f_x(x^*, y^*) + ig_x(x^*, y^*) = g_y(x^*, y^*) - if_y(x^*, y^*).$$

As a consequence $f_x(x^*, y^*) = g_y(x^*, y^*) = -k/m$ and $f_y(x^*, y^*) = g_x(x^*, y^*) = 0$ and the Jacobian matrix of $F(x, y)$ at (x^*, y^*) is given by (9). As the eigenvalues of $F'(x^*, y^*)$ are negative, (x^*, y^*) is an asymptotically stable fixed point of the system (7). $\square$

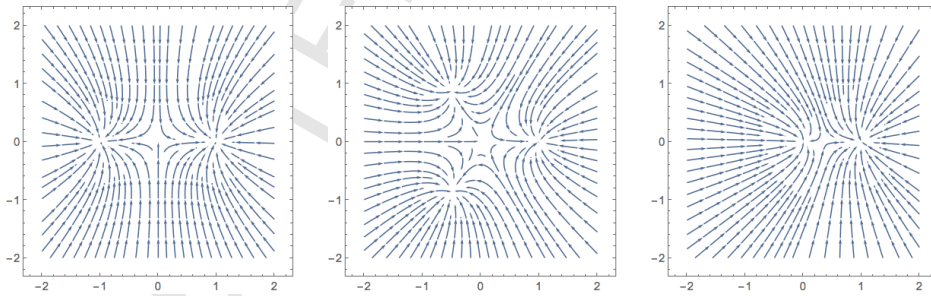


FIGURE 1. Phase portraits of the continuous Newton's method applied to the polynomials $p(z) = z^2 - 1$, $p(z) = z^3 - 1$ and $p(z) = z(z - 1)^2$.

Remark 1. *Note that $F'(x^*, y^*)$ is a diagonal matrix with a double eigenvalue of $\lambda = -k/m$. So from a local point of view, (x^*, y^*) behaves as a shrink star node whose typical phase portrait is shown in Figure 1. In fact, when the trajectories $(x(t), y(t))$ are close enough to the equilibrium point (x^*, y^*) , these trajectories behaves as the functions*

$$\begin{cases} x(t) &= x^* + C_1 e^{-kt/m} \\ y(t) &= y^* + C_2 e^{-kt/m}. \end{cases}$$

So the convergence of the trajectories to the equilibrium point is slower when the multiplicity m increases. However the convergence of the trajectories to the equilibrium point is faster when the parameter k increases.

2.1. Numerical examples. We illustrate the theoretical results given previously with some numerical examples. In addition, we introduce other experiments to analyze the numerical properties of the continuous Newton's method and its variants.

Example 1. In this section we have seen that when $0 < k_1 < k_2$ the solutions for k_2 in (6) flow to the roots of $p(z) = 0$ in a faster way than the solutions for k_1 . We have considered different polynomials: $p_1(z) = z^2 - 1$, $p_2(z) = z^3 - 1$, $p_3(z) = z(z - 1)^2$ and we have solved the corresponding system of differential equations (6) with $\omega_0 = 2 + 3i$. We have use the function `NDSolve` in the mathematical computer program *Mathematica* for the calculations. We have computed the solutions for $k = 0.5, 1, 2, 3$ and shown the results in Figures 2, 3 and 4. In all cases, the solutions of the corresponding problems converge to the real root $z^* = 1$. So in these figures we have plotted the real and imaginary parts of the solution (the constants functions 1 and 0 respectively).

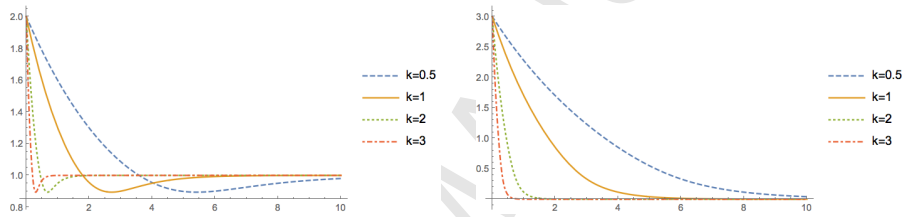


FIGURE 2. Graphics of the real and imaginary parts of the solutions of (6) for $p(z) = z^2 - 1$ and $k = 0.5, 1, 2, 3$.

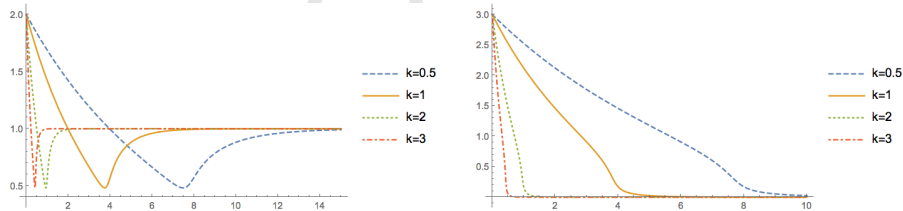


FIGURE 3. Graphics of the real and imaginary parts of the solutions of (6) for $p(z) = z^3 - 1$ and $k = 0.5, 1, 2, 3$.

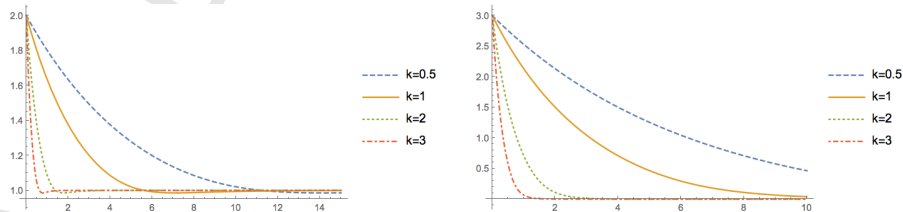


FIGURE 4. Graphics of the real and imaginary parts of the solutions of (6) for $p(z) = z(z - 1)^2$ and $k = 0.5, 1, 2, 3$.

Example 2. In this example we present a strategy to numerically solve problem (6). Instead of calculating the limits of the solution when $t \rightarrow \infty$ to obtain roots of

the considered polynomial, let us make a change of variable that allow us to work on a finite interval, actually on $[0, 1)$. If we define

$$y(t) = \omega \left(\log \left(\frac{1}{1-t} \right) \right)$$

and we take into account Lemma 4, we have

$$(10) \quad y(0) = \omega_0, \quad y'(t) = -kp(\omega_0) \frac{(1-t)^{k-1}}{p'(y(t))}.$$

If $\lim_{t \rightarrow 1^-} y(t) = u$, then $\lim_{t \rightarrow \infty} \omega(t) = u$.

We have reproduced the previous numerical experiment, for the same polynomials, initial condition and values of k but for the method (10). In all cases the interval of integration is $[0, 1)$. The results are shown in Figures 5, 6 and 7. As in the previous example, the solutions of the corresponding problems converge to the real root $z^* = 1$.

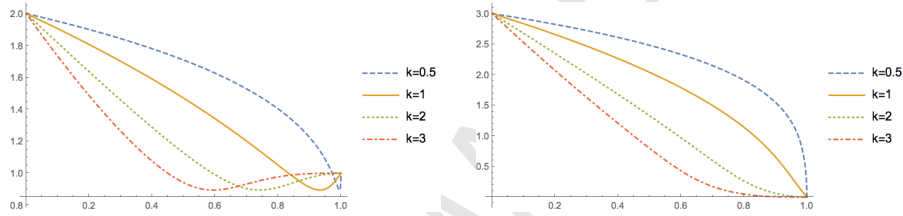


FIGURE 5. Graphics of the real and imaginary parts of the solutions of (10) for $p(z) = z^2 - 1$ and $k = 0.5, 1, 2, 3$.

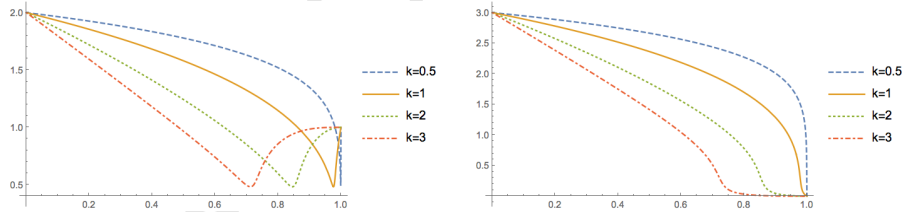


FIGURE 6. Graphics of the real and imaginary parts of the solutions of (10) for $p(z) = z^3 - 1$ and $k = 0.5, 1, 2, 3$.

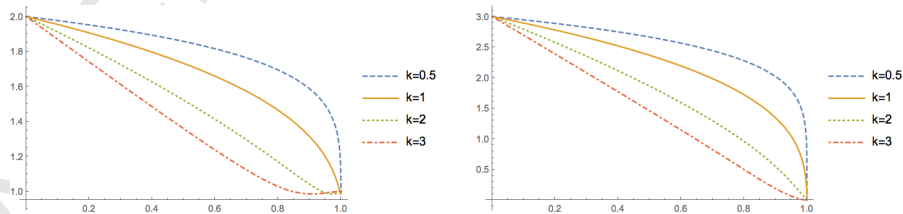


FIGURE 7. Graphics of the real and imaginary parts of the solutions of (10) for $p(z) = z(z-1)^2$ and $k = 0.5, 1, 2, 3$.

3. ANOTHER MODIFICATION OF THE CONTINUOUS NEWTON'S METHOD

We go a step beyond in the generalization of the continuous Newton's method by considering initial value problems in the form

$$(11) \quad \varpi(0) = \varpi_0, \quad \varpi'(t) = -k(t) \frac{p(\varpi(t))}{p'(\varpi(t))}$$

for a given (non constant) function $k(t)$. As a first step to determine suitable functions $k(t)$ that improve the convergence and numerical properties of the solutions of this problem, we consider the particular case $k(t) = \rho^t$, with $\rho > 1$. In this way, we obtain a modified continuous Newton's method in the form

$$(12) \quad \varpi(0) = \varpi_0, \quad \varpi'(t) = -\rho^t \frac{p(\varpi(t))}{p'(\varpi(t))}, \quad \rho > 1.$$

We can see that the sequence $\{p(\varpi(n))\}_{n \geq 0}$, where $\varpi(t)$ is a solution of (12), converges to 0 with order ρ . Actually, let us introduce the parameter $\alpha = e^{-1/\log \rho}$. Note that $\alpha < 1$ for $\rho > 1$. So

$$(13) \quad p(\varpi(t)) = p(\varpi_0) \alpha^{\rho^t - 1}$$

and

$$\lim_{t \rightarrow \infty} p(\varpi(t)) = 0.$$

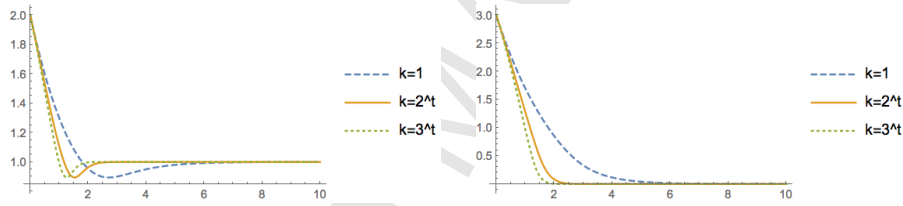


FIGURE 8. Graphics of the real and imaginary parts of the solutions of (12) for $p(z) = z^2 - 1$ and $k(t) = 1, 2^t, 3^t$.

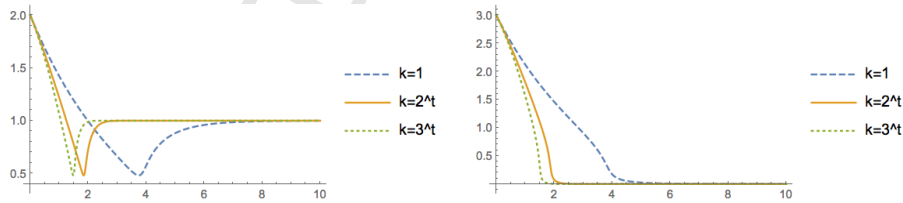


FIGURE 9. Graphics of the real and imaginary parts of the solutions of (12) for $p(z) = z^3 - 1$ and $k(t) = 1, 2^t, 3^t$.

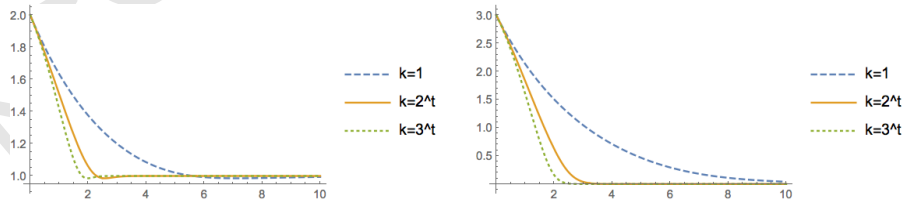


FIGURE 10. Graphics of the real and imaginary parts of the solutions of (12) for $p(z) = z(z-1)^2$ and $k(t) = 1, 2^t, 3^t$.

Then, if we denote $\varpi_n = \varpi(n)$, $n \in \mathbb{N}$, we conclude

$$p(\varpi_{n+1}) - p(\varpi_n) = p(\varpi_0) \alpha^{\rho^n - 1} \left(\alpha^{(\rho-1)\rho^n} - 1 \right),$$

that guarantees the order of convergence ρ .

In Figures 8, 9 and 10 we have plotted the solutions of (12) for $k(t) = 1$ (actually the continuous Newton's method), $k(t) = 2^t$ and $k(t) = 3^t$. If we compare these graphics with the obtained in Figures 2, 3 and 4, we appreciate that the solutions have a similar behavior. We see that taking high values of k in (6) or taking high values of ρ in (12) produces an acceleration in the flow of the corresponding solutions $\omega(t)$ and $\varpi(t)$ to the roots of the considered polynomial p . In addition, because of Lemma 4 and (13) we realize the influence of this conduct in the values $p(\omega(t))$ and $p(\varpi(t))$, as we illustrate in Tables 1–6.

TABLE 1. Numerical values of $(|1 - \omega(t)|, |p(\omega(t))|)$ for the cases $p(z) = z^2 - 1$ with $\omega_0 = 3 + 2i$ and $k = 1, 2, 3$.

t	$k = 1$	$k = 2$	$k = 3$
0	(3.16228, 13.4164)	(3.16228, 13.4164)	(3.16228, 13.4164)
1	(1.72648, 4.93562)	(0.12576, 0.24573)	(0.00083, 0.00166)
2	(0.85176, 1.81571)	(0.00225, 0.00450)	$(1.3 \times 10^{-7}, 2.1 \times 10^{-7})$
3	(0.34620, 0.667964)	(0.00004, 0.00008)	$(2.4 \times 10^{-10}, 4.8 \times 10^{-10})$
4	(0.12576, 0.24573)	$(7.5 \times 10^{-7}, 1.5 \times 10^{-6})$	$(5.3 \times 10^{-11}, 1.1 \times 10^{-10})$

TABLE 2. Numerical values of $(|1 - \omega(t)|, |p(\omega(t))|)$ for the cases $p(z) = z^3 - 1$ with $\omega_0 = 3 + 2i$ and $k = 1, 2, 3$.

t	$k = 1$	$k = 2$	$k = 3$
0	(3.16228, 47.8539)	(3.16228, 47.8539)	(3.16228, 47.8539)
1	(2.16143, 17.6045)	(0.45853, 0.87647)	(0.00197, 0.00591)
2	(1.45856, 6.47633)	(0.00538, 0.01605)	$(2.4 \times 10^{-7}, 7.3 \times 10^{-7})$
3	(0.96688, 2.38251)	(0.00010, 0.00029)	$(4.2 \times 10^{-10}, 1.3 \times 10^{-9})$
4	(0.45853, 0.87647)	$(1.8 \times 10^{-6}, 5.4 \times 10^{-6})$	$(1.6 \times 10^{-10}, 5.0 \times 10^{-10})$

4. CONCLUSIONS

In this paper we have studied the behavior of the following variant of the continuous Newton's method

$$\omega(0) = \omega_0, \quad \omega'(t) = -k \frac{p(\omega(t))}{p'(\omega(t))}, \quad k > 0.$$

and we have characterized the related fixed points in terms of the corresponding Jacobian matrix. We have proved that the main properties of the original continuous Newton's method are inherited by the modified method. In addition, we have seen that as bigger is k as faster is the convergence of the solution $\omega(t)$ to a root of p .

TABLE 3. Numerical values of $(|1 - \omega(t)|, |p(\omega(t))|)$ for the cases $p(z) = z(z - 1)^2$ with $\omega_0 = 3 + 2i$ and $k = 1, 2, 3$.

t	$k = 1$	$k = 2$	$k = 3$
0	(3.16228, 36.0555)	(3.16228, 36.0555)	(3.16228, 36.0555)
1	(2.22280, 13.2641)	(0.71380, 0.66038)	(0.06694, 0.004450)
2	(1.54795, 4.87958)	(0.11034, 0.01209)	(0.00074, 5.4×10^{-7})
3	(1.06262, 1.79510)	(0.01490, 0.00022)	(8.2×10^{-6} , 6.7×10^{-11})
4	(0.71380, 0.66038)	(0.00201, 4.0×10^{-6})	(9.2×10^{-8} , 8.5×10^{-15})

TABLE 4. Numerical values of $(|1 - \varpi(t)|, |p(\varpi(t))|)$ for the cases $p(z) = z^2 - 1$ with $\varpi_0 = 3 + 2i$ and $k = 1, 2^t, 3^t$.

t	$k = 1$	$k = 2^t$	$k = 3^t$
0	(3.16228, 13.4164)	(3.16228, 13.4164)	(3.16228, 13.4164)
1	(1.72648, 4.93562)	(1.28578, 3.17016)	(0.97861, 2.17276)
2	(0.85176, 1.81571)	(0.09008, 0.17700)	(1.3×10^{-7} , 0.00923)
3	(0.34620, 0.667964)	(0.00028, 0.00055)	(0.00462, 8.1×10^{-10})
4	(0.12576, 0.24573)	(2.8×10^{-9} , 5.8×10^{-9})	(4.7×10^{-10} , 9.5×10^{-10})

TABLE 5. Numerical values of $(|1 - \varpi(t)|, |p(\varpi(t))|)$ for the cases $p(z) = z^3 - 1$ with $\varpi_0 = 3 + 2i$ and $k = 1, 2^t, 3^t$.

t	$k = 1$	$k = 2^t$	$k = 3^t$
0	(3.16228, 47.8539)	(3.16228, 47.8539)	(3.16228, 47.8539)
1	(2.16143, 17.6045)	(1.81912, 11.3074)	(1.56691, 7.74986)
2	(1.45856, 6.47633)	(0.27819, 0.63133)	(0.01109, 0.03292)
3	(0.96688, 2.38251)	(0.00066, 0.00197)	(8.9×10^{-10} , 2.7×10^{-9})
4	(0.45853, 0.87647)	(6.8×10^{-9} , 2.1×10^{-8})	(1.6×10^{-11} , 4.8×10^{-11})

We have studied what happens if we change the parameter k by a non-constant function:

$$\varpi(0) = \varpi_0, \quad \varpi'(t) = -k(t) \frac{p(\varpi(t))}{p'(\varpi(t))}.$$

A first approach to the numerical properties of this kind of methods is made, by considering the particular case $k(t) = \rho^t$, with $\rho > 1$.

The numerical results we have developed reveal us that the studied methods (6) and (12) have a very good behavior for approaching roots of the polynomial p for big values of the involved parameters k and ρ and small values of t . For higher values of t the methods needed to solve these initial value problems seem to have some numerical instabilities, so a deeper numerical and theoretical analysis

TABLE 6. Numerical values of $(|1 - \varpi(t)|, |p(\varpi(t))|)$ for the cases $p(z) = z(z - 1)^2$ with $\omega_0 = 3 + 2i$ and $k = 1, 2^t, 3^t$.

t	$k = 1$	$k = 2^t$	$k = 3^t$
0	(3.16228, 36.0555)	(3.16228, 36.0555)	(3.16228, 36.0555)
1	(2.22280, 13.2641)	(1.89645, 8.51956)	(1.65330, 5.83912)
2	(1.54795, 4.87958)	(0.62260, 0.47567)	(0.15763, 0.02480)
3	(1.06262, 1.79510)	(0.03860, 0.00148)	(0.00004, 1.9×10^{-11})
4	(0.71380, 0.66038)	(0.00012, 1.4×10^{-8})	(8.5×10^{-11} , 7.2×10^{-21})

should be elaborated. In the same way, other choices of functions $k(t)$ in (11) need a theoretical study to determine their influence in the numerical and dynamical properties of the resultant initial value problem.

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