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On the local and semilocal convergence of a parameterized multi-step Newton method

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ABSTRACT

This paper is devoted to a family of Newton-like methods with frozen derivatives used to approximate a locally unique solution of an equation. We perform a convergence study and an analysis of the efficiency. This analysis gives us the opportunity to select the most efficient method in the family without the necessity of their implementation. The method can be applied to many type of problems, including the discretization of ordinary differential equations, integral equations, integro-differential equations or partial differential equations. Moreover, multi-step iterative methods are computationally attractive.

1. Introduction and review

Iterative methods for solving nonlinear systems of equations are an important research topic in Numerical Analysis. Nonlinear systems of equations appear for instance when discretizing ordinary differential equations, integral equations, integro-differential equations or partial differential equations. In particular, we mention a relevant application in signal processing: image denoising (see, e.g. [1,2]). Let $f : \Omega \to \mathbb{R}$ be a noisy image, to improve it, we solve the following optimization problem

$$\begin{aligned} &\text{Minimize } u : R(u) \\ &\text{subject to } \|u-f\|^2_{L^2(\Omega)} = E\left(\int_\Omega (u-f)^2 dx\right), \end{aligned} \tag{1}$$

where E is the expectation of the random variable X and $R(u)$ is the regularization functional as $R(u) = \|u\|$ or $\|\Delta(u)\|$, where Δ is the Laplacian (see [3]). However, using these functionals the edges of the images cannot be satisfactorily recovered. In [4], the TV-norm is proposed:

$$TV(u) = \int_\Omega |\nabla u(x)| dx, \tag{2}$$

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being ∇ the gradient. Thus, if there is not a good estimation to the variance of the noise, we can consider the unconstrained optimization problem. The Euler–Lagrange equation becomes

$$-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + \lambda(u - f) = 0, \tag{3}$$

being λ a positive parameter which establishes the relative importance of the smoothness of u and the quality of the approximation to the given signal f (see [1]). In [2] a linearization based on a dual variable is proposed introducing in Eq. (3) the following new variable with ϵ a regularized parameter:

$$w = \frac{\nabla u}{\sqrt{|\nabla u|^2 + \epsilon}}.$$

Then, we get

$$\begin{aligned} &-\nabla \cdot w + \lambda(u - f) = 0, \\ &w\sqrt{|\nabla u|^2 + \epsilon} - \nabla u = 0. \end{aligned} \tag{4}$$

Using the discretization showed in [1] for a regular mesh $h = 1/m$, $m \in \mathbb{N}$, $x_i = i \cdot h$, $i = 0, \ldots, m$, if in each iteration n we approximate the divergence and the gradient operators by

$$\nabla \cdot v(x_i) = \frac{v_i - v_{i-1}}{h},$$

we obtain the following problem:

$$\begin{aligned} &-\frac{w_i - w_{i-1}}{h} + \lambda(u_i - f_i) = 0, \quad w_1 = w_m = 0, \\ &w_i \cdot \sqrt{\left(\frac{u_i - u_{i-1}}{h}\right)^2 + \epsilon} - \frac{u_i - u_{i-1}}{h} = 0, \quad u_0 = f_0,\ u_m = f_m, \end{aligned} \tag{5}$$

for $i = 1, \ldots, m-1$. Then, we can define the following nonlinear operator:

$$\begin{aligned} &F_{2i-1}(u, w, \lambda_h, \epsilon_h) = -w_i - w_{i-1} + \lambda_h(u_i - f_i) = 0, \\ &F_{2i}(u, w, \lambda_h, \epsilon_h) = w_i \cdot \sqrt{(u_i - u_{i-1})^2 + \epsilon_h} - u_i - u_{i-1} = 0, \end{aligned} \tag{6}$$

with $\lambda_h = h\lambda$, $\epsilon_h = h^2\epsilon$, $w_0 = w_m = 0$, $u_0 = f_0$ and $u_m = f_m$. This nonlinear system is approximated in Matlab using the classical Newton method (see [1,2,5]). This is a basic iterative method for solving nonlinear systems of equations. This method is a second order iterative scheme using only first derivatives. The classical higher order methods, like Chebyshev or Halley iterative methods, compute high order derivatives. This fact increases too much the computational cost in general. However, it is possible to increase the order making several Newton-type steps. An appropriated procedure is to consider a freeze Jacobian since in this case the computational cost to solve the associated linear systems can be also reduced. In this paper, we study a novel multi-step iterative method for solving systems of nonlinear equations by introducing a parameter θ to generalize the multi-step Newton method while keeping its order of convergence and computational cost. By an appropriate selection of θ, the new method can both have faster convergence and have larger radius of convergence [6].

Let $F : \Omega \subset B_1 \to B_2$ be a continuously differentiable operator in the sense of Fréchet, where Ω is a nonempty convex set and B_1, B_2 are Banach spaces. Consider the equation

$$F(x) = 0. \tag{7}$$

Let $y_0^{(0)} \in \Omega$. We study the method defined for each $n = 0, 1, 2, \ldots$ by:

$$\text{Base part} \rightarrow \begin{cases} F'(y_0^{(n)})\phi_1 = F(y_0^{(n)}) \\ y_1^{(n)} = y_0^{(n)} - (1 + \theta - \theta^2)\phi_1 \\ F'(y_0^{(n)})\phi_2 = F(y_0^{(n)} - \frac{1}{\theta}\phi_1) \\ y_2^{(n)} = y_1^{(n)} - \theta^2\phi_2 \end{cases} \tag{8}$$

$$\text{Multi-step part} \rightarrow \begin{cases} \text{For } i = 1, \ldots, m-2 \\ F'(y_0^{(n)})\phi_{i+2} = F(y_{i+1}^{(n)}) \\ y_{i+2}^{(n)} = y_{i+1}^{(n)} - \phi_{i+2} \end{cases} \tag{9}$$

Set $y^{(n+1)} = y_0^{(n+1)} = y_m^{(n)}$ and $y^{(0)} = y_0^{(0)}$.

This new method produces relevant results as we can see in the following example shown in [6]. They applied it in comparison with the known iterative method resulting when $\theta = 1$ (see [7]) to solve a system of nonlinear equations in a

Table 1
Comparison between the method with $\theta = 1, 2$ for the test problem, Eq. (10), with domain $\Lambda = [1, 1] \times [0, 1.3]$, [6].

m	$\theta = 2$		
	E_u	E_v	E_w
1	2.3489	1.8274	0.49851
2	1.1081	0.54759	0.96708
7	0.10085	0.067689	0.029542
13	6.2449e−05	7.1607e−05	1.8598e−05
17	2.4756e−07	3.291e−07	7.4927e−08
19	1.9145e−08	2.1149e−08	5.7041e−09
22	5.151e−10	6.2691e−10	1.1519e−10
23	2.9278e−10	1.2219e−10	9.2533e−11
24	5.3785e−11	6.9398e−11	7.4369e−11
m	$\theta = 1$		
	E_u	E_v	E_w
1	0.86782	1.1797	0.23024
2	1.5424	1.9725	0.96708
3	5.615	3.8987	0.95367
4	15.611	20.172	1.5002
5	348.08	205.49	3.8896
6	2.355e+05	5.6693e+05	9506.1
7	7.8999e+14	2.5429e+14	1.1504e+12
8	3.528e+42	5.2674e+42	1.3026e+31
9	1.8566e+127	1.1924e+127	2.1703e+86

complex generalized Zakharov system of partial differential equations (see more details in [6] and the references therein). The problem test is [6,8]

$$\begin{aligned} & i\partial_t\psi + \partial_{xx}\psi + 2\psi w - 2|\psi|^2 = 0, \\ & \partial_{tt}w - \partial_{xx}w + \partial_{xx}|\psi|^2 = 0, \end{aligned} \tag{10}$$

with the domain $\Lambda = [-1, 1] \times [0, 1.3]$, subject to initial and boundary conditions. The errors are computed over the entire grid as

$$E_\varphi = \max_{(x,t)\in\Lambda} |\varphi(x, t) - \varphi_{num}(x, t)|,$$

where Λ is the grid of values for (x, t) used in the discretization, and $\varphi_{num}(x, t)$, are the computed numerical values of the functions $\xi(x, t)$ with $\varphi = u, v, \omega$. The errors obtained are summarized in Table 1. As we can see, the novel method improves the results.

Remark 1. When the finite difference and pseudospectral methods are compared with respect to cost and accuracy we conclude that the method of choice is dependent on the accuracy required. Specifically:

1. If high accuracy is required, without concern for cost, the standard pseudospectral method is the method of choice, giving the least error.
2. The local form of the frozen derivatives formulation poses more and more problems as the order of the differential equation increases and the order of the spatial approximation to the derivatives becomes wider. Because pseudospectral methods are inherently global this is not an issue.
3. For moderate accuracy the frozen derivatives seem more efficient.

See [6,9].

Our first contribution is the study of its convergence. We consider both local and semilocal convergence. In the first case we find hypothesis on the solution and in the second case on the initial guess. We also perform an analysis on the efficiency. We will be able to select the best method in the family before to compute the approximations. The best method will depend not only on the problem but also on the number of equations [1]. Our study will be performed in a general setting in order to consider a bigger number of problems. Indeed, we consider operators between Banach spaces and the derivatives in the Fréchet sense. In any case, as a particular case, we can derive the convergence for systems of real equations.

2. Local convergence analysis

We shall introduce some scalar functions and some parameters to show the local convergence analysis of the proposed method.

Set $I_0 = \mathbb{R} \bigcup \{0\}$. Let $\lambda_0 : I_0 \to \mathbb{R}^+$ be a continuous and increasing function with $\lambda_0(0) = 0$.

Suppose that equation

$$\lambda_0(t) = 1 \tag{11}$$

has at least one positive solution.

Denote by ρ_0 the smallest such solution. Let $\theta \in \mathbb{R} \setminus \{0\}$ or $\theta \in \mathbb{C} \setminus \{0\}$ be a given parameter and set $I_1 = [0, \rho_0)$.

Let $\lambda : I_1 \to \mathbb{R}^+$ and $\mu : I_1 \to \mathbb{R}^+$ be continuous and increasing functions with $\lambda(0) = 0$.

Define scalar functions h_i, $\bar{h}_i$, $i = 0, 1, \ldots, m-2$ on the interval I_1 by

$$h_0(t) = \frac{\int_0^1 \lambda((1-\tau)t)d\tau + |1 - \frac{1}{\theta}| \int_0^1 \mu(\tau t)d\tau}{1 - \lambda_0(t)},$$

$$h_1(t) = \frac{\int_0^1 \lambda((1-\tau)t)d\tau + |\theta(1-\theta)| \int_0^1 \mu(\tau t)d\tau}{1 - \lambda_0(t)},$$

$$h_2(t) = h_1(t) + \frac{|\theta|^2 \int_0^1 \mu(\tau h_0(t)t)d\tau h_0(t)}{1 - \lambda_0(t)},$$

for $i = 1, 2, \ldots, m-2$

$$h_{i+2}(t) = \left(1 + \frac{\int_0^1 \mu(\tau t)d\tau}{1 - \lambda_0(t)}\right)^i h_2(t),$$

$$\bar{h}_0(t) = h_0(t) - 1,$$

$$\bar{h}_1(t) = h_1(t) - 1,$$

$$\bar{h}_2(t) = h_2(t) - 1$$

and

$$\bar{h}_{i+2}(t) = h_{i+2}(t) - 1.$$

Suppose that

$$\frac{|1-\theta|\mu(0)}{|\theta|} - 1 < 0. \tag{12}$$

We have by the definition of function $\bar{h}_0(t)$, (11) and (12) that $\bar{h}_0(0) < 0$ and $\lim_{t \to \rho_0^-} \bar{h}_0(t) = +\infty$.

Then, the intermediate value theorem guarantees that the equation $\bar{h}_0(t) = 0$ has solutions on the interval $I_2 = [0, \rho_0)$. Denote by $\bar{\rho}_0$ the smallest such solution.

Suppose that

$$(1 + \mu(0))^{m-1} \mu(0) |\theta(1-\theta)| - 1 < 0. \tag{13}$$

It follows by the definition of functions $\bar{h}_i$, $i = 1, 2, \ldots, m-2$, (11) and (13) that $\bar{h}_i(0) < 0$ and $\lim_{t \to \rho_0^-} \bar{h}_i(t) = +\infty$.

Denote by ρ_i the smallest solutions on interval I_2 of equations $\bar{h}_i(t) = 0$, respectively.

Define the radius of convergence ρ by

$$\rho = \min\{\bar{\rho}_0, \rho_i\}, \;\; i = 1, 2, \ldots, m-2. \tag{14}$$

It follows that for each $t \in [0, \rho)$

$$0 \le \lambda_0(t) < 1 \tag{15}$$

and

$$0 \le h_i(t) < 1. \tag{16}$$

Let $B(v, \xi)$, $\bar{B}(v, \xi)$ stand, for the open and closed balls in B_1, respectively with center $v \in B_1$ and of radius $\xi > 0$.

The local convergence analysis of the proposed method uses an aforementioned notation and the conditions:

(a_1) $F : \Omega \subset B_1 \to B_2$ is continuously differentiable operator in the Fréchet sense and there exists $p \in \Omega$ such that $F(p) = 0$ and $F'(p)^{-1} \in \mathrm{L}(B_2, B_1)$.

(a_2) There exists a function $\lambda_0 : I_0 \to \mathbb{R}_+$ continuous and increasing with $\lambda_0(0) = 0$ such that for each $x \in \Omega$

$$\|F'(p)^{-1}(F'(x) - F'(p))\| \le \lambda_0(\|x - p\|).$$

Set $\Omega_0 = \Omega \bigcap B(p, \rho_0)$, where ρ_0 is given by (11).

(a_3) There exist functions $\lambda : I_1 \to \mathbb{R}_+$, $\mu : I_1 \to \mathbb{R}_+$ continuous and increasing with $\lambda(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \leq \lambda(\|x - y\|)$$

and

$$\|F'(p)^{-1}F'(x)\| \leq \mu(\|x - p\|).$$

(a_4) $\bar{B}(p, \rho) \subseteq \Omega$ and conditions (12) and (13) hold, where ρ exists and is given in (11).

(a_5) There exists $\rho^* \geq \rho$ such that

$$\int_0^1 \lambda_0(\tau\rho^*)d\tau < 1.$$

Set $\Omega_1 = \Omega \bigcap \bar{B}(p, \rho^*)$.

We can now show the local convergence analysis of the proposed method.

Theorem 1. *Suppose that the conditions (a_i), $i = 1, 2, \ldots, 5$ hold. Then, the proposed method for $y_0 \in B(p, \rho)\backslash\{p\}$ converges to p which is the only solution of equation $F(x) = 0$ in Ω_1.*

Proof. Let $u \in B(p, \rho)$. Using (11), (14), (15), (a_1) and (a_2), we have that

$$\|F'(p)^{-1}(F'(u) - F'(p))\| \leq \lambda_0(\|u - p\|) \leq \lambda_0(\rho) < 1. \tag{17}$$

The Banach lemma on invertible operators and (17) assure that $F'(u)^{-1} \in \mathrm{L}(B_2, B_1)$ and

$$\|F'(u)^{-1}F'(p)\| \leq \frac{1}{1 - \lambda_0(\|u - p\|)}. \tag{18}$$

It also follows from (18) for $u = y_0^{(0)}$ that $F'(y_0^{(0)})^{-1} \in \mathrm{L}(B_2, B_1)$, (18) holds for $u = p$ and

$$y_1^{(0)} = y_0^{(0)} - (1 + \theta - \theta^2)F'(y_0^{(0)})^{-1}F(y_0^{(0)}), \tag{19}$$

is well defined.

By (a_1), we can write

$$F(u) = F(u) - F(p) = \int_0^1 F'(p + \tau(u - p))d\tau(u - p). \tag{20}$$

Notice that

$$\|p + \tau(u - p) - p\| = \tau\|u - p\| < \rho, \tag{21}$$

so $p + \tau(u - p) \in B(p, \rho)$ for each $\tau \in [0, 1]$.

Then, by the second condition in (a_3) and (20), we get that

$$\begin{aligned}\|F'(p)^{-1}F(u)\| &= \|\int_0^1 F'(p)^{-1}F'(p + \tau(u - p))d\tau(u - p)\| \\ &\leq \int_0^1 \mu(\tau(\|u - p\|))d\tau\|u - p\|.\end{aligned} \tag{22}$$

We also have by the first condition in (a_3), (a_1) and (18) (for $u = y_0^{(0)}$) that

$$\begin{aligned}\|y_0^{(0)} - p - F'(y_0^{(0)})^{-1}F(y_0^{(0)})\| &\leq \|F'(y_0^{(0)})^{-1}F'(p)\| \\ &\quad \|\int_0^1 F'(p)^{-1}(F'(p + \tau(y_0^{(0)} - p)) - F'(y_0^{(0)}))(y_0^{(0)} - p)d\tau\| \\ &\leq \frac{\int_0^1 \lambda((1 - \tau)\|y_0^{(0)} - p\|)d\tau\|y_0^{(0)} - p\|}{1 - \lambda_0(\|y_0^{(0)} - p\|)}.\end{aligned} \tag{23}$$

By (14), (16) (for $i = 1$), (18) (for $u = y_0^{(0)}$), (19), (22) and (23), we obtain in turn that

$$\begin{aligned}\|y_1^{(0)} - p\| &= \|(y_0^{(0)} - p - F'(y_0^{(0)})^{-1}F(y_0^{(0)})) - \theta(1 - \theta)F'(y_0^{(0)})^{-1}F(y_0^{(0)})\| \\ &\leq \|y_0^{(0)} - p - F'(y_0^{(0)})^{-1}F(y_0^{(0)})\| + |\theta(1 - \theta)|\ \|F'(y_0^{(0)})^{-1}F'(p)\|\ \|F'(p)^{-1}F(y_0^{(0)})\|\end{aligned} \tag{24}$$

$$\begin{aligned} &\leq \frac{\int_0^1 \lambda((1-\tau)\|y_0^{(0)}-p\|)d\tau\,\|y_0^{(0)}-p\| + |\theta(1-\theta)|\int_0^1 \mu(\tau\|y_0^{(0)}-p\|)d\tau\,\|y_0^{(0)}-p\|}{1-\lambda_0(\|y_0^{(0)}-p\|)} \\ &= h_1(\|y_0^{(0)}-p\|)\|y_0^{(0)}-p\| \leq \|y_0^{(0)}-p\| < \rho, \end{aligned}$$

so $y_1^{(0)} \in B(p,\rho)$.

We need to show that $v = y_0^{(0)} - \frac{1}{\theta}F'(y_0^{(0)})^{-1}F(y_0^{(0)}) \in B(p,\rho)$.

As in (24) but using (16) (for $i = 0$), we get in turn that

$$\begin{aligned} \|v-p\| &= \|y_0 - p - \frac{1}{\theta}F'(y_0)^{-1}F(y_0)\| \\ &= \|(y_0-p-F'(y_0)^{-1}F(y_0)) + (1-\frac{1}{\theta})F'(y_0)^{-1}F(y_0)\| \\ &\leq \|y_0-p-F'(y_0)^{-1}F(y_0)\| + |1-\frac{1}{\theta}|\ \|F'(y_0)^{-1}F'(p)\|\ \|F'(p)^{-1}F(y_0)\| \\ &\leq \frac{[\int_0^1 \lambda((1-\tau)\|y_0-p\|)d\tau + |1-\frac{1}{\theta}|\int_0^1 \mu(\tau\|y_0-p\|)d\tau]\|y_0-p\|}{1-\lambda_0(\|y_0-p\|)} \\ &= h_0(\|y_0^{(0)}-p\|)\|y_0^{(0)}-p\| \leq \|y_0^{(0)}-p\| < \rho, \end{aligned} \tag{25}$$

so $v \in B(p,\rho)$. It also follows that

$$y_2^{(0)} = y_1^{(0)} - \theta^2 F'(y_0^{(0)})^{-1}F(v), \tag{26}$$

is well defined.

We have using (25) that

$$\|p + \tau(y_0^{(0)} - \frac{1}{\theta}F^{-1}(y_0^{(0)})^{-1}F(y_0^{(0)})) - p\| \leq |\tau|\|v\| \leq \|v\| < \rho, \tag{27}$$

so $\bar{v} = p + \tau(y_0^{(0)} - \frac{1}{\theta}F'(y_0^{(0)})^{-1}F(y_0^{(0)})) \in B(p,\rho)$.

Then, using (14), (16) (for $i = 2$), (22) (for $u = \bar{v}$) and (25), we have in turn that

$$\begin{aligned} \|y_2^{(0)}-p\| &= \|(y_1^{(0)}-p) - \theta^2 F'(y_0^{(0)})^{-1}F(v)\| \\ &\leq \|y_1^{(0)}-p\| + \theta^2\|F'(y_0^{(0)})^{-1}F'(p)\|\ \|F'(p)^{-1}F(v)\| \\ &\leq (h_1(\|y_0^{(0)}-p\|) + \frac{\theta^2\int_0^1 \mu(\tau h_0(\|y_0^{(0)}-p\|)\|y_0^{(0)}-p\|)d\tau\, h_0(\|y_0^{(0)}-p\|)}{1-\lambda_0(\|y_0^{(0)}-p\|)})\|y_0^{(0)}-p\| \\ &= h_2(\|y_0^{(0)}-p\|)\|y_0^{(0)}-p\| \leq \|y_0^{(0)}-p\| < \rho, \end{aligned} \tag{28}$$

so $y_2^{(0)} \in B(p,\rho)$.

Similarly, by (14), (16) (for $i = 3$), (18) (for $u = y_0$), (22) (for $u = y_2^{(0)}$), (28), and

$$y_3^{(0)} = y_2^{(0)} - F'(y_0^{(0)})^{-1}F(y_2^{(0)}), \tag{29}$$

we have in turn that

$$\begin{aligned} \|y_3^{(0)}-p\| &= \|(y_2^{(0)}-p) - F'(y_0^{(0)})^{-1}F(y_2^{(0)})\| \\ &\leq \|y_2^{(0)}-p\| + \frac{\int_0^1 \mu(\tau\|y_2^{(0)}-p\|)d\tau\,\|y_2^{(0)}-p\|}{1-\lambda_0(\|y_0^{(0)}-p\|)} \\ &= h_3(\|y_0^{(0)}-p\|)\|y_0^{(0)}-p\| \leq \|y_0^{(0)}-p\| < \rho, \end{aligned} \tag{30}$$

so $y_3^{(0)} \in B(p,\rho)$.

Then, since for each $i = 1, 2, \ldots, m-2$,

$$y_{i+2}^{(0)} = y_{i+1}^{(0)} - F'(y_0^{(0)})^{-1}F(y_{i+1}^{(0)}), \tag{31}$$

we obtain in turn that

$$\begin{aligned} \|y_{i+2}^{(0)}-p\| &= \|(y_{i+1}^{(0)}-p) - F'(y_0^{(0)})^{-1}F(y_{i+1}^{(0)})\| \\ &\leq \|y_{i+1}^{(0)}-p\| + \frac{\int_0^1 \mu(\tau\|y_{i+1}^{(0)}-p\|)d\tau\,\|y_{i+1}^{(0)}-p\|}{1-\lambda_0(\|y_0^{(0)}-p\|)} \end{aligned} \tag{32}$$

$$\leq (1+\frac{\int_0^1 \mu(\tau\|y_{i+1}^{(0)}-p\|)d\tau}{1-\lambda_0(\|y_0^{(0)}-p\|)})\|y_{i+1}^{(0)}-p\|$$
$$= h_{i+2}(\|y_0^{(0)}-p\|)\|y_0^{(0)}-p\| \leq \|y_0^{(0)}-p\| < \rho,$$

so $y_{i+1}^{(0)} \in B(p,\rho)$.

In particular, by the definition of the method for $i = m-2$, estimate (32) gives

$$\|y^{(1)}-p\| \leq c\|y^{(0)}-p\|, \tag{33}$$

where $c = h_m(\|y^{(0)}-p\|) \in [0,1)$.

By simply replacing $y^{(0)}, y^{(1)}$ by $y^{(n)}, y^{(n+1)}$, respectively in the preceding estimates we get

$$\|y^{(n+1)}-p\| \leq c\|y^{(n)}-p\| < \rho, \tag{34}$$

so $y^{(n+1)} \in B(p,\rho)$.

By letting $n \to +\infty$ in (34), we obtain $\lim_{n\to+\infty} y^{(n)} = p$.

Finally, to show the uniqueness part, let $q \in \Omega_1$ with $F(q) = 0$. Define $T = \int_0^1 F'(q+\tau(p-q))d\tau$. Using (a_5), we obtain in turn that

$$\|F'(p)^{-1}(T-F'(p))\| \leq \int_0^1 \lambda_0(\tau\|p-q\|)d\tau$$
$$\leq \int_0^1 \lambda_0(\tau\rho^*)d\tau < 1,$$

so $T^{-1} \in L(B_2, B_1)$, and from the identity

$$0 = F(p) - F(q) = T(p-q),$$

we obtain $p = q$. $\square$

3. Semilocal convergence analysis

We shall introduce some scalar parameters to assist us with the semilocal convergence analysis that follows.

Let $\Psi_0 : I_0 \to I_0$ be a continuous and increasing function with $\Psi_0(0) = 0$. Suppose that equation

$$a_0\Psi_0(t) = 1, \tag{35}$$

for some $a_0 > 0$ has at least one positive solution. Denote by r_0 the smallest such solution.

Let $\Psi : [0, r_0) \to I_0$ and $\Psi_1 : [0, r_0) \to I_0$ be continuous and increasing functions with $\Psi(0) = 0$.

Let $\eta \geq 0$ be a parameter. Define:

$$b_0 = \int_0^1 \Psi_0(\tau|1+\theta-\theta^2|a_0\eta)d\tau + |\theta(1-\theta)|,$$
$$a_0^1 = \int_0^1 \Psi_0(\tau|\frac{1}{\theta}|a_0\eta)d\tau|\frac{1}{\theta}|a_0\eta + |\frac{1}{\theta}|a_0\eta + a_0,$$
$$\bar{\eta} = (\theta^2 a_0^1 + |1+\theta-\theta^2|)a_0\eta,$$
$$a_1 = a_1(t) = \frac{a_0}{1-a_0\Psi(t)},$$
$$a = a(t) = a_0 \max\{\theta^2 a_0^1, \int_0^1 \Psi_0(|1+\theta-\theta^2|a_0\eta + \tau\theta^2 a_0 a_0^1\eta)d\tau\theta^2 a_0^1, \int_0^1 \Psi_0(t+\tau\eta)d\tau\}$$
$$\delta_0 = \delta_0(t) = \int_0^1 \Psi_0(t+\tau a^{m-1}\eta)d\tau a^{m-1}$$
$$b_0 = |1+\theta-\theta^2|a_1\delta_0,$$
$$b_1 = [\int_0^1 \Psi(\tau b_0\eta)d\tau + |\theta(1-\theta)|\Psi_1(t)]b_0$$
$$b = b(t) = \max\left\{b_1, b_1 a_1, \sqrt{\int_0^1 \Psi(t+\tau\eta)d\eta + b_1}\right\}.$$

We shall use the following conditions in the semilocal convergence analysis of the proposed method:

$(h_1) F : \Omega \subseteq B_1 \to B_2$ is continuously differentiable in the sense of Fréchet and there exists $y_0 \in \Omega$ such that $F'(y_0)^{-1} \in L(B_2, B_1)$ with $\|F'(y_0)^{-1}\| \leq a_0$.

(h_2) There exists function $\Psi_0 : I_0 \to I_0$ continuous and increasing such that for each $x \in \Omega$

$$\|F'(x) - F'(y_0)\| \leq \Psi_0(\|x - x_0\|).$$

Set $\Omega_2 = \Omega \bigcap B(y_0, r_0)$ and $I_3 = [0, r_0)$, where r_0 is given by (35).

(h_3) There exist functions $\Psi : I_3 \to I_0$ and $\Psi_1 : I_3 \to I_0$ continuous and increasing such that for each $x, y \in \Omega_2$

$$\|F'(x) - F'(y)\| \leq \Psi(\|x - y\|)$$

and

$$\|F'(x)\| \leq \Psi_1(\|x - y_0\|).$$

(h_4) Equation

$$(\frac{a}{1-a} + \frac{b}{1-b} + |1+\theta-\theta^2|a_0 + b_0)\eta - t = 0$$

has at least one positive solution. Denote by r the smallest such solution.

(h_5) The following hold $a(r) < 1$, $b(r) < 1$,

$$(\theta^2 a_0^1 + |1+\theta-\theta^2|)a_0\eta < r,$$

$$|\frac{1}{\theta}|a_0\eta < r$$

and

$$(\frac{a}{1-a} + |1+\theta-\theta^2|a_0 + |\frac{1}{\theta}|a_1 b_1)\eta < r.$$

(h_6) $\bar{B}(y_0, r) \subseteq \Omega$.

(h_7) There exists $r^* \geq r$ such that

$$a_0 \int_0^1 \Psi_0((1-\tau)r^* + \tau r)d\tau < 1.$$

Set $\Omega_3 = \Omega \bigcap B(y_0, r^*)$.

The main result on the semilocal convergence of the proposed method is provided under the previously introduced notation and conditions.

Theorem 2. *Suppose that the conditions (h_1)–(h_7) hold. Then, the proposed method starting at y_0 generates a sequence converging to a solution $p \in \bar{B}(y_0, r)$ of equation $F(x) = 0$. The limit p is the unique solution of the equation $F(x) = 0$ in Ω_3.*

Proof. Iterates $y_1^{(0)}, y_2^{(0)}, \ldots, y_{i+2}^{(0)}$, $i = 1, 2, \ldots, m-2$, are well defined by condition (h_1) and are given by

$$\begin{aligned}
y_1^{(0)} &= y_0^{(0)} - (1+\theta-\theta^2)F'(y_0^{(0)})^{-1}F(y_0^{(0)}), \\
y_2^{(0)} &= y_1^{(0)} - \theta^2 F'(y_0^{(0)})^{-1}F(y_0^{(0)} - \frac{1}{\theta}F'(y_0^{(0)})^{-1}F(y_0^{(0)})), \\
y_3^{(0)} &= y_2^{(0)} - F'(y_0^{(0)})^{-1}F(y_2^{(0)}), \\
&\ \vdots \\
y_{i+2}^{(0)} &= y_{i+1}^{(0)} - F'(y_0^{(0)})^{-1}F(y_{i+1}^{(0)}).
\end{aligned} \tag{36}$$

By the first substep of the proposed method, we can write (by (h_5))

$$\begin{aligned}
\|y_1^{(0)} - y_0^{(0)}\| &= \|(1+\theta-\theta^2)F'(y_0^{(0)})^{-1}F(y_0^{(0)})\| \\
&\leq |1+\theta-\theta^2|\ \|F'(y_0^{(0)})^{-1}\|\ \|F(y_0^{(0)})\| \\
&\leq |1+\theta-\theta^2|\ a_0\ \eta < r,
\end{aligned} \tag{37}$$

so $y_1^{(0)} \in B(y_0, r)$ and

$$\begin{aligned}
F(y_1^{(0)}) &= F(y_1^{(0)}) - (1+\theta-\theta^2)F(y_0^{(0)}) - F'(y_0^{(0)})(y_1^{(0)} - y_0^{(0)}) \\
&= F(y_1^{(0)}) - F(y_0^{(0)}) - F'(y_0^{(0)})(y_1^{(0)} - y_0^{(0)}) - \theta(1-\theta)F(y_0^{(0)}).
\end{aligned} \tag{38}$$

Using (36), (h_2), (37) and (38), we have in turn that

$$\begin{aligned}\|F(y_1^{(0)})\| &= \| \int_0^1 (F'(y_0^{(0)} + \tau(y_1^{(0)} - y_0^{(0)})) - F'(y_0^{(0)}))d\tau(y_1^{(0)} - y_0^{(0)}) \\ &\quad -\theta(1-\theta)F(y_0^{(0)}) \| \\ &\le \int_0^1 \Psi_0(\tau\|y_1^{(0)} - y_0^{(0)}\|)d\tau\|y_1^{(0)} - y_0^{(0)}\| + |\theta(1-\theta)|\ \|F(y_0^{(0)})\| \\ &\le (\int_0^1 \Psi_0(\tau|1+\theta-\theta^2|a_0\eta)d\tau + |\theta(1-\theta)|)\eta = b_0\eta.\end{aligned} \tag{39}$$

Let $w = y_0^{(0)} - \frac{1}{\theta}F'(y_0^{(0)})^{-1}F(y_0^{(0)})$. We get by (h_5)

$$\|w - y_0^{(0)}\| = \|\frac{1}{\theta}F'(y_0^{(0)})^{-1}F(y_0^{(0)})\| \le |\frac{1}{\theta}|a_0\eta < r, \tag{40}$$

so $w \in B(y_0^{(0)}, r)$ and

$$\begin{aligned}\|F'(y_0^{(0)})^{-1}F(w)\| &= \|F'(y_0^{(0)})^{-1}(F(w) - F(y_0^{(0)}) + F(y_0^{(0)}))\| \\ &\le \| \int_0^1 F'(y_0^{(0)} + \tau(w - y_0^{(0)}))d\tau(w - y_0^{(0)}) - F'(y_0^{(0)})(w - y_0^{(0)}) \\ &\quad +F'(y_0^{(0)})(w - y_0^{(0)}) + F(y_0^{(0)}) \| \\ &\le a_0 \int_0^1 \Psi_0(\tau\|w - y_0^{(0)}\|)d\tau\|w - y_0^{(0)}\| + \|w - y_0^{(0)}\| + a_0\ \|F(y_0^{(0)})\| \\ &\le a_0 \int_0^1 \Psi_0(\tau|\frac{1}{\theta}|a_0\eta)d\tau|\frac{1}{\theta}|a_0\eta + |\frac{1}{\theta}|a_0\eta + a_0\eta \\ &= a_0^1\eta.\end{aligned} \tag{41}$$

By (36) and (41), we get

$$\|y_2^{(0)} - y_1^{(0)}\| = \|\theta^2F'(y_0^{(0)})^{-1}F(w)\| \le \theta^2a_0^1a_0\eta \le a\eta \tag{42}$$

and by (h_5)

$$\|y_2^{(0)} - y_0^{(0)}\| \le \|y_2^{(0)} - y_1^{(0)}\| + \|y_1^{(0)} - y_0^{(0)}\| \le (\theta^2\bar{a}_0 + |1+\theta-\theta^2|)a_0\eta = \bar{\eta} < r, \tag{43}$$

so $y_2^{(0)} \in B(y_0^{(0)}, r)$.

Similarly, we have from

$$\begin{aligned}F(y_2^{(0)}) &= F(y_2^{(0)}) - F(y_1^{(0)}) + F(y_1^{(0)}) \\ &= \int_0^1 F'(y_1^{(0)} + \tau(y_2^{(0)} - y_1^{(0)}))d\tau(y_2^{(0)} - y_1^{(0)}) + F(y_1^{(0)})\end{aligned}$$

that

$$\begin{aligned}\|F(y_2^{(0)})\| &\le \int_0^1 \Psi_1(\|y_1^{(0)} - y_0^{(0)}\| + \tau\|y_2^{(0)} - y_1^{(0)}\|)d\tau\|y_2^{(0)} - y_1^{(0)}\| + \|F(y_1^{(0)})\| \\ &\le \int_0^1 \Psi_1(|1+\theta-\theta^2|a_0\eta + \tau\theta^2a_0^1a_0\eta)d\tau\theta^2a_0^1a_0\eta \\ &\le \frac{a^2\eta}{a_0},\end{aligned}$$

$$\begin{aligned}\|y_3^{(0)} - y_2^{(0)}\| &= \|F'(y_0^{(0)})^{-1}F(y_2^{(0)})\| \le a_0\frac{a^2\eta}{a_0} = a^2\eta, \\ \|y_3^{(0)} - y_0^{(0)}\| &\le \|y_3^{(0)} - y_2^{(0)}\| + \|y_2^{(0)} - y_0^{(0)}\| \le a^2\eta + \bar{\eta} < r, \\ \|F(y_3^{(0)})\| &\le \int_0^1 \Psi_0(\|y_2^{(0)} - y_0^{(0)}\| + \tau\|y_3^{(0)} - y_2^{(0)}\|)d\tau\|y_3^{(0)} - y_2^{(0)}\| \\ &\le \int_0^1 \Psi_0(r + \tau a^2\eta)d\tau a^2\eta \le \frac{a^3}{a_0}\eta, \\ \|y_4^{(0)} - y_3^{(0)}\| &\le a_0\int_0^1 \Psi_0(r + \tau a^2\eta)\frac{a^3}{a_0}\eta = a^3\eta,\end{aligned}$$

and

$$\begin{aligned}\|y_4^{(0)} - y_0^{(0)}\| &\leq \|y_4^{(0)} - y_3^{(0)}\| + \|y_3^{(0)} - y_2^{(0)}\| + \|y_2^{(0)} - y_1^{(0)}\| + \|y_1^{(0)} - y_0^{(0)}\| \\ &\leq a^3\eta + a^2\eta + a\eta + |1+\theta-\theta^2|a_0\eta \\ &= a\eta\frac{1-a^3}{1-a} + |1+\theta-\theta^2|a_0\eta \\ &< \frac{a\eta}{1-a} + |1+\theta-\theta^2|a_0\eta < r,\end{aligned}$$

so $y_4^{(0)} \in B(y_0^{(0)}, r)$. Similarly, we get

$$\|F(y_i^{(0)})\| \leq \frac{a^i}{a_0}\eta, \;\; i = 3, 4, \ldots, m, \tag{44}$$

$$\|y_{i+2}^{(0)} - y_{i+1}^{(0)}\| \leq a^{i+1}\eta, \;\; i = 3, 4, \ldots, m-2 \tag{45}$$

and

$$\|y_{i+2}^{(0)} - y_0^{(0)}\| \leq (\frac{a}{1-a} + |1+\theta-\theta^2|a_0)\eta < r. \tag{46}$$

That is $y_0^{(1)} = y_m^{(0)} \in B(x_0, r)$. By the definition of the method

$$\begin{aligned}F(y_0^{(1)}) &= F(y_m^{(0)}) - F(y_{m-1}^{(0)}) - F'(y_0^{(0)})(y_m - y_{m-1}^{(0)}) \\ &= \int_0^1 [F'(y_{m-1}^{(0)} + \tau(y_m^{(0)} - y_{m-1}^{(0)})) - F'(y_0^{(0)})]d\tau(y_m^{(0)} - y_{m-1}^{(0)}),\end{aligned}$$

so

$$\begin{aligned}\|F(y_0^{(1)})\| &\leq \int_0^1 \Psi_0(\|y_m^{(0)} - y_0^{(0)}\| + \tau\|y_m^{(0)} - y_{m-1}^{(0)}\|)d\tau\|y_m^{(0)} - y_{m-1}^{(0)}\| \\ &\leq \int_0^1 \Psi_0(r + \tau a^{m-1}\eta)d\tau a^{m-1}\eta = \delta_0\eta.\end{aligned}$$

Next, we shall show that $F'(x)^{-1} \in \mathrm{L}(B_2, B_1)$ provided that $x \in B(y_0, r)$. We obtain in turn that since $r < r_0$ (by (35))

$$\|F'(y_0^{(0)})^{-1}\| \;\|F'(x) - F'(y_0^{(0)})\| \leq a_0\Psi_0(\|x - y_0^{(0)}\|) \leq a_0\Psi(r) < 1,$$

so $F'(x)^{-1} \in \mathrm{L}(B_2, B_1)$ and

$$\|F'(x)^{-1}\| \leq \frac{a_0}{1 - a_0\Psi_0(r)} = a_1. \tag{47}$$

It follows from (36) that

$$\begin{aligned}\|y_1^{(1)} - y_0^{(1)}\| &\leq |1+\theta-\theta^2| \;\|F'(y_0^{(1)})^{-1}\| \;\|F(y_0^{(1)})\| \\ &\leq |1+\theta-\theta^2|a_1\delta_0\eta = b_0\eta.\end{aligned}$$

We can write

$$F(y_1^{(1)}) = (F(y_1^{(1)}) - F(y_1^{(0)}) - F'(y_0^{(1)})(y_1^{(1)} - y_0^{(1)})) - \theta(1-\theta)F'(y_0^{(1)})(y_1^{(1)} - y_0^{(1)})$$

to obtain

$$\begin{aligned}\|F(y_1^{(1)})\| &\leq \int_0^1 \Psi(\tau\|y_1^{(1)} - y_0^{(1)}\|)d\tau\|y_1^{(1)} - y_0^{(1)}\| \\ &\quad + |\theta(1-\theta)|\|F'(y_0^{(1)})\|\|y_1^{(1)} - y_0^{(1)}\| \\ &\leq \int_0^1 \Psi(\tau b_0\eta)d\tau b_0\eta + |\theta(1-\theta)|\Psi_1(r)b_0\eta = b_1\eta.\end{aligned} \tag{48}$$

Let $\bar{w} = y_1^{(1)} - \frac{1}{\theta}F'(y_0^{(1)})^{-1}F(y_0^{(1)})$. We must show $\bar{w} \in B(y_0, r)$. Indeed, we have by (h_5)

$$\begin{aligned}\|\bar{w} - y_0^{(0)}\| &\leq \|y_0^{(1)} - y_0^{(0)}\| + |\frac{1}{\theta}| \;\|F'(y_0^{(1)})^{-1}\| \;\|F(y_1^{(1)})\| \\ &\leq (\frac{a}{1-a} + |1+\theta-\theta^2|a_0 + |\frac{1}{\theta}|a_1b_1)\eta < r.\end{aligned}$$

Then, we get

$$\begin{aligned}
\|y_1^{(2)} - y_1^{(1)}\| &\leq \theta^2 \|F'(y_0^{(1)})^{-1} F(\bar{w})\| \\
&\leq \theta^2 \|F'(y_0^{(1)})^{-1} F(\bar{w}) - F(y_0^{(1)}) + F(y_0^{(1)})\| \\
&\leq \theta^2 [\ \| F'(y_0^{(1)})^{-1} (\int_0^1 F'(y_0^{(1)} + \tau(\bar{w} - y_0^{(1)})) d\tau (\bar{w} - y_0^{(1)}) \\
&\quad - F'(y_0^{(1)})(\bar{w} - y_0^{(1)}) + F'(y_0^{(1)})(\bar{w} - y_0^{(1)}) + F(y_0^{(1)}))\ \|\ ] \\
&\leq \theta^2 [a_1 \int_0^1 \Psi(\tau \|\bar{w} - y_0^{(1)}\|) d\tau \|\bar{w} - y_0^{(1)}\| + \|\bar{w} - y_0^{(1)}\| + a_1 \|F(y_0^{(1)})\|] \\
&\leq \theta^2 [a_1 \int_0^1 \Psi(\tau |\frac{1}{\theta}| a_1 b_1 \eta) d\tau |\frac{1}{\theta}| a_1 b_1 \eta + |\frac{1}{\theta}| a_1 b_1 \eta + a_1 \delta_0 \eta] = a\eta,
\end{aligned} \tag{49}$$

so

$$\|y_0^{(1)} - y_0^{(0)}\| \leq (\frac{a}{1-a} + |1 + \theta - \theta^2| a_0)\eta < r,$$

$$\begin{aligned}
\|y_1^{(1)} - y_0\| &= \|y_1^{(1)} - y_0^{(1)}\| + \|y_0^{(1)} - y_0\| \\
&\leq a_0 \eta + (\frac{a}{1-a} + |1 + \theta - \theta^2| a_0)\eta < r,
\end{aligned}$$

and

$$\begin{aligned}
\|y_1^{(2)} - y_0^{(0)}\| &\leq \|y_1^{(2)} - y_1^{(1)}\| + \|y_1^{(1)} - y_0^{(0)}\| \\
&\leq a\eta + a_0 \eta + \|x_1 - x_0\| \\
&\leq a\eta + a_0 \eta + (\frac{a}{1-a} + |1 + \theta - \theta^2| a_0)\eta < r,
\end{aligned}$$

so $y_0^{(1)}, y_1^{(1)}, y_1^{(2)} \in B(y_0, r)$.

Similarly, we can write

$$F(y_1^{(2)}) = F(y_1^{(2)}) - F(y_1^{(1)}) + F(y_1^{(1)})$$

so

$$\begin{aligned}
\|F(y_1^{(2)})\| &\leq \int_0^1 \Psi_1(\|y_1^{(1)} - y_0^{(0)}\| + \tau \|y_1^{(2)} - y_1^{(1)}\|) d\tau \|y_1^{(2)} - y_1^{(1)}\| + \|F(y_1^{(1)})\| \\
&\leq \int_0^1 \Psi_1(r + \tau b\eta) d\tau b\eta + b_1 \eta \leq \frac{b^2 \eta}{a_1},
\end{aligned}$$

$$\begin{aligned}
\|y_1^{(3)} - y_1^{(2)}\| &\leq \|F'(y_1^{(2)})^{-1}\|\ \|F(y_1^{(2)})\| \\
&\leq a_1 \frac{b^2 \eta}{a_1} = b^2 \eta,
\end{aligned}$$

$$\begin{aligned}
\|y_1^{(3)} - y_0\| &\leq \|y_1^{(3)} - y_1^2\| + \|y_1^{(2)} - y_0^{(0)}\| \\
&\leq b^2 \eta + b\eta + b_0 \eta + (\frac{a}{1-a} + |1 + \theta - \theta^2| a_0)\eta < r,
\end{aligned}$$

so $y_1^{(3)} \in B(y_0, r)$. Hence, we obtain in an analogous way that

$$\|F(y_1^{(i)})\| \leq \frac{b^i}{a_1}\eta, \ i = 3, 4, \ldots, m, \tag{50}$$

$$\|y_1^{(s+2)} - y_1^{(s+1)}\| \leq a^{s+1}\eta, \ s = 3, 4, \ldots, m-2 \tag{51}$$

and

$$y_1^{(i)} \in B(y_0, r). \tag{52}$$

It follows from these estimates and the definition of a and b_1 that sequence $\{x_k\}$ is complete in the Banach space B_1. Hence, it converges to some $p \in \bar{B}(p, r)$. But sequence $\{F(x_n)\}$ is also bounded from above by the sequence $\{\|x_n - x_{n-1}\|\}$ leading to

$$\|F(p)\| = \lim_{n \to +\infty} \|F(x_n)\| \leq \lim_{n \to +\infty} \|x_n - x_{n-1}\| = 0,$$

so $F(p) = 0$.

Table 2
Values of n for optimal efficiency.

m	$\mu_1 = 63$			$\mu_1 = 12.5$		
	$\mu_0 = \mu_1$	$\mu_0 = 2\mu_1$	$\mu_0 = 4\mu_1$	$\mu_0 = \mu_1$	$\mu_0 = 4\mu_1$	$\mu_0 = 8\mu_1$
2	3	2	1	3	1	1
3	3	2	1	3	2	1
4	4	3	2	4	2	1
5	4	3	2	4	2	1
6	5	3	2	4	2	2
7	5	4	2	5	2	2
8	6	4	3	5	3	2
9	6	4	3	5	3	2
10	7	5	3	6	3	2

Finally, to show the uniqueness part, let $T = \int_0^1 F'(q + \tau(p - q))d\tau$ for $q \in \Omega_3$ with $F(q) = 0$. Then, we obtain in turn by (h_2) and (h_7)

$$\begin{aligned}\|F'(y_0)^{-1}(T - F'(y_0))\| &\leq a_0 \int_0^1 \Psi_0((1-\tau)\|q - y_0\| + \tau\|p - y_0\|)d\tau \\ &\leq a_0 \int_0^1 \Psi_0((1-\tau)r^* + \tau r)d\tau < 1,\end{aligned}$$

so $T^{-1} \in \mathrm{L}(B_2, B_1)$. Moreover, by the identity

$$0 = F(p) - F(q) = T(p - q),$$

we conclude that $p = q$. □

4. Optimal computational efficiency

For the study of the efficiency, we suppose that in Eq. (7),

$$F : \Omega \subset B_1 = \mathbb{R}^m \to B_2 = \mathbb{R}^m,$$

with $\Omega \neq \emptyset$. We will use the computational efficiency index (CEI) shown in [1] which is defined as:

$$CEI(n, \mu_0, \mu_1, m) = \rho_n^{(a_0(n,m)\mu_0 + a_1(m)\mu_1 + p(n,m))^{-1}},$$

where ρ_n is the local order of convergence in the step n, $a_0(n, m)$ and $a_1(m)$ are the number of evaluations of the scalar functions of F and F' respectively, the function $p(n, m)$ is the number of products needed per iteration and the ratios μ_0 and μ_1 between products and evaluations are introduced in order to express the value $a_0(n, m)\mu_0 + a_1(m)\mu_1 + p(n, m)$ only in terms of products. We will assume an LU decomposition in order to solve the linear systems. In Eq. (8) we evaluate $2m$ component functions and m^2 evaluations of scalar functions of the derivatives, the LU decompositions consist in $(m^3 - m)/3$ products/quotients and $2m^2$ in the solution of four triangular linear systems. Also, we add $3m$ products corresponding to parameter. Then, $p(1, m) = (m^3 - m)/3 + 2m^2 + 3m$. After n iterations we have that $a_0(n, m) = n\,m$, $a_1(m) = m^2$, $p(n, m) = (m^3 - m)/3 + 3m + n\,m^2$. We only have to know the local order of convergence. For this, we use the following theorem.

Theorem 3 (*[6]*). *Let $F : \Omega \subset B_1 = \mathbb{R}^m \to \mathbb{R}^m$ has at least third order Fréchet derivative on an open convex neighborhood Ω of $x^* \in \mathbb{R}^m$ with $F(x^*) = 0$ and $det(F'(x^*)) \neq 0$. Then, the multi-step iterative method Eqs.* (8) and (9) *has, for $n \geq 2$, local convergence order at least $n + 1$.*

Therefore, using the same notation of [1], the *CEI* value is

$$CEI(n, \mu_0, \mu_1, m) = (n + 1)^{\mathcal{C}^{-1}},$$

with $\mathcal{C} = Mn + N$, where $M = m(m + \mu_0)$ and $N = \frac{m}{3}(m^2 + 3\mu_1 m + 8)$. We calculate the optimal point of *CEI* for the iterative method solving $\frac{d}{dn}(\ln CEI) = 0$, then

$$\frac{-1}{(Mn + N)^2}\left(M\ln(n + 1) - \frac{Mn + N}{n + 1}\right) = 0 \to (n + 1)\ln(n + 1) - n - \frac{N}{M} = 0.$$

Finally,

$$(n + 1)\ln(n + 1) - n - \frac{m^2 + 3\mu_1 m + 8}{3(m + \mu_0)} = 0. \tag{53}$$

We can solve Eq. (53) in terms of n if m, μ_0 and μ_1 are given. In Table 2 some values of n are shown as a function of m, μ_0 and μ_1 that are positive integer solutions of Eq. (53).

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