



Existence of solutions for quasilinear second order differential inclusions with nonlinear boundary conditions

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Received 22 December 1998

Abstract

In this paper we consider a quasilinear second-order differential inclusion with a convex-valued multivalued term and nonlinear, multivalued boundary conditions. Using the Leray–Schauder fixed-point theorem and techniques from multivalued analysis and from nonlinear analysis, we prove the existence of a solution. Our formulation of the problem is general and includes as special cases the Dirichlet, the Neumann, the periodic problems, as well as certain Sturm–Liouville-type problems. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 34B15

Keywords: Multifunction; Caratheodory function; Approximate selector; Maximal monotone operator; Surjective operator; Sobolev space; Compact embedding; Leray–Schauder fixed point theorem

1. Introduction

In this paper we consider the following second-order quasilinear multivalued boundary value problem:

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &\in F(t, x(t), x'(t)) \quad \text{a.e. on } T = [0, b], \\ (\|x'(0)\|^{p-2}x'(0), -\|x'(b)\|^{p-2}x'(b)) &\in \xi(x(0), x(b)), \quad 2 \leq p < +\infty. \end{aligned} \quad (1)$$

Here $F: T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ is a multifunction with convex values and $\xi: \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map.

This problem with $p=2$ (semilinear case), was studied recently by Halidias and Papageorgiou [5], who proved existence and strong relaxation theorems under various conditions on the multivalued

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term $F(t, x, y)$. For the “convex problem” (i.e. F was assumed to be convex valued), their approach was based on the multivalued Leray–Schauder fixed point theorem, which requires that the compact set-valued map has convex values. This forced them to assume (for the convex problem), that $\text{Gr } \xi \subseteq \mathbb{R}^N \times \mathbb{R}^N$ was convex (see [5, hypothesis $H(\xi)_2$, p. 133]). So there was a dichotomy between the “nonconvex problem” (i.e., F has in general nonconvex values), whose proof was based on the single-valued Leray–Schauder theorem and the convex problem. In this paper we remove the convexity hypothesis on the graph of ξ and treat the convex problem using the single-valued Leray–Schauder fixed-point theorem. The price that we have to pay to achieve this generalization, is that we have to strengthen the measurability hypothesis on $F(t, x, y)$. Moreover, here we deal with the quasilinear problem, instead of the semilinear problem (i.e., $p = 2$), which was studied by Halidias and Papageorgiou [5]. Our approach is based on an approximation technique which uses the fact that an upper semicontinuous, convex valued multifunction has an approximate continuous selection.

Second-order differential inclusions have been studied recently by Erbe and Krawcewicz [1], Frigon [2,3], Frigon and Granas [4], Kravvaritis and Papageorgiou [8], Marano [9] and Pruszko [10]. The most detailed study is that of Frigon [2], who examines both convex and nonconvex problems, but with Dirichlet boundary conditions. Erbe and Krawcewicz [1], consider only the convex problem, using certain single-valued nonlinear boundary conditions. Frigon [3] considers the scalar convex problem under Sturm–Liouville boundary conditions, while Frigon and Granas [4] consider the nonconvex problem with Dirichlet boundary conditions. Kravvaritis and Papageorgiou [8] consider the nonconvex Dirichlet problem and prove the existence of extremal solutions. Their work was extended by Halidias and Papageorgiou [5]. Marano [9] considers the convex Dirichlet problem, while Pruszko [10] considers both the convex and nonconvex Dirichlet problems. All these works deal with the semilinear inclusion (i.e., $p = 2$).

2. Preliminaries

The purpose of this section is to fix our terminology and notation and briefly recall some definitions and facts from multivalued analysis and from nonlinear functional analysis. For details, we refer to the books of Hu and Papageorgiou [7] and Zeidler [11,12].

By $P_{f(c)}(\mathbb{R}^N)$ (resp. $P_{k(c)}(\mathbb{R}^N)$) we denote the collection of nonempty, closed (and convex) subsets (resp. compact (and convex) subsets) of $\mathbb{R}^N$. Let $T = [0, b]$. We say that $F: T \rightarrow P_f(\mathbb{R}^N)$ is measurable, if for all $x \in \mathbb{R}^N$, the $\mathbb{R}_+$ -valued function $t \rightarrow d(x, F(t)) = \inf\{\|x - v\|: v \in F(t)\}$ is Lebesgue measurable. In fact, due to the completeness of the Lebesgue σ -field $\mathcal{L}(T)$ with respect to the Lebesgue measure on T , the above definition of measurability of $F(\cdot)$ is equivalent to saying that $\text{Gr } F = \{(t, x) \in T \times \mathbb{R}^N: x \in F(t)\} \in \mathcal{L}(T) \times B(\mathbb{R}^N)$, with $B(\mathbb{R}^N)$ being the Borel σ -field of $\mathbb{R}^N$. Given $F: T \rightarrow P_f(\mathbb{R}^N)$ and $1 \leq p \leq \infty$, by S_F^p we denote the set of L^p -selectors of F , i.e. $S_F^p = \{f \in L^p(T, \mathbb{R}^N): f(t) \in F(t) \text{ a.e. on } T\}$. In general this set may be empty. It is easy to check that $S_F^p \neq \emptyset$ if and only if $t \rightarrow \inf\{\|v\|: v \in F(t)\} \in L^p(T)$ (see [7, Lemma II.3.2, p. 175]). The set $S_F^p \subseteq L^p(T, \mathbb{R}^N)$ is closed and is convex if and only if for almost all $t \in T$, $F(t)$ is convex. Moreover, S_F^p is bounded if and only if $|F(t)| = \sup\{\|v\|: v \in F(t)\} \leq \phi(t)$, a.e. on T , with $\phi \in L^p(T)$.

If Y, Z are Hausdorff topological spaces and $G: Y \rightarrow 2^Z \setminus \{\emptyset\}$ is a multifunction, we say that $G(\cdot)$ is upper semicontinuous (usc), if for all $C \subseteq Z$ closed, the set $G^-(C) = \{y \in Y: G(y) \cap C \neq \emptyset\}$ is

closed. A $P_f(\mathbb{R}^N)$ -valued multifunction G which is usc, has a closed graph (i.e., $\text{Gr } G = \{(y, z) \in Y \times Z : z \in G(y)\}$ is closed). The converse is true, if for example $\overline{G(Y)}$ is compact in Z .

Let X be a reflexive Banach space and X^* its topological dual. A possibly multivalued map $A : D \subseteq X \rightarrow 2^{X^*}$ is said to be “monotone”, if for any $x, y \in D$, we have $(x^* - y^*, x - y) \geq 0$ for all $x^* \in A(x), y^* \in A(y)$. If $(x^* - y^*, x - y) = 0$, implies $x = y$, then we say that A is “strictly monotone”. A monotone map for which the inequalities $(x^* - y^*, x - y) \geq 0$ for all $y \in D$ and all $y^* \in A(y)$, imply $x^* \in A(x)$, is said to be “maximal monotone”. It is clear from this definition that A is maximal monotone, if its graph $\text{Gr } A = \{[x, x^*] \in X \times X^* : x^* \in A(x)\}$ is maximal with respect to inclusion among the graphs of monotone maps. When A is maximal monotone, then for every $x \in D, A(x)$ is closed and convex and the set $\text{Gr } A$ is demiclosed, i.e., if $[x_n, x_n^*] \in \text{Gr } A, n \geq 1$, and $x_n \rightarrow x$ in $X, x_n^* \xrightarrow{w} x^*$ in X^* or $x_n \xrightarrow{w} x$ in $X, x_n^* \rightarrow x^*$ in X^* , then $[x, x^*] \in \text{Gr } A$. Let $D = X$ and assume that $A : X \rightarrow X^*$ is single-valued. We say that $A(\cdot)$ is “demicontinuous”, if it is sequentially continuous from X into X_w^* , where X_w^* denotes the space X^* furnished with the weak topology. A monotone, demicontinuous map $A : X \rightarrow X^*$, is maximal monotone. A map $A : D \subseteq X \rightarrow 2^{X^*}$ is said to be “coercive”, if D is bounded or D is unbounded and $\inf\{\|x^*\|_* : x^* \in A(x)\} \rightarrow \infty$ as $\|x\| \rightarrow \infty, x \in D$ (here by $\|\cdot\|$ and $\|\cdot\|_*$, we denote the norms of X and X^* , respectively). A maximal monotone, coercive map $A : D \subseteq X \rightarrow 2^{X^*}$ is surjective. In particular then a monotone, demicontinuous and coercive map $A : X \rightarrow X^*$ is surjective.

If Y, Z are Banach spaces, then an operator $L : Y \rightarrow Z$ (not necessarily linear) is said to be: (a) “completely continuous”, if for all $y_n \xrightarrow{w} y$ in Y as $n \rightarrow \infty$, we have $L(y_n) \rightarrow L(y)$ in Z as $n \rightarrow \infty$; and (b) “compact”, if L is continuous and maps bounded sets into relatively compact sets. In general these two notions are not comparable. However, if Y is reflexive, then complete continuity of L implies compactness. Moreover, if L is linear and Y is reflexive, then the two notions are equivalent.

Our approach ultimately relies on a fixed-point argument, which is based on the “Leray–Schauder fixed-point theorem”. For the convenience of the reader we recall this result here (see [11, Theorem 6A, p. 245]).

Theorem 1. If Y is a Banach space $K : Y \rightarrow Y$ is compact and there exists $r > 0$ such that if $y = \lambda K(y)$ with $0 < \lambda < 1$, we have $\|y\| \leq r$ (a priori bound), then K has a fixed point, i.e., there exists $y \in Y$ such that $y = K(y)$.

If $K \subseteq \mathbb{R}^k$ and $v \in \mathbb{R}^k$, by $\sigma(v, K)$ we denote the support function of K evaluated at $v \in \mathbb{R}^k$, i.e. $\sigma(v, K) = \sup\{(v, x)_{\mathbb{R}^k} : x \in K\}$. Evidently, if $K = \bigcup_{i \in I} K_i$, I arbitrary index set, then $\sigma(v, K) = \sup_{i \in I} \sigma(v, K_i)$. Finally recall that if $2 \leq p < \infty$ and $a, c \in \mathbb{R}$, then $2^{2-p}|a - c|^p \leq (|a|^{p-2}a - |c|^{p-2}c)(a - c)$.

3. Auxiliary results

It is well known that a convex-valued, usc multifunction, need not have a continuous selector, but it has a continuous approximate selector. In the next proposition we prove a measurably parametrized version of this result.

Proposition 2. If (Ω, Σ, μ) is a complete finite measure space, $F: \Omega \times \mathbb{R}^k \rightarrow P_{fc}(\mathbb{R}^k)$ is a measurable multifunction such that for all $\omega \in \Omega$, $F(\omega, \cdot)$ is usc and $\varepsilon > 0$ is given, then there exists a Caratheodory function $g_\varepsilon: \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ (i.e. for all $x \in \mathbb{R}^k$, $\omega \rightarrow g_\varepsilon(\omega, x)$ is measurable and for all $\omega \in \Omega$, $x \rightarrow g_\varepsilon(\omega, x)$ is continuous), such that $g_\varepsilon(\omega, x) \in \overline{\text{conv}}F(\omega, \varepsilon \bar{B}_\varepsilon(x)) + \bar{B}_1$ for all $(\omega, x) \in \Omega \times \mathbb{R}^k$, where $\bar{B}_\varepsilon(x) = \{y \in \mathbb{R}^k: \|y - x\| \leq \varepsilon\}$ and $\bar{B}_1 = \{y \in \mathbb{R}^k: \|y\| \leq 1\}$.

Proof. Let $C(\omega) = \{\eta \in C(\mathbb{R}^k): \eta(x) \in \overline{\text{conv}}F(\omega, \varepsilon \bar{B}_\varepsilon(x)) + B_1\}$. From Theorem I.4.42, p. 107, of Hu and Papageorgiou [7], we know that $C(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Let $F_1(\omega, x) = \overline{\text{conv}}F(\omega, \bar{B}_\varepsilon(x)) + \varepsilon \bar{B}_1$. By virtue of Corollary I.2.20, p. 42 of Hu and Papageorgiou [7], we see that $\overline{\text{conv}}F(\omega, \bar{B}_\varepsilon(x)) \in P_{kc}(\mathbb{R}^k)$. Then for every $v \in \mathbb{R}^k$ we have that

$$\begin{aligned} \sigma(v, \overline{\text{conv}}F(\omega, \bar{B}_\varepsilon(x))) &= \sup_{y \in \varepsilon \bar{B}_1} \sigma(v, F(\omega, x + y)) \\ &\Rightarrow \omega \rightarrow \sigma(v, \overline{\text{conv}}F(\omega, \bar{B}_\varepsilon(x))) \text{ is measurable} \\ &\quad (\text{see [7, Proposition II.2.23, p. 161]}) \\ &\Rightarrow \omega \rightarrow F_1(\omega, x) \text{ is measurable (see [7, Proposition II.2.39, p. 166]).} \end{aligned}$$

Next, fix $\omega \in \Omega$. Since $F(\omega, \cdot)$ is usc, we have that the function $[x, y] \rightarrow \sigma(v, F(\omega, x + y))$ is upper semicontinuous (see [7, Proposition I.2.29, p. 46]) and then so is $x \rightarrow \sup[\sigma(v, F(\omega, x + y)): y \in \varepsilon \bar{B}_1] = \sigma(v, \overline{\text{conv}}F(\omega, \bar{B}_\varepsilon(x)))$ (see [7, Proposition I.3.3, p. 83]). Because $\overline{\text{conv}}F(\omega, \bar{B}_\varepsilon(x)) \in P_{kc}(\mathbb{R}^k)$ we infer that $x \rightarrow F_1(\omega, x)$ is usc (see [7, Proposition I.2.32, p. 47]). Then using the fact that for every $v \in \mathbb{R}^k$, $x \rightarrow d(v, F_1(\omega, x))$ is lower semicontinuous (see [7, I.2.64, p. 61]), for $\{x_n\}_{n \geq 1} \subseteq \mathbb{R}^k$ a dense subset, we have

$$\begin{aligned} \text{Gr } C &= \{(\omega, \eta) \in \Omega \times C(\mathbb{R}^k): \eta(x) \in F_1(\omega, x)\} \\ &= \{(\omega, \eta) \in \Omega \times C(\mathbb{R}^k): d(\eta(x), F_1(\omega, x)) = 0\} \\ &= \bigcup_{n \geq 1} \{(\omega, \eta) \in \Omega \times C(\mathbb{R}^k): d((\eta(x_n), F_1(\omega, x_n))) = 0\} \\ &\Rightarrow \text{Gr } C \in \Sigma \times B(C(\mathbb{R}^k)). \end{aligned}$$

Note that $C(\mathbb{R}^k)$ equipped with the usual compact-open topology is a separable Frechet space. So we can apply Theorem II.2.14, p. 158, of Hu and Papageorgiou [7] (the Yankov-von Neumann–Aumann selection theorem) and obtain $\eta_\varepsilon: \Omega \rightarrow C(\mathbb{R}^k)$ a measurable map such that $\eta_\varepsilon(\omega) \in C(\omega)$ for all $\omega \in \Omega$. Then if we set $g_\varepsilon(\omega, x) = \eta_\varepsilon(\omega)(x)$, this is the desired Caratheodory approximate selector of F . $\square$

In what follows by $\phi: \mathbb{R}^N \rightarrow \mathbb{R}^N$ we denote the function $\phi(x) = \|x\|^{p-2}x$. Given $h \in L^q(T, \mathbb{R}^N)$ ($1/p + 1/q = 1$), we consider the following auxiliary problem:

$$\begin{aligned} -(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) &= h(t) \quad \text{a.e. on } T, \\ (\phi(x'(0)), -\phi(x'(b))) &\in \xi(x(0), x(b)). \end{aligned} \tag{2}$$

Proposition 3. If $\xi: \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is maximal monotone map with $(0, 0) \in \xi(0, 0)$, then problem (2) has a unique solution $x \in C^1(T, \mathbb{R}^N)$ with $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N)$.

Proof. For $v, w \in \mathbb{R}^N$ consider the following two-point boundary value problem:

$$\begin{aligned} -(\|x'(t)\|^{p-2}x'(t))' + \|x(t)\|^{p-2}x(t) &= h(t) \quad \text{a.e. on } T, \\ x(0) &= v, x(b) = w. \end{aligned} \quad (3)$$

Let $\gamma(t) = (1 - t/b)v + t/bw$ (so $\gamma(0) = v, \gamma(b) = w$). Set $y(t) = x(t) - \gamma(t)$ (thus $y(0) = y(b) = 0$). Then problem (3) written in terms of y , becomes a homogeneous Dirichlet problem:

$$\begin{aligned} (\|(y + \gamma)'(t)\|^{p-2}(y + \gamma)'(t))' + \|(y + \gamma)(t)\|^{p-2}(y + \gamma)(t) &= h(t) \quad \text{a.e. on } T, \\ y(0) &= y(b) = 0. \end{aligned} \quad (4)$$

We solve (4). To this end let $A_1 : W_0^{1,p}(T, \mathbb{R}^N) \rightarrow W^{-1,q}(T, \mathbb{R}^N)$ be defined by

$$\begin{aligned} \langle A_1(u), z \rangle &= \int_0^b \|u' + \gamma'\|^{p-2}(u' + \gamma', z')_{\mathbb{R}^N} dt \\ &\quad + \int_0^b \|u + \gamma\|^{p-2}(u + \gamma, z)_{\mathbb{R}^N} dt \quad \text{for all } u, z \in W_0^{1,p}(T, \mathbb{R}^N). \end{aligned}$$

Here by $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the dual pair $(W_0^{1,p}(T, \mathbb{R}^N), W^{-1,q}(T, \mathbb{R}^N))$. Then for $u, z \in W_0^{1,p}(T, \mathbb{R}^N)$, we have

$$\begin{aligned} \langle A_1(u) - A_1(z), u - z \rangle &= \int_0^b \|u' + \gamma'\|^{p-2}(u' + \gamma', u' - z')_{\mathbb{R}^N} dt + \int_0^b \|u + \gamma\|^{p-2}(u + \gamma, u - z)_{\mathbb{R}^N} dt \\ &\quad - \int_0^b \|z' + \gamma'\|^{p-2}(z' + \gamma', u' - z')_{\mathbb{R}^N} dt - \int_0^b \|z + \gamma\|^{p-2}(z + \gamma, u - z)_{\mathbb{R}^N} dt. \end{aligned}$$

Note that

$$\begin{aligned} &\int_0^b (\|u' + \gamma'\|^{p-2}(u' + \gamma', u' - z')_{\mathbb{R}^N} - \|z' + \gamma'\|^{p-2}(z' + \gamma', u' - z')_{\mathbb{R}^N}) dt \\ &\geq \int_0^b (\|u' + \gamma'\|^p - \|u' + \gamma'\|^{p-1}\|z' + \gamma'\| - \|u' + \gamma'\| \|z' + \gamma'\|^{p-1} + \|z' + \gamma'\|^p) dt \\ &= \int_0^b (\|u' + \gamma'\|^{p-1}(\|u' + \gamma'\| - \|z' + \gamma'\|) - \|z' + \gamma'\|^{p-1}(\|u' + \gamma'\| - \|z' + \gamma'\|)) dt \\ &= \int_0^b (\|u' + \gamma'\| - \|z' + \gamma'\|)(\|u' + \gamma'\|^{p-1} - \|z' + \gamma'\|^{p-1}) dt \\ &\geq 2^{2-p} \int_0^b \|\|u' + \gamma'\| - \|z' + \gamma'\|\|^p dt. \end{aligned} \quad (5)$$

Similarly, we obtain

$$\begin{aligned} &\int_0^b \|u + \gamma\|^{p-2}(u + \gamma, u - z)_{\mathbb{R}^N} dt - \int_0^b \|z + \gamma\|^{p-2}(z + \gamma, u - z)_{\mathbb{R}^N} dt \\ &\geq 2^{2-p} \int_0^b \|\|u + \gamma\| - \|z + \gamma\|\|^p dt \end{aligned} \quad (6)$$

From (5) and (6), it follows that A_1 is monotone. Moreover, A_1 is strictly monotone. Indeed if

$$\begin{aligned} \langle A_1(u) - A_1(z), u - z \rangle &= 0, \\ \Rightarrow \int_0^b \|u' + \gamma'\| - \|z' + \gamma'\| \, dt + \int_0^b \|u + \gamma\| - \|z + \gamma\| \, dt &= 0 \\ \Rightarrow \|(u' + \gamma')(t)\| &= \|(z' + \gamma')(t)\| = \eta_1(t) \\ \text{and } \|(u + \gamma)(t)\| &= \|(z + \gamma)(t)\| = \eta_2(t) \end{aligned}$$

We have

$$\begin{aligned} 0 &= \int_0^b \eta_1(t)^{p-2} \|u'(t) - z'(t)\|^2 \, dt + \int_0^b \eta_2(t)^{p-2} \|u(t) - z(t)\|^2 \, dt, \\ \Rightarrow u' &= z' \quad \text{and} \quad u = z, \\ \Rightarrow A_1 &\text{ is strictly monotone.} \end{aligned}$$

Also, it is straightforward to check that A_1 is demicontinuous. Finally we have

$$\begin{aligned} \langle A_1(u), u \rangle &= \int_0^b \|u' + \gamma'\| (u' + \gamma', u')_{\mathbb{R}^N} \, dt + \int_0^b \|u + \gamma\|^{p-2} (u + \gamma, u)_{\mathbb{R}^N} \, dt \\ &\geq \|u' + \gamma'\|_p^p - \|u' + \gamma'\|_p^{p-1} \|\gamma'\|_p + \|u + \gamma\|_p^p - \|u + \gamma\|_p^{p-1} \|\gamma\|_p \\ &\geq \|u + \gamma\|_{1,p}^p - \|u + \gamma\|_{1,p}^{p-1} M_1, \end{aligned} \quad (7)$$

where $M_1 > 0$. Here by $\|\cdot\|_{1,p}$ we denote the norm of the Sobolev space $W_0^{1,p}(T, \mathbb{R}^N)$.

Therefore from (7) we see that $A_1(\cdot)$ is coercive. Recall that a coercive, demicontinuous, monotone operator (hence maximal monotone operator), is surjective (see Section 2). So there exists $u \in W_0^{1,p}(T, \mathbb{R}^N)$ such that $A_1(u) = h$. Moreover, by virtue of the strict monotonicity of A_1 , this u is unique.

Let $\theta \in C_0^\infty(T, \mathbb{R}^N)$. We have

$$\langle A_1(u), \theta \rangle = (h, \theta)_{pq},$$

where by $(\cdot, \cdot)_{pq}$ we denote the duality brackets of the pair $(L^p(T, \mathbb{R}^N), L^q(T, \mathbb{R}^N))$. We have

$$\begin{aligned} \int_0^b \|u' + \gamma'\|^{p-2} (u' + \gamma', \theta')_{\mathbb{R}^N} \, dt + \int_0^b \|u + \gamma\|^{p-2} (u + \gamma, \theta)_{\mathbb{R}^N} \, dt &= \int_0^b (h, \theta)_{\mathbb{R}^N} \, dt \\ \Rightarrow \int_0^b (-\|u' + \gamma'\|^{p-2} (u' + \gamma'))', \theta)_{\mathbb{R}^N} \, dt + \int_0^b \|u + \gamma\|^{p-2} (u + \gamma, \theta)_{\mathbb{R}^N} \, dt &= \int_0^b (h, \theta)_{\mathbb{R}^N} \, dt \\ \text{(by Green's identity)} \\ \Rightarrow -(\|u'(t) + \gamma'(t)\|^{p-2} (u'(t) + \gamma'(t)))' + \|u(t) + \gamma(t)\|^{p-2} (u(t) + \gamma(t)) &= h(t) \quad \text{a.e. on } T; \\ \text{i.e. } u &\text{ solves (4).} \end{aligned}$$

Set $x = u + \gamma \in C^1(T, \mathbb{R}^N)$, with $\|x'\|^{p-2} x' \in W^{1,q}(T, \mathbb{R}^N)$. Clearly x solves (3) uniquely. Let $s(v, w)$ be this unique solution and $\theta(v, w) = (-\varphi(s(v, w)'(0)), \varphi(s(v, w)'(b))$. Then $\varepsilon + \theta$ is maximal monotone, coercive and so $0 \in (\varepsilon + \theta)(v, w)$. Thus $x = s(v, w)$ is the unique solution of (2). $\square$

Now, we introduce the hypotheses for the data of our problem (1).

$H(F): F: T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow P_{kc}(\mathbb{R}^N)$ is a multifunction such that

- (i) $(t, x, y) \rightarrow F(t, x, y)$ is measurable;
- (ii) for all $t \in T, (x, y) \rightarrow F(t, x, y)$ is usc;
- (iii) for almost all $t \in T$, all $x, y \in \mathbb{R}^N$ and all $v \in F(t, x, y)$ we have

$$(v, x)_{\mathbb{R}^N} \geq -a\|x\|^p - b\|x\|^r \|y\|^{p-r} - c(t)\|x\|^s$$

with $a, b \geq 0, 1 \leq r, s < p, c \in L^1(T)$;

- (iv) there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{\mathbb{R}^N} = 0$, then we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$ we have

$$\inf[(v, x)_{\mathbb{R}^N} + \|y\|^p: \|x - x_0\| + \|y - y_0\| \leq \delta, \quad v \in F(t, x, y)] \geq c,$$

- (v) for almost all $t \in T$ and all $x, y \in \mathbb{R}^N$ we have

$$|F(t, x, y)| = \sup\{\|v\|: v \in F(t, x, y)\} \leq \gamma_1(t, \|x\|) + \gamma_2(t, \|x\|)\|y\|^{p-1},$$

where for all $r > 0, \sup_{0 \leq v \leq r} \gamma_1(t, v) \leq \psi_{1r}(t)$ a.e. on T with $\psi_{1r} \in L^q(T)$ and $\sup_{0 \leq v \leq r} \gamma_2(t, r) \leq \psi_{2r}(t)$ a.e. on T with $\psi_{2r} \in L^\infty(T)$.

Remark. Hypothesis $H(F)$ (iv) is a suitable extension to the present measurable and multivalued situation of the classical Nagumo–Hartman condition (see [6, p. 433]). Note that hypothesis $H(F)$ (i) is stronger than the corresponding hypothesis of Halidias and Papageorgiou [5] (see hypothesis $H(F)_2$ (i), p. 132). Recall that if $F(t, x, y)$ is measurable in $t \in T$ and usc in (x, y) , then it need not be jointly measurable. For a counterexample, we refer to Hu and Papageorgiou [7, Example II.7.2, pp. 227].

Our hypotheses on ξ are the following:

$H(\xi): \xi: \mathbb{R}^N \times \mathbb{R}^N \rightarrow 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is maximal monotone map such that $(0, 0) \in \xi(0, 0)$ and one of the following holds:

- (i) for every $[a', b'] \in \xi(a, b)$, we have $(a', a)_{\mathbb{R}^N} \geq 0, (b', b)_{\mathbb{R}^N} \geq 0$; or
- (ii) $\text{dom } \xi = \{[a, b] \in \mathbb{R}^N \times \mathbb{R}^N: a = b\}$.

By virtue of hypotheses $H(F)$ (i) and (ii) and Proposition (4), given $\varepsilon > 0$ we can find a Caratheodory function $g_\varepsilon: T \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that for all $(t, x, y) \in T \times \mathbb{R}^N \times \mathbb{R}^N$ we have $g_\varepsilon(t, x, y) \in F(t, \bar{B}_\varepsilon(x), \bar{B}_\varepsilon(y)) + \varepsilon \bar{B}_1$. We consider the following single-valued approximation of problem (1):

$$\begin{aligned} (\|x'(t)\|^{p-2} x'(t))' &= g_\varepsilon(t, x(t), x'(t)), \\ (\phi(x'(0)), -\phi(x'(b))) &\in \xi(x(0), x(b)). \end{aligned} \tag{8}$$

Proposition 4. If hypotheses $H(F)$ and $H(\xi)$ hold, then problem (8) has a solution $x(\cdot) \in C^1(T, \mathbb{R}^N)$ with $\|x'(\cdot)\|^{p-2} x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N)$.

Proof. Let $D = \{x \in C^1(T, \mathbb{R}^N) : \|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N), (\phi(x'(0)), -\phi(x'(b))) \in \xi(x(0), x(b))\}$ and let $A : D \subseteq L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N)$ ($1/q + 1/p = 1$) be defined by $A(x)(\cdot) = -(\|x'(\cdot)\|^{p-2}x'(\cdot))'$. $\square$

Claim 5. A is maximal monotone.

Arguing as in the proof of Proposition 4, for every $x, y \in D$ we have

$$(A(x) - A(y), x - y)_{pq} \geq \int_0^b (\|x'(t)\|^{p-1} - \|y'(t)\|^{p-1})(\|x'(t)\| - \|y'(t)\|) dt \geq 0.$$

So $A(\cdot)$ is monotone (in fact it can be shown to be strictly monotone). To prove the maximality of A , it suffices to show that if $J : L^p(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N)$ is defined by $J(x)(\cdot) = \|x(\cdot)\|^{p-2}x(\cdot)$, then $R(A + J) = L^q(T, \mathbb{R}^N)$. Indeed suppose that $A + J$ is surjective and let $y \in L^p(T, \mathbb{R}^N)$, $v \in L^q(T, \mathbb{R}^N)$ be such that

$$(A(x) - v, x - y)_{pq} \geq 0 \quad \text{for all } x \in D. \quad (9)$$

Since $R(A + J) = L^q(T, \mathbb{R}^N)$, we can find $x_1 \in D$ such that $A(x_1) + J(x_1) = v + J(y)$. In (9) we set $x = x_1 \in D$ and obtain

$$0 \leq (A(x_1) - A(x_1) - J(x_1) + J(y), x_1 - y)_{pq} = (J(y) - J(x_1), x_1 - y)_{pq}.$$

But $J(\cdot)$ is strictly monotone. Hence $(J(y) - J(x_1), x_1 - y) = 0$ and $x_1 = y$. Therefore, $y \in D$ and $v = A(x_1)$. This proves the maximality of $A(\cdot)$. Hence it remains to show that $R(A + J) = L^q(T, \mathbb{R}^N)$. But this follows immediately from Proposition 4.

Since $J(\cdot)$ is strictly monotone, we see that $(A + J)^{-1} : L^q(T, \mathbb{R}^N) \rightarrow D \subseteq W^{1,p}(T, \mathbb{R}^N)$ is a well-defined single-valued map. In what follows, we set $K = (A + J)^{-1}$.

Claim 6. $K : L^q(T, \mathbb{R}^N) \rightarrow D \subseteq W^{1,p}(T, \mathbb{R}^N)$ is compact.

Because of the reflexivity of $L^q(T, \mathbb{R}^N)$, in order to prove the claim, we need to show that if $u_n \xrightarrow{w} u$ in $L^q(T, \mathbb{R}^N)$, then $x_n = K(u_n) \rightarrow K(u) = x$ in $W^{1,p}(T, \mathbb{R}^N)$ as $n \rightarrow \infty$ (see Section 2). We have

$$\begin{aligned} A(x_n) + J(x_n) &= u_n, n \geq 1 \\ &\Rightarrow (A(x_n), x_n)_{pq} + (J(x_n), x_n)_{pq} = (u_n, x_n)_{pq}, \\ &\Rightarrow \|x'_n\|_p^p + \|x_n\|_p^p \leq \|u_n\|_q \|x_n\|_p, \\ &\Rightarrow \{x_n\}_{n \geq 1} \subseteq W^{1,p}(T, \mathbb{R}^N) \text{ is bounded} \end{aligned}$$

Thus by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T, \mathbb{R}^N)$. Because $W^{1,p}(T, \mathbb{R}^N)$ is embedded compactly in $L^p(T, \mathbb{R}^N)$, we also have that $x_n \rightarrow x$ in $L^p(T, \mathbb{R}^N)$ as $n \rightarrow \infty$. Hence,

$$\begin{aligned} \lim(A(x_n) + J(x_n), x_n - x)_{pq} &= \lim(u_n, x_n - x)_{pq} = 0 \\ &\Rightarrow A(x_n) \xrightarrow{w} A(x) \text{ in } L^q(T, \mathbb{R}^N) \\ &\Rightarrow \|x'_n\|^{p-2} x'_n \rightarrow \|x'\|^{p-2} x' \text{ in } W^{1,q}(T, \mathbb{R}^N) \\ &\Rightarrow x'_n \rightarrow x' \text{ in } L^q(T, \mathbb{R}^N). \end{aligned} \quad (10)$$

Hence, $x_n \rightarrow x$ in $W^{1,p}(T, \mathbb{R}^N)$ as $n \rightarrow \infty$. We have $[x_n, u_n - J(x_n)] \in \text{Gr } A$, $n \geq 1$, and because A is maximal monotone (Claim 5), $\text{Gr } A \subseteq L^p(T, \mathbb{R}^N) \times L^q(T, \mathbb{R}^N)$ is demiclosed and so $[x, u - J(x)] \in \text{Gr } A \rightarrow A(x) + J(x) = u \rightarrow x = K(u)$. Therefore, we conclude that $K(u_n) \rightarrow K(u)$ in $W^{1,p}(T, \mathbb{R}^N)$ as $n \rightarrow \infty$, which proves the claim.

Let $N_\varepsilon : W^{1,p}(T, \mathbb{R}^N) \rightarrow L^q(T, \mathbb{R}^N)$ be the Nemitsky operator corresponding to g_ε ; i.e. $N_\varepsilon(x)(\cdot) = g_\varepsilon(\cdot, x(\cdot), x'(\cdot))$. By Krasnoselskii's theorem, $N_\varepsilon(\cdot)$ is continuous and bounded. Set $L_\varepsilon = -N_\varepsilon + J$. So L_ε is continuous and bounded. Then problem (8) can be written as the following abstract fixed-point problem:

$$x = KL_\varepsilon(x). \quad (11)$$

We will solve (12) using Theorem 1. Note that $KL_\varepsilon : W^{1,p}(T, \mathbb{R}^N) \rightarrow W^{1,p}(T, \mathbb{R}^N)$ is compact.

Claim 7. $C = \{x \in D \subseteq W^{1,p}(T, \mathbb{R}^N) : x = \lambda KL_\varepsilon(x), 0 < \lambda < 1\}$ is bounded in $W^{1,p}(T, \mathbb{R}^N)$ for all $\varepsilon > 0$ small.

Let $x \in C$. By definition we have

$$\begin{aligned} x &= \lambda KL_\varepsilon(x) \text{ with } 0 < \lambda < 1 \\ \Rightarrow (A + J) \left(\frac{1}{\lambda} x \right) &= L_\varepsilon(x) \\ \Rightarrow -(\|x'(t)\|^{p-2} x'(t))' &= -\lambda^{p-1} g_\varepsilon(t, x(t), x'(t)) + (\lambda^{p-1} - 1) \|x(t)\|^{p-2} x(t) \quad \text{a.e. on } T \\ (\phi(x'(0)), -\phi(x'(b))) &\in \xi(x(0), x(b)). \end{aligned} \quad (12)$$

We take the inner product of the differential equation with $x(t)$ and then integrate over $T = [0, b]$. We have

$$\int_0^b (-\|x'(t)\|^{p-2} x'(t), x(t))_{\mathbb{R}^N} dt = \lambda^{p-1} \int_0^b (-g_\varepsilon(t, x(t), x'(t), x(t)))_{\mathbb{R}^N} dt + (\lambda^{p-1} - 1) \|x\|_p^p.$$

Using Green's identity and hypothesis $H(\xi)$, we have

$$\begin{aligned} \int_0^b (-\|x'(t)\|^{p-2} x'(t))', x(t))_{\mathbb{R}^N} dt &= -(\phi(x'(b)), x(b))_{\mathbb{R}^N} + (\phi(x'(0)), x(0))_{\mathbb{R}^N} + \|x'\|_p^p \geq \|x'\|_p^p \\ \Rightarrow \|x'\|_p^p &\leq \lambda^{p-1} \int_0^b (-g_\varepsilon(t, x(t), x'(t), x(t)))_{\mathbb{R}^N} dt \quad (\text{since } 0 < \lambda < 1). \end{aligned} \quad (13)$$

By construction we have $g_\varepsilon(t, x(t), x'(t)) \in F(t, \bar{B}_\varepsilon(x(t)), \bar{B}_\varepsilon(x'(t))) + \varepsilon \bar{B}_1$ a.e. on T .

Let $\Gamma_\varepsilon(t) = \{(u, z, e) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N : g_\varepsilon(t, x(t), x'(t)) \in F(t, u, z) + \varepsilon e, \|u - x(t)\| \leq \varepsilon, \|z - x'(t)\| \leq \varepsilon, \|e\| \leq 1\}$. Evidently, $\Gamma_\varepsilon(t) \neq \emptyset$ a.e. on T and by setting it to be equal to $\{0\}$ on the exceptional Lebesgue-null set, we can have $\Gamma_\varepsilon(t) \neq \emptyset$ for all $t \in T$. Since g_ε is Caratheodory, is jointly measurable. This combined with hypothesis $H(F)(i)$ implies that $\text{Gr } \Gamma_\varepsilon \in \mathcal{L} \times B(\mathbb{R}^N) \times B(\mathbb{R}^N)$, with $\mathcal{L}$ being the Lebesgue σ -field of T . So we can apply the Yankov-von Neumann–Aumann selection theorem (see [7, Theorem II.2.14, p. 158]) and obtain $u : T \rightarrow \mathbb{R}^N, z : T \rightarrow \mathbb{R}^N$ and $e : T \rightarrow \mathbb{R}^N$ measurable maps such that $(u(t), z(t), e(t)) \in \Gamma_\varepsilon(t)$ for all $t \in T$. Evidently, $u, z, e \in L^p(T, \mathbb{R}^N)$. We have

$$g_\varepsilon(t, x(t), x'(t)) \in F(t, u(t), z(t)) + \varepsilon e(t) \quad \text{a.e. on } T$$

with $\|u(t) - x(t)\| \leq \varepsilon, \|z(t) - x'(t)\| \leq \varepsilon, \|e(t)\| \leq \varepsilon$ for all $t \in T$.

Using hypothesis $H(F)(iii)$, we obtain

$$\begin{aligned} (g_\varepsilon(t, x(t), x'(t)), x(t))_{\mathbb{R}^N} &= (g_\varepsilon(t, x(t), x'(t)), x(t) - u(t) + u(t))_{\mathbb{R}^N} \\ &= (g_\varepsilon(t, x(t), x'(t)), x(t) - u(t))_{\mathbb{R}^N} + (g_\varepsilon(t, x(t), x'(t)), u(t))_{\mathbb{R}^N} \\ &\geq -\|g_\varepsilon(t, x(t), x'(t))\| \varepsilon - a\|u(t)\|^p - b\|u(t)\|^r \|z(t)\|^{p-r} \\ &\quad - c(t)\|u(t)\|^s - \|e(t)\| \|u(t)\|. \end{aligned}$$

Note that

$$\begin{aligned} \|u(t)\| &\leq \|u(t) - x(t)\| + \|x(t)\| \leq \varepsilon + \|x(t)\| \\ \Rightarrow \|u(t)\|^p &\leq 2^{p-1} \varepsilon^p + 2^{p-1} \|x(t)\|^p \quad \text{and} \quad \|u(t)\|^r \leq 2^{r-1} \varepsilon^r + 2^{r-1} \|x(t)\|^r. \end{aligned}$$

Similarly, we obtain $\|z(t)\|^{p-r} \leq 2^{p-r-1} \varepsilon^{p-r} + 2^{p-r-1} \|x'(t)\|^{p-r}$ and $\|u(t)\|^s \leq 2^{s-1} \varepsilon^s + 2^{s-1} \|x(t)\|^s$. Using these facts we have

$$\begin{aligned} (-g_\varepsilon(t, x(t), x'(t)), x(t))_{\mathbb{R}^N} &\leq \|g_\varepsilon(t, x(t), x'(t))\| \varepsilon + k(t) \varepsilon + b_1 \|x(t)\|^r \|x'(t)\|^{p-r} \\ &\quad + b_2 \|x(t)\|^r + b_3 \|x'(t)\|^{p-r} + b_4 \|x(t)\|^s + \varepsilon \|x(t)\| \end{aligned} \quad (14)$$

for some $k \in L^1(T)$ and $b_k > 0$, $k \in \{1, 2, 3, 4\}$.

Next, we will show that for all $t \in T$, we have that $\|x(t)\| \leq M$ with $M > 0$ being as in hypothesis $H(F)(iv)$. To this end let $r(t) = \|x(t)\|^p$ and suppose that $t_0 \in T$ is a point where $r(\cdot)$ attains its maximum. Proceeding by contradiction, assume that $r(t_0) > M^p$. First consider the case $0 < t_0 < b$. We have

$$\begin{aligned} 0 &= r'(t_0) = p\|x(t_0)\|^{p-2} (x'(t_0), x(t_0))_{\mathbb{R}^N} \\ &\Rightarrow (x'(t_0), x(t_0))_{\mathbb{R}^N} = 0. \end{aligned}$$

So by virtue of hypothesis $H(F)(iv)$, we can find $\delta > 0$ and $c > 0$ such that for almost all $t \in T$ we have

$$\inf[(v, x)_{\mathbb{R}^N} + \|y\|^p : \|x - x(t_0)\| + \|y - y(t_0)\| \leq \delta, v \in F(t, x, y)] \geq c.$$

Because $x \in C$, $\|x'(\cdot)\|^{p-2} x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N)$ and so $\|x'(\cdot)\|^{p-2} x'(\cdot) \in C(T, \mathbb{R}^N)$ (since $W^{1,q}(T, \mathbb{R}^N) \subseteq C(T, \mathbb{R}^N)$) and so $\phi^{-1}(\|x'(\cdot)\|^{p-2} x'(\cdot)) = x'(\cdot) \in C(T, \mathbb{R}^N)$, because $\phi^{-1}(\cdot)$ is continuous). Hence $x \in C^1(T, \mathbb{R}^N)$ and we can find $\delta_1 > 0$ such that for all $t \in (t_0, t_0 + \delta_1]$ we have $\|x(t) - x(t_0)\| + \|x'(t) - x'(t_0)\| \leq \delta/3$. Let $0 < \varepsilon \leq \delta/3$. Then for $t \in (t_0, t_0 + \delta_1]$ we have

$$\begin{aligned} &\|u(t) - x(t_0)\| + \|z(t) - x'(t_0)\| \\ &\leq \|u(t) - x(t)\| + \|x(t) - x(t_0)\| + \|z(t) - x'(t)\| + \|x'(t) - x'(t_0)\| \\ &\leq 3 \frac{\delta}{3} = \delta. \end{aligned}$$

Since $g_\varepsilon(t, x(t), x'(t)) - \varepsilon e(t) \in F(t, u(t), z(t))$ a.e. on T , for almost all $t \in T$, we have

$$\begin{aligned} \lambda^{p-1} (g_\varepsilon(t, x(t), x'(t)), x(t))_{\mathbb{R}^N} + \lambda^{p-1} \|x'(t)\|^p &\geq \lambda^{p-1} c + \lambda^{p-1} \varepsilon (e(t), x(t))_{\mathbb{R}^N} \\ &\Rightarrow ((\|x'(t)\|^{p-2} x'(t))', x(t))_{\mathbb{R}^N} + (\lambda^p - 1) \|x(t)\|^p \\ &\quad + \lambda^{p-1} \|x'(t)\|^p \geq \lambda^{p-1} c + \lambda^{p-1} \varepsilon (e(t), x(t))_{\mathbb{R}^N} \\ &\geq \lambda^{p-1} (c - \varepsilon \|x(t)\|). \end{aligned}$$

Let $\varepsilon > 0$ be such that $\varepsilon < c/\|x\|_\infty$. We have

$$((\|x'(t)\|^{p-2}x'(t))', x(t))_{\mathbb{R}^N} + \lambda^{p-1}\|x'(t)\|^p \geq \lambda^{p-1}c_1 > 0 \quad (c_1 = c - \varepsilon\|x\|_\infty > 0).$$

Integrate over $[t_0, t]$, $t \leq t_0 + \delta_1$ and on the first integral use Green's identity. We obtain

$$\begin{aligned} \|x'(t)\|^{p-2}(x'(t), x(t))_{\mathbb{R}^N} &\geq \lambda^{p-1}c_1(t - t_0) > 0 \quad (\text{recall that } (x'(t_0), x(t_0))_{\mathbb{R}^N} = 0) \\ &\Rightarrow r'(t) > 0 \quad \text{for all } t \in (t_0, t_0 + \delta_1] \\ &\Rightarrow r(t) > r(t_0) \quad \text{for all } t \in (t_0, t_0 + \delta_1], \end{aligned}$$

a contradiction to the choice of t_0 .

If $t_0 = 0$, then $r'(0) \leq 0$, hence $\|x(0)\|^{p-2}(x'(0), x(0))_{\mathbb{R}^N} \leq 0 \Rightarrow (x'(0), x(0))_{\mathbb{R}^N} \leq 0$. On the other hand since $(\phi(x'(0)), -\phi(x'(b))) \in \xi(x(0), x(b))$, by hypothesis $H(\xi)$, we have $\|x'(0)\|^{p-2}(x'(0), x(0))_{\mathbb{R}^N} \geq 0 \rightarrow (x'(0), x(0))_{\mathbb{R}^N} \geq 0 \Rightarrow (x'(0), x(0))_{\mathbb{R}^N} = 0 \rightarrow r'(0) = 0$ and we proceed as before. Similarly if $t_0 = b$. Therefore we infer that $\|x(t)\| \leq M$ for all $t \in T$. Also note that by virtue of hypothesis $H(F)(v)$, we have for $0 < \varepsilon \leq 1$

$$\begin{aligned} \|g_\varepsilon(t, x(t), x'(t))\| &\leq |F(t, u(t), z(t))| + \varepsilon\|e(t)\| \\ &\leq \gamma_1(t, M + 1) + \gamma_2(t, M + 1)(\|x'(t)\| + 1) + 1 \\ &\leq \gamma_1(t, M + 1) + \gamma_2(t, M + 1)\|x'(t)\| + \beta_5 \quad \text{a.e. on } T, \text{ for some } \beta_5 > 0. \end{aligned} \quad (15)$$

Now going back to (13) and using (14) and (15), we obtain

$$\lambda^{p-1}\|x'\|_p^p \leq \|x'\|_p^p \leq \lambda^{p-1}(\beta_6\varepsilon + \beta_7\|x\|_p^p + \beta_8\|x\|_p^r\|x'\|_p^{p-r} + \beta_9\|x\|_p^s + \beta_{10})$$

for some $\beta_k > 0, k \in \{6, 7, 8, 9, 10\}$. Hence from (16) it follows that

$$\|x'\|_p \leq M_1 \quad \text{for some } M_1 > 0 \text{ independent of } \lambda \in (0, 1) \text{ and } x \in C.$$

This proves that C is bounded in $W^{1,p}(T, \mathbb{R}^N)$ (Claim 5).

This permits the use of Theorem 1, which gives a solution for problem (8). $\square$

4. Main theorem and special cases

In this section we prove an existence theorem for problem (1) and we indicate some special cases of interest that are included in our formulation. In particular, we derive as special cases the Dirichlet, Neumann and periodic problems. By a solution of problem (1) we mean a function $x \in C^1(T, \mathbb{R}^N)$ such that $\phi(x'(\cdot)) \in W^{1,q}(T, \mathbb{R}^N)$, $(\|x'(t)\|^{p-2}x'(t))' = f(t)$ a.e. on T with $f \in \mathcal{S}_{F(\cdot, x(\cdot), x'(\cdot))}^q$ and $(\phi(x'(0)), -\phi(x'(b))) \in \xi(x(0), x(b))$. Recall that $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is defined by $\phi(r) = \|r\|^{p-2}r$.

Theorem 8. *If hypotheses $H(F)$ and $H(\xi)$ hold, then problem (1) has a solution $x(\cdot) \in C^1(T, \mathbb{R}^N)$ with $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N)$.*

Proof. Let $\varepsilon_n = 1/n$, $n \geq 1$, and let x_n be a solution of (8) with $g_{\varepsilon_n}(t, x, y)$ on the right-hand side. Proposition 5 guarantees the existence of x_n . From the proof of Proposition 5 we know that for all $n \geq 1$

$$\|x_n(t)\| \leq M \quad \text{for all } t \in T \text{ and } \|x'_n\|_p \leq M_1.$$

Using these bounds together with hypothesis $H(F)(v)$ and Eq. (8), we infer that $\{(\|x'_n(\cdot)\|^{p-2}x'_n(\cdot))'\}_{n \geq 1}$ is bounded, hence $\{\|x'_n(\cdot)\|^{p-2}x'_n(\cdot)\}_{n \geq 1} \subseteq W^{1,q}(T, \mathbb{R}^N)$ is bounded. Thus we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T, \mathbb{R}^N)$ and $\|x'_n(\cdot)\|^{p-2}x'_n(\cdot) \xrightarrow{w} u$ in $W^{1,q}(T, \mathbb{R}^N)$ as $n \rightarrow \infty$. Because $W^{1,r}(T, \mathbb{R}^N)$, $1 < r < \infty$, is embedded compactly in $C(T, \mathbb{R}^N)$, we have that $x_n \rightarrow x$ and $\|x'_n(\cdot)\|^{p-2}x'_n(\cdot) \rightarrow u$ in $C(T, \mathbb{R}^N)$ as $n \rightarrow \infty$. Since $\phi^{-1} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous (recall that ϕ is a homeomorphism), we have that $\phi^{-1}(\|x'_n(t)\|^{p-2}x'_n(t)) = x'_n(t) \rightarrow \phi^{-1}(u(t))$ as $n \rightarrow \infty$ for all $t \in T$. Because $x'_n \xrightarrow{w} x'$ in $L^p(T, \mathbb{R}^N)$ as $n \rightarrow \infty$, we infer that $u(t) = \|x'(t)\|^{p-2}x'(t)$ for all $t \in T$. Also, since $\text{Gr } \xi$ is closed (because ξ is maximal monotone, see Section 2) and $(\phi(x'_n(0)), -\phi(x'_n(b))) \in \xi(x_n(0), x_n(b))$, we infer that $(\phi(x'(0)), -\phi(x'(b))) \in \xi(x(0), x(b))$. Moreover, from the proof of Proposition 5, we have that

$$g_{e_n}(t, x_n(t), x'_n(t)) \in F(t, u_n(t), z_n(t)) + \frac{1}{n}e_n(t) \quad \text{a.e. on } T \quad (16)$$

with $\|x_n(t) - u_n(t)\| \leq 1/n$, $\|x'(t) - z_n(t)\| \leq 1/n$ and $\|e_n(t)\| \leq 1$ for all $t \in T$. So $u_n(t) \rightarrow x(t)$, $z_n(t) \rightarrow x'(t)$ and $1/ne_n(t) \rightarrow 0$ as $n \rightarrow \infty$ for all $t \in T$. Because $F(t, \cdot, \cdot)$ is usc, we have

$$\limsup F(t, u_n(t), z_n(t)) \subseteq F(t, x(t), x'(t)) \quad \text{for all } t \in T. \quad (17)$$

Also if $\hat{g}_{e_n}(x_n)(\cdot) = g_{e_n}(\cdot, x_n(\cdot), x'_n(\cdot)) \in L^q(T, \mathbb{R}^N)$, $n \geq 1$, we have that $\{\hat{g}_{e_n}(x_n)(\cdot)\}_{n \geq 1} \subseteq L^q(T, \mathbb{R}^N)$ is bounded and so we may assume that $\hat{g}_{e_n}(x_n) \xrightarrow{w} f$ in $L^q(T, \mathbb{R}^N)$. Invoking Proposition VII.3.9, p. 694, of Hu and Papageorgiou [7], we have that

$$f(t) \in \text{conv} \limsup \{\hat{g}_{e_n}(x_n)(t)\}_{n \geq 1} \quad \text{a.e. on } T. \quad (18)$$

From (16)–(18) it follows that

$$f(t) \in F(t, x(t), x'(t)) \quad \text{a.e. on } T, \text{ i.e. } f \in S_{F(\cdot, x(\cdot), x'(\cdot))}^q.$$

Therefore in the limit we have that

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &= f(t) \quad \text{a.e. on } T, \\ (\phi(x(0)), -\phi(x(b))) &\in \xi(x(0), x(b)) \end{aligned} \quad (19)$$

with $f \in S_{F(\cdot, x(\cdot), x'(\cdot))}^q$, hence x is a solution of (1). $\square$

Next, we present some special cases of interest that are incorporated in our general formulation.

(a) Suppose $K_1, K_2 \in P_{fc}(\mathbb{R}^N)$ and $0 \in K_1 \cap K_2$.

Let

$$\delta_{K_1 \times K_2}(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \in K_1, x_2 \in K_2, \\ +\infty & \text{otherwise.} \end{cases}$$

Then from convex analysis (see, for example, [7, Section VI.5]), we know that $\partial \delta_{K_1 \times K_2}(x_1, x_2) = N_{K_1 \times K_2}(x_1, x_2) = N_{K_1}(x_1) \times N_{K_2}(x_2)$, with $N_{K_i}(x_i)$ the normal cone to K_i at x_i , $i = 1, 2$. Thus, problem

(1) becomes

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &\in F(t, x(t), x'(t)) \quad \text{a.e on } T, \\ x(0) &\in K_1, \quad x(b) \in K_2, \\ (x'(0), x(0))_{\mathbb{R}^N} &= \sigma(x'(0), K_1), \\ (-x'(b), x(b))_{\mathbb{R}^N} &= \sigma(-x'(b), K_2). \end{aligned} \quad (20)$$

Note that since $0 \in K_i$, if $x^* \in N_{K_i}(x)$, then $(x^*, x)_{\mathbb{R}^N} = \sigma(x^*, K_i) \geq 0$, $i = 1, 2$. So hypothesis $H(\xi)$ is satisfied and we can apply Theorem 8.

Theorem 9. If $K_1, K_2 \subseteq \mathbb{R}^N$ are as above and hypotheses $H(F)$ hold, then problem (20) has a solution $x(\cdot) \in C^1(T, \mathbb{R}^N)$ with $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(T, \mathbb{R}^N)$.

(b) In the previous example, let $K_1 = K_2 = \{0\}$. Then $N_{K_1}(0) = N_{K_2}(0) = \mathbb{R}^N$ and so problem (1) is the Dirichlet problem, i.e.,

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &\in F(t, x(t), x'(t)) \quad \text{a.e on } T, \\ x(0) &= x(b) = 0. \end{aligned} \quad (21)$$

Theorem 10. If hypotheses $H(F)$ hold, then problem (21) has a solution.

(c) In example (a) let $K_1 = K_2 = \mathbb{R}^N$. Then $N_{K_1}(x_1) = N_{K_2}(x_2) = \{0\}$ for all $x_1, x_2 \in \mathbb{R}^N$ and so problem (1) is the Neumann problem, i.e.,

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &\in F(t, x(t), x'(t)) \quad \text{a.e on } T, \\ x'(0) &= x'(b) = 0. \end{aligned} \quad (22)$$

Theorem 11. If hypotheses $H(F)$ hold, then problem (22) has a solution.

(d) Let

$$K = \{[x, y] \in \mathbb{R}^N \times \mathbb{R}^N : x = y\}$$

and

$$\delta_K(x, y) = \begin{cases} 0 & \text{if } x = y, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\partial\delta_K(x, y) = N_K(x, y) = K^\perp = \{[v, w] \in \mathbb{R}^N \times \mathbb{R}^N : v = -w\}$. Thus problem (1) is the periodic problem, i.e.,

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &\in F(t, x(t), x'(t)) \quad \text{a.e on } T, \\ x'(0) &= x'(b), \quad x(0) = x(b). \end{aligned} \quad (23)$$

Theorem 12. If hypotheses $H(F)$ hold, then problem (23) has a solution.

(c) Consider the following Sturm–Liouville-type problem:

$$\begin{aligned} (\|x'(t)\|^{p-2}x'(t))' &\in F(t, x(t), x'(t)) \quad \text{a.e on } T, \\ x(0) - \beta x'(0) &= 0, \\ x(b) + \gamma x'(b) &= 0, \quad \beta, \gamma > 0. \end{aligned} \tag{24}$$

Let $\xi(x, y) = [1/(\beta^{p-1})\phi(x), 1/(\gamma^{p-1})\phi(y)]$, where as before $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is defined by $\phi(r) = \|r\|^{p-2}r$. Note that if $(v, w) \in \xi(x, y)$, then $(v, x)_{\mathbb{R}^N} = 1/(\beta^{p-1})\|x\|^p \geq 0$ and $(w, y)_{\mathbb{R}^N} = 1/(\gamma^{p-1})\|y\|^p \geq 0$. Thus, hypothesis $H(\xi)$ is satisfied and we can apply Theorem 8, to obtain:

Theorem 13. *If hypotheses $H(F)$ hold, then problem (24) has a solution.*

Remark. It will be interesting to know if we can have Theorem 8 under a weaker measurability hypothesis on F . Namely assume only that $t \rightarrow F(t, x, y)$ is measurable (or even more general that $t \rightarrow F(t, x, y)$ has a measurable selector) and that $(x, y) \rightarrow F(t, x, y)$ is usc (see [2,3,5]). It seems that in our approach, the crucial Proposition 3 needs the full strength of the joint measurability hypothesis $H(F)(i)$.

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