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A collocation method for nonlinear Cauchy singular integral equations

Peter Junghanns *, Katja Müller

Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany

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Abstract

The application of a collocation method with respect to the Chebyshev nodes of second kind together with a Newton iteration to a class of nonlinear Cauchy singular integral equations is discussed. The investigation of the convergence of the Newton method is based on the stability of the respective collocation method applied to linear Cauchy singular integral equations, which is proved by using Banach algebra techniques. Numerical results are presented. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In a series of applications (see, for example, [6,8,9]) nonlinear Cauchy singular integral equations of the form

$$F(x,u(x)) + \frac{1}{\pi}\int_{-1}^{1} \frac{u(y)}{y-x}\,\mathrm{d}y + d = 0, \quad x \in (-1,1) \tag{1.1}$$

occur, where $F : [-1,1] \times \mathbb{R} \to \mathbb{R}$ is given and the function $u : [-1,1] \to \mathbb{R}$ satisfying

$$u(-1) = u(1) = 0 \tag{1.2}$$

as well as the real number d are sought. In [4] one of the authors studied in addition to other methods a Newton collocation method for the numerical solution of such equations, which is based on

* Corresponding author.

E-mail address: peter.junghanns@mathematik.tu-chemnitz.de (P. Junghanns)

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the so-called classical collocation method for linear Cauchy singular integral equations. In general, this method needs the computation of the parameters of Gaussian rules with respect to generalized Jacobi weights. The aim of the present paper is to decrease the computational complexity by using only such parameters with respect to Chebyshev weights. For this end we generalize recent results ([10–12]) concerning projection methods for linear Cauchy singular integral equations basing on weighted Chebyshev polynomials to the case, where the original equation contains additional unknown parameters (like d in (1.1)). This is done in Section 2. In Section 3 we apply these results in connection with a Newton iteration method to problem (1.1), (1.2). Convergence and error estimates are proved. Finally, in Section 4 some numerical results are presented and in Section 5 possibilities for further investigations are mentioned.

2. A class of linear Cauchy singular integral equations

In [10–12] projection methods basing on weighted Chebyshev polynomials are studied for Cauchy singular integral equations of the form

$$a(x)u(x) + \frac{b(x)}{\pi \mathrm{i}} \int_{-1}^{1} \frac{u(y)}{y-x}\,\mathrm{d}y = f(x), \quad x \in (-1,1), \tag{2.1}$$

where $a, b, f : [-1,1] \to \mathbb{C}$ are given functions. By $\mathrm{i} = \sqrt{-1}$ the imaginary unit is denoted. We define the Chebyshev weights $\sigma(x) = (1-x^2)^{-1/2}$ and $\varphi(x) = (1-x^2)^{1/2}$ of the first and second kind, respectively, as well as the weighted Lebesque space $\boldsymbol{L}^2_\sigma := \boldsymbol{L}^2_\sigma(-1,1)$ of all (classes of) measurable functions $u : (-1,1) \to \mathbb{C}$ for which

$$\|u\|^2_\sigma := \int_{-1}^{1} |u(x)|^2 \sigma(x)\,\mathrm{d}x$$

is finite, equipped with the inner product

$$\langle u, v \rangle_\sigma := \int_{-1}^{1} u(x)\overline{v(x)}\sigma(x)\,\mathrm{d}x,$$

which turns $\boldsymbol{L}^2_\sigma$ into a Hilbert space. Write Eq. (2.1) in the short form

$$Au := (aI + bS)u = f, \tag{2.2}$$

where $aI : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$, $u \mapsto au$ and

$$S : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma, \quad u \mapsto \frac{1}{\pi \mathrm{i}} \int_{-1}^{1} \frac{u(y)}{y - \cdot}\,\mathrm{d}y$$

denote the multiplication operator by the function a and the Cauchy singular integral operator, respectively. Remark that $S : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$ is a linear continuous operator ([3, Theorem I.4.1]). Having in mind a linearized equation of (1.1) we are interested in solving more general equations than (2.2) of the following kind. Let $\boldsymbol{V}$ be a finite-dimensional subspace of $\boldsymbol{L}^2_\sigma$ and look for a solution $(u, v) \in \boldsymbol{L}^2_\sigma \times \boldsymbol{V}$ of the equation

$$Au + v = (aI + bS)u + v = f, \tag{2.3}$$

where $a, b \in \boldsymbol{PC}$ and $f \in \boldsymbol{L}^2_\sigma$ are given functions. Here $\boldsymbol{PC} := \boldsymbol{PC}[-1,1]$ denotes the algebra of all piecewise continuous functions defined as the closure (in the supremum norm) of the set of

those functions being continuous on $[-1,1]$ with the possible exception of a finite number of jumps in $(-1,1)$, where the value of the function coincides with the left-sided limit. For definiteness we agree that $a(-1+0)=a(-1)$ for $a \in \boldsymbol{PC}$.

Let T_n and U_n be the orthonormal polynomials of degree n (with positive leading coefficients) with respect to the inner products $\langle .,. \rangle_\sigma$ and $\langle .,. \rangle_\varphi$, respectively, that is $\langle T_n, T_m \rangle_\sigma = \langle U_n, U_m \rangle_\varphi = \delta_{mn}$, where δ_{mn} is the Kronecker symbol. Remember the trigonometric representations

$$T_0 = \frac{1}{\sqrt{\pi}}, \quad T_n(\cos s) = \sqrt{\frac{2}{\pi}} \cos ns, \quad n = 1,2,\ldots$$

and

$$U_n(\cos s) = \sqrt{\frac{2}{\pi}} \frac{\sin (n+1)s}{\sin s}, \quad n = 0,1,2,\ldots,$$

as well as the well-known relation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{U_n(y)}{y-x} \sqrt{1-y^2}\, \mathrm{d}y = -T_{n+1}(x), \quad -1 < x < 1, \ \ n = 0,1,\ldots,$$

which we can write in the form

$$S\varphi U_n = \mathrm{i}\, T_{n+1}, \quad n = 0,1,2,\ldots\ . \tag{2.4}$$

Since the multiplication operator φI is an isometric isomorphism from $\boldsymbol{L}^2_\varphi$ onto $\boldsymbol{L}^2_\sigma$, the functions

$$\tilde{u}_n := \varphi U_n, \quad n = 0,1,2,\ldots$$

form an orthonormal basis in $\boldsymbol{L}^2_\sigma$, because the same is true for U_n in the space $\boldsymbol{L}^2_\varphi$. Using the relation

$$x\, U_n(x) + T_{n+1}(x) = U_{n+1}(x), \quad n = 0,1,2,\ldots,$$

together with (2.4) one can see that the operator $V := \psi I - \mathrm{i}\, \varphi S$, where $\psi(x) = x$, is the shift operator

$$V : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma, \quad u \mapsto \sum_{n=0}^{\infty} \langle u, \tilde{u}_n \rangle_\sigma \tilde{u}_{n+1},$$

the left-sided inverse of which is given by $V^{(-1)} = V^* = \psi I + \mathrm{i}\, \varphi S$.

Obviously, the set $\boldsymbol{X} := \boldsymbol{L}^2_\sigma \times \boldsymbol{V}$ equipped with the norm $\|(u,v)\|_{\boldsymbol{X}} := \sqrt{\|u\|^2_\sigma + \|v\|^2_\sigma}$ is a Banach space. Let $\{\tilde{v}_0, \ldots, \tilde{v}_{m-1}\}$ be a basis in $\boldsymbol{V}$. We define the isomorphism $J_m : \boldsymbol{X} \to \boldsymbol{L}^2_\sigma$ by

$$J_m \left(u, \sum_{k=0}^{m-1} \alpha_k \tilde{v}_k \right) := \sum_{j=0}^{\infty} \langle u, \tilde{u}_j \rangle_\sigma \tilde{u}_{j+m} + \sum_{k=0}^{m-1} \alpha_k \tilde{u}_k = V^m u + \sum_{k=0}^{m-1} \alpha_k \tilde{u}_k.$$

Then the inverse operator $J^{-1} : \boldsymbol{L}^2_\sigma \to \boldsymbol{X}$ is given by

$$J_m^{-1} f = \left(\sum_{j=0}^{\infty} \langle f, \tilde{u}_{j+m} \rangle_\sigma \tilde{u}_j, \sum_{k=0}^{m-1} \langle f, \tilde{u}_k \rangle_\sigma \tilde{v}_k \right) = \left((V^*)^m f, \sum_{k=0}^{m-1} \langle f, \tilde{u}_k \rangle_\sigma \tilde{v}_k \right).$$

Thus, setting $w = J_m(u,v)$ Eq. (2.3) is equivalent to

$$(B^{(m)} + K)w := (aI + bS)(V^*)^m w + Kw = f, \quad w \in \boldsymbol{L}^2_\sigma, \tag{2.5}$$

where the operator $K : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$ is defined by

$$Kw := \sum_{k=0}^{m-1} \langle w, \tilde{u}_k \rangle_\sigma \tilde{v}_k.$$

We remark that in [2, Chapter III, Section 1] an analogous idea is discussed for the finite section method applied to a one-sided invertible operator belonging to an algebra of continuous functions of an only one-sided invertible operator. Define the sequence $\{P_n\}_{n=1}^\infty$ of Fourier projections $P_n : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$ by

$$P_n u := \sum_{k=0}^{n-1} \langle u, \tilde{u}_k \rangle_\sigma \tilde{u}_k.$$

We will investigate the collocation method

$$(Au_n)(x^\varphi_{jn}) + v_n(x^\varphi_{jn}) = f(x^\varphi_{jn}), \quad j = 1, \ldots, n, \;\; u_n \in \operatorname{im} P_{n-m}, \; v_n \in \boldsymbol{V}, \tag{2.6}$$

where x^φ_{jn} are the Chebyshev nodes of second kind,

$$x^\varphi_{jn} := \cos \frac{j\pi}{n+1}.$$

If L^φ_n defines the respective Lagrange interpolation operator, i.e.

$$L^\varphi_n f = \sum_{j=1}^{n} f(x^\varphi_{jn}) l^\varphi_{jn},$$

where

$$l^\varphi_{jn}(x) := \frac{U_n(x)}{(x - x^\varphi_{jn}) U'_n(x^\varphi_{jn})}$$

and $M_n := \varphi L^\varphi_n \varphi^{-1} I$, then the above collocation equation can be written in the form

$$M_n(B^{(m)} + K) P_n w_n = M_n f, \quad w_n \in \operatorname{im} P_n. \tag{2.7}$$

A sequence $\{A_n\}_{n=1}^\infty$ with $A_n \in \mathscr{L}(\operatorname{im} P_n)$ is said to be stable, if there is an n_0 such that $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ is invertible for all $n \geqslant n_0$ and $\sup\{\|A_n^{-1} P_n\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} : n \geqslant n_0\} < \infty$. $\mathscr{L}(\boldsymbol{L}^2_\sigma)$ denotes the Banach space of all linear and bounded operators from $\boldsymbol{L}^2_\sigma$ into $\boldsymbol{L}^2_\sigma$. Let $A_n := M_n(B^{(m)} + K)P_n$ and assume that $\{A_n\}$ is stable. If w^* and w^*_n are the solutions of (2.5) and (2.7), respectively, then, because of the estimate

$$\|P_n w^* - w^*_n\|_\sigma \leqslant \|A_n^{-1} P_n\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} \|A_n P_n w^* - M_n f\|_\sigma$$

and the strong convergence $P_n \to I$, we have $w^*_n \to w^*$ in $\boldsymbol{L}^2_\sigma$ if $A_n P_n \to B + K$ strongly and $M_n f \to f$ in $\boldsymbol{L}^2_\sigma$. The following lemma gives a sufficient condition on $f \in \boldsymbol{L}^2_\sigma$ such that the last relation holds. Let, for some $\varepsilon \geqslant 0$, $\boldsymbol{R}_\varepsilon$ denote the space of all functions $f : (-1, 1) \to \mathbb{C}$, which are locally bounded and Riemann integrable and satisfy

$$|f(x)| \leqslant \text{const}(1 - x^2)^{-\varepsilon}, \quad x \in (-1, 1).$$

Equipped with the norm

$$\|f\|_{\boldsymbol{R}_\varepsilon} = \sup\{|f(x)|\,(1-x^2)^\varepsilon : x \in (-1,1)\}$$

$\boldsymbol{R}_\varepsilon$ becomes a Banach space.

Lemma 2.1 (Junghanns and Weber [11, Corollary 4.6]). *If $f \in \boldsymbol{R}_\varepsilon$ for some $\varepsilon < \frac{1}{4}$, then*

$$\lim_{n\to\infty} \|M_n f - f\|_\sigma = 0.$$

Proof. By definition of M_n we have $\|M_n f - f\|_\sigma = \|L_n^\varphi \varphi^{-1} f - \varphi^{-1} f\|_\varphi$, and the assertion follows from [1, Proposition III.2.1]. □

Corollary 2.2. *Let $\tilde{v}_k \in \boldsymbol{R}_\varepsilon$, $k = 0,\dots,m-1$, for some $\varepsilon < \frac{1}{4}$. Then*

$$\lim_{n\to\infty} \|M_n K - K\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} = 0.$$

Proof. The assertion follows immediately from the compactness of the operator $K : \boldsymbol{L}^2_\sigma \to \boldsymbol{R}_\varepsilon$, Lemma 2.1, and the strong convergence $P_n^* = P_n \to I$ in $\boldsymbol{L}^2_\sigma$. □

In case $m = 0$ the following theorem was proved in [12].

Theorem 2.3 (Junghanns and Weber [12, Theorem 6.4]). *For piecewise continuous coefficients $a, b \in \boldsymbol{PC}$ the collocation method $\{M_n(aI + bS)P_n\}$ is stable in $\boldsymbol{L}^2_\sigma$ if and only if the operators $aI \pm bS$ are invertible in $\boldsymbol{L}^2_\sigma$.*

The proof of this statement is based on the following considerations and the general Theorem 2.5. Let $W_n : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$ be defined by

$$W_n u = \sum_{k=0}^{n-1} \langle u, \tilde{u}_{n-1-k}\rangle_\sigma \tilde{u}_k.$$

Then $W_n = W_n^*$ converges weakly to zero and $W_n P_n = W_n$, $W_n^2 = P_n$. By $\mathscr{A}$ we denote the unital Banach algebra of all sequences $\{A_n\}$, $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$, for which $A_n P_n$, $A_n^* P_n$, $\tilde{A}_n P_n$, and $\tilde{A}_n^* P_n$ converge strongly in $\boldsymbol{L}^2_\sigma$, where $\tilde{A}_n = W_n A_n W_n$, equipped with componentwise algebraic operations and the norm

$$\|\{A_n\}\|_{\mathscr{A}} := \sup\{\|A_n P_n\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} : n = 1,2,\dots\}$$

(comp., for example, [13, p. 268]). Let $\mathscr{K}(\boldsymbol{L}^2_\sigma)$ denote the ideal in $\mathscr{L}(\boldsymbol{L}^2_\sigma)$ of all compact operators. Then the set

$$\mathscr{I} = \left\{\{P_n T_1 P_n + W_n T_2 W_n + C_n\} : T_i \in \mathscr{K}(\boldsymbol{L}^2_\sigma), \lim_{n\to\infty} \|C_n\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} = 0\right\}$$

is a two-sided closed ideal in $\mathscr{A}$ ([14, Proposition 2], [13, Proposition 7.6]). Thus, the quotient algebra $\mathscr{A}/\mathscr{I}$ is again a Banach algebra.

Lemma 2.4 (Junghanns and Weber [12, Proposition 4.3]). *If $a,b \in \boldsymbol{PC}$ and $A = aI + bS$, then $\{M_nAP_n\} \in \mathscr{A}$, where $M_nAP_n \to A$ and $W_nM_nAW_n \to aI - bS$ strongly in $\boldsymbol{L}^2_\sigma$.*

Theorem 2.5 (Silbermann [14, Proposition 3], [13, Theorem 7.7]). *If the sequence $\{A_n\}$ belongs to $\mathscr{A}$, where $A_nP_n \to A$ and $\tilde{A}_nP_n \to \tilde{A}$ strongly in $\boldsymbol{L}^2_\sigma$, then the sequence $\{A_n\}$ is stable if and only if A and $\tilde{A}$ are invertible in $\mathscr{L}(\boldsymbol{L}^2_\sigma)$ and the coset $\{A_n\} + \mathscr{I}$ is invertible in $\mathscr{A}/\mathscr{I}$.*

To consider the case $m > 0$ we present the following lemma.

Lemma 2.6. *For $a,b \in \boldsymbol{PC}$ and for each $m = 1,2,\ldots$ the operator $B^{(m)} = (aI + bS)(V^*)^m$ can be written in the form*

$$a_mI + b_mS + K_m, \tag{2.8}$$

where a_m and b_m belong to $\boldsymbol{PC}$ and $K_m : \boldsymbol{L}^2_\sigma \to \boldsymbol{R}_0$ is compact.

Proof. Using relations (2.4) as well as

$$T_{n+1} = \tfrac{1}{2}(U_{n+1} - U_{n-1}), \quad n = 0,1,2,\ldots, \ U_{-1} \equiv 0$$

and

$$T_{n+2} - T_n = -2\varphi\tilde{u}_n, \quad n = 1,2,\ldots, \ T_2 - \sqrt{2}T_0 = -2\varphi\tilde{u}_0,$$

we get

$$S\varphi S\tilde{u}_n = \mathrm{i}\,S\varphi\,T_{n+1} = \frac{\mathrm{i}}{2}\,S\varphi(U_{n+1} - U_{n-1}) = \begin{cases} \frac{1}{2}(T_n - T_{n+2}), & n = 1,2,\ldots, \\ -\frac{1}{2}\,T_2, & n = 0, \end{cases}$$

$$= \begin{cases} \varphi\tilde{u}_n, & n = 1,2,\ldots, \\ \varphi\tilde{u}_0 - \frac{1}{\sqrt{2}}\,T_0, & n = 0. \end{cases}$$

Since the operator $S\varphi S : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$ is continuous, it follows $S\varphi S = \varphi I + K_0$, where $K_0u = -1/\sqrt{2\pi}\langle u, \tilde{u}_0\rangle_\sigma$. Consequently,

$$(aI + bS)V^* = (aI + bS)(\psi I + \mathrm{i}\varphi S)$$

$$= (a\psi + \mathrm{i}b\varphi)I + (\mathrm{i}a\varphi + b\psi)S + b(S\psi I - \psi S) + \mathrm{i}bK_0.$$

Since the operators $K_0 : \boldsymbol{L}^2_\sigma \to \boldsymbol{R}_0$ and $b(S\psi I - \psi S) : \boldsymbol{L}^2_\sigma \to \boldsymbol{R}_0$ are compact, we get representation (2.8) for $m = 1$. Now assume that (2.8) holds for $m = k$ and consider $B^{(k+1)} = (a_kI + b_kS + K_k)V^*$. In the same manner as above we get

$$B^{(k+1)} = (a_k\psi + \mathrm{i}\,b_k\varphi)I + (\mathrm{i}\,a_k\varphi + b_k\psi)S + b_k(S\psi I - \psi S) + \mathrm{i}b_k\,K_0 + K_kV^*,$$

which proves the lemma by induction. □

Theorem 2.7. *Let $a,b \in \boldsymbol{PC}$, $m = 1,2,\ldots$, and $\tilde{v}_k \in \boldsymbol{R}_\varepsilon$, $k = 0,\ldots,m-1$, for some $\varepsilon < \frac{1}{4}$. Then the sequence $\{M_n[(aI+bS)(V^*)^m+K]P_n\}$ is stable in $\boldsymbol{L}^2_\sigma$ if and only if the operators $(aI+bS)(V^*)^m+K$ and $(aI - bS)V^m$ are invertible in $\mathscr{L}(\boldsymbol{L}^2_\sigma)$.*

Proof. Using the K_m from (2.8) we get

$$M_n(K_m + K)P_n = P_n(K_m + K)P_n + P_n(I - M_n)(K_m + K)P_n$$
$$=: P_n(K_m + K)P_n + C_{mn},$$

with $\lim_{n\to\infty} \| C_{mn} \|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} = 0$ (see Corollary 2.2). Thus,

$$\{M_n(K_m + K)P_n\} \in \mathscr{I}, \quad W_n M_n(K_m + K)W_n \to 0 \quad \text{strongly in } \boldsymbol{L}^2_\sigma,$$

and, in view of Lemma 2.6 and Lemma 2.4, the sequence $\{M_n(B^{(m)} + K)P_n\}$ belongs to $\mathscr{A}$, where

$$M_n(B^{(m)} + K)P_n \to a_m I + b_m S + K_m + K = B^{(m)} + K$$

and

$$W_n M_n(B^{(m)} + K)W_n = W_n M_n(aI + bS)W_n(W_n V^* W_n)^m + W_n M_n K W_n$$
$$= W_n M_n(aI + bS)W_n(P_n V P_n)^m + W_n M_n K W_n$$
$$\to (aI - bS)V^m = a_m I - b_m S \text{ stronly in } \boldsymbol{L}^2_\sigma.$$

The equation $(aI - bS)V^m = a_m I - b_m S$ is a consequence of the representation $B^{(m)} + K = a_m I + b_m S + K_m + K$, of the strong convergence $W_n M_n(K_m + K)W_n \to 0$, and of Lemma 2.4. It remains to prove that $\{M_n(B^{(m)} + K)P_n\} + \mathscr{I} = \{M_n B^{(m)} P_n\} + \mathscr{I}$ is invertible in $\mathscr{A}/\mathscr{I}$ if $(aI + bS)(V^*)^m + K$ and $(aI - bS)V^m = a_m I - b_m S$ are invertible in $\mathscr{L}(\boldsymbol{L}^2_\sigma)$. But, this follows from $\{M_n B^{(m)} P_n\} + \mathscr{I} = \{M_n(a_m I + b_m S)P_n\} + \mathscr{I}$, Theorem 2.3, and Theorem 2.5 taking into account that the invertibility of $B^{(m)} + K$ implies, that the singular integral operator $a_m I + b_m S$ has Fredholm index 0 and is, consequently, invertible (see [3] and Section 4 below). □

3. Nonlinear Cauchy singular integral equations

Now we want to consider Eq. (1.1) under condition (1.2). We will assume the existence of a continuous function u^* and a complex constant d^* satisfying (1.1) and (1.2).

Our aim is to solve the operator equation

$$H(u, d) = 0, \quad (u, d) \in \boldsymbol{X} := \boldsymbol{L}^2_\sigma \times \mathbb{C}, \tag{3.1}$$

approximately, where H is defined by

$$H(u, d) = F(.\,, u) + \mathrm{i}Su + d.$$

For this end we use the collocation method with respect to the Chebyshev nodes of second kind and look for an approximate solution $(u_n, d_n) \in \boldsymbol{X}_n := \operatorname{im} P_{n-1} \times \mathbb{C}$ as explained in Section 2. This leads to the discretisation

$$F(x^\varphi_{jn}, u_n(x^\varphi_{jn})) + \frac{1}{\pi} \int_{-1}^{1} \frac{u_n(y)}{y - x^\varphi_{jn}} \,\mathrm{d}y + d_n = 0, \quad j = 1, \ldots, n, \tag{3.2}$$

for short

$$H_n(u_n, d_n) := M_n H(u_n, d_n) = 0,$$

which is in fact a system of n nonlinear equations. To solve this system we use a Newton method, which means that (u_n, d_n) is approximated by $\{(u_n^{(m)}, d_n^{(m)})\}_{m=0}^{\infty} \subset \boldsymbol{X}_n$, where

$$u_n^{(m+1)} = u_n^{(m)} + \Delta u_n^{(m)}, \quad d_n^{(m+1)} = d_n^{(m)} + \Delta d_n^{(m)}$$

and $(\Delta u_n^{(m)}, \Delta d_n^{(m)})$ is calculated from

$$F_u(x_{jn}^{\varphi}, u_n^{(m)}(x_{jn}^{\varphi}))\Delta u_n^{(m)}(x_{jn}^{\varphi}) + \frac{1}{\pi}\int_{-1}^{1} \frac{\Delta u_n^{(m)}(y)}{y - x_{jn}^{\varphi}}\,\mathrm{d}y + \Delta d_n^{(m)}$$

$$= -F(x_{jn}^{\varphi}, u_n^{(m)}(x_{jn}^{\varphi})) - \frac{1}{\pi}\int_{-1}^{1} \frac{u_n^{(m)}(y)}{y - x_{jn}^{\varphi}}\,\mathrm{d}y + d_n^{(m)}, \quad j = 1, \ldots, n, \tag{3.3}$$

for short

$$[H_n'(u_n^{(m)}, d_n^{(m)})](\Delta u_n^{(m)}, \Delta d_n^{(m)}) = -H_n(u_n^{(m)}, d_n^{(m)}).$$

Thus, we need an initial approximation $(u_n^{(0)}, d_n^{(0)})$ from which we can start the iteration process by calculating

$$a_n^{(0)}(x_{jn}^{\varphi}) = F_u(x_{jn}^{\varphi}, u_n^{(0)}(x_{jn}^{\varphi}))$$

and

$$f_n^{(0)}(x_{jn}^{\varphi}) = -F(x_{jn}^{\varphi}, u_n^{(0)}(x_{jn}^{\varphi})) - \frac{1}{\pi}\int_{-1}^{1} \frac{u_n^{(0)}(y)}{y - x_{jn}^{\varphi}}\,\mathrm{d}y$$

and by solving the resulting linear system of equations

$$a_n^{(0)}(x_{jn}^{\varphi})\Delta u_n^{(0)}(x_{jn}^{\varphi}) + \frac{1}{\pi}\int_{-1}^{1} \frac{\Delta u_n^{(0)}(y)}{y - x_{jn}^{\varphi}}\,\mathrm{d}y + d_n^{(1)} = f_n^{(0)}(x_{jn}^{\varphi}), \quad j = 1, \ldots, n,$$

which is of the same structure as the linear equations considered in Section 2 and can be solved with $\mathrm{O}(n^2)$ complexity using the fast algorithm explained in [5, Section 3.3] and [12, Section 9]. With the help of $\Delta u_n^{(0)}(x_{jn}^{\varphi})$ we get $u_n^{(1)}(x_{jn}^{\varphi})$ and the new values $a_n^{(1)}(x_{jn}^{\varphi})$ and $f_n^{(1)}(x_{jn}^{\varphi})$ for the next Newton step. Continuing this way we obtain the sequence $\{(u_n^{(m)}, d_n^{(m)})\}_{m=0}^{\infty}$ in order to approximate the solution (u_n, d_n) of (3.2).

But for theoretical investigations we use a modified Newton method, which means we use the derivative H_n' with respect to the initial approximation $(u_n^{(0)}, d_n^{(0)})$ in every Newton step and, instead of (3.3), we have to solve the system

$$F_u(x_{jn}^{\varphi}, u_n^{(0)}(x_{jn}^{\varphi}))\Delta u_n^{(m)}(x_{jn}^{\varphi}) + \frac{1}{\pi}\int_{-1}^{1} \frac{\Delta u_n^{(m)}(y)}{y - x_{jn}^{\varphi}}\,\mathrm{d}y + \Delta d_n^{(m)}$$

$$= -F(x_{jn}^{\varphi}, u_n^{(m)}(x_{jn}^{\varphi})) - \frac{1}{\pi}\int_{-1}^{1} \frac{u_n^{(m)}(y)}{y - x_{jn}^{\varphi}}\,\mathrm{d}y + d_n^{(m)}, \quad j = 1, \ldots, n, \tag{3.4}$$

for short

$$[H_n'(u_n^{(0)}, d_n^{(0)})](\Delta u_n^{(m)}, \Delta d_n^{(m)}) = -H_n(u_n^{(m)}, d_n^{(m)}).$$

Now the Operator $A_n : \boldsymbol{X} \to \operatorname{im} P_n$ defined by

$$A_n(u_n, d_n) := M_n[(a_n I + \mathrm{i}S)P_{n-1}u_n + d_n] \quad \text{with } a_n(x) = F_u(x, u_n^{(0)}(x))$$

only depends on the initial approximation $u_n^{(0)}$ and system (3.4) can be written as

$$A_n(\Delta u_n^{(m)}, d_n^{(m+1)}) = \tilde{f}_n^{(m)}. \tag{3.5}$$

If we now set $w_n^{(m)} = J_1(\Delta u_n^{(m)}, d_n^{(m+1)})$ using the isomorphism J_1 (with $\tilde{v}_0 \equiv 1$) defined in Section 2 and define the operators B_n and K by

$$B_n := (a_n I + \mathrm{i}\, S)V^*, \quad Kw = \langle w, \tilde{u}_0 \rangle_\sigma \tilde{u}_0,$$

Eq. (3.5) can be written alternatively as

$$M_n(B_n + K)P_n w_n^{(m)} = \tilde{f}_n^{(m)}. \tag{3.6}$$

For the proof of convergence we shall use the following theorem on the modified Newton iteration method which was proved in [4].

Theorem 3.1 (Junghanns [4, Theorem 4.2]). *Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be Banach spaces and $P : \boldsymbol{U} \to \boldsymbol{V}$ an operator, whose Frechet derivative $P'(u) \in \mathscr{L}(\boldsymbol{U}, \boldsymbol{V})$ exists for all $u \in \boldsymbol{U}$, and $P'(u_0)$ is invertible for some $u_0 \in \boldsymbol{U}$. Further, assume that there are constants M_0, N_0, C_0 and $\eta \in (0, 1]$ such that*

$$\| [P'(u_0)]^{-1} \|_{\mathscr{L}(\boldsymbol{V},\boldsymbol{U})} \leqslant M_0,$$

$$\| [P'(u_0)]^{-1} P(u_0) \|_{\boldsymbol{U}} \leqslant N_0,$$

$$\| P'(u) - P'(u_0) \|_{\mathscr{L}(\boldsymbol{U},\boldsymbol{V})} \leqslant C_0 \| u - u_0 \|_{\boldsymbol{U}}^{\eta}, \quad \forall u \in U,$$

$$h_0 := M_0 N_0^{\eta} C_0 \leqslant \tfrac{1}{4}.$$

Then, there exists a neighbourhood $\boldsymbol{U}_0 = \{u \in \boldsymbol{U} : \| u - u_0 \| \leqslant \rho\}$ of u_0 such that $\rho \leqslant 2N_0$ and the equation

$$P(u) = 0$$

has a unique solution u^* in $\boldsymbol{U}_0$, where the sequence $\{u_k\}_{k=1}^{\infty}$ defined by

$$u_{k+1} = u_k - [P'(u_0)]^{-1} P(u_k)$$

converges to u^*.

For checking the three conditions of this theorem for the operator $H_n : \boldsymbol{X}_n \to \operatorname{im} P_n$ we first have to choose a suitable initial approximation $u_n^{(0)} \in \operatorname{im} P_{n-1}$ (depending on the level n of discretisation) for the iteration.

Remark 3.2. We do not really need an initial approximation $d_n^{(0)} \in \mathbb{R}$ since, due to the linearity of H_n with respect to d_n, neither the operator A_n nor the right-hand side $\tilde{f}_n^{(m)}$ (compare (3.5)) depend on $d_n^{(m)}$. So it is sufficient to choose a suitable $d_n^{(0)}$ for checking the second condition of Theorem 3.1. Moreover, one can easily see that there exist polynomials $p_n^* \in \boldsymbol{\Pi}_{n-1}$ such that $\lim_{n\to\infty} \| \varphi p_n^* - u^* \|_{\infty} = 0$.

Using this property we choose $u_n^{(0)} = \varphi p_n^*$ and prove the following lemma.

Lemma 3.3. *Let, for* $a(x) := F_u(x, u^*(x))$, $b(x) \equiv \mathrm{i}$, $\tilde{v}_0(x)$, *and* $m = 1$ *all conditions of Theorem* 2.7 *be satisfied. If the function* $F_u : [-1,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous, then there exists an* $n_0 \in \mathbb{N}$ *such that the operators* $M_n(B_n + K)P_n \in \mathscr{L}(\operatorname{im} P_n)$ *with* $B_n = (a_n I + \mathrm{i}S)V^*$ *and* $a_n(x) = F_u(x, \varphi(x)p_n^*(x))$ (p_n^* *from Remark* 3.2) *are invertible for all* $n \geqslant n_0$ *and*

$$M_0 := \sup\{\|[M_n(B_n + K)P_n]^{-1}P_n\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)}\} < \infty.$$

Proof. We define $B := (aI + \mathrm{i}S)V^*$, start by splitting the operator

$$M_n(B_n + K)P_n = M_n(B_n - B)P_n + M_n(B + K)P_n$$

and prove that

$$\lim_{n\to\infty} \|M_n(B_n - B)P_n\|_{\mathscr{L}(\boldsymbol{L}^2_\sigma)} = 0. \tag{3.7}$$

Then, since for $M_n(B+K)P_n$ Theorem 2.7 is in force, we can conclude the stability of $M_n(B_n+K)P_n$. Thus, let us continue with the proof of (3.7). Since

$$M_n(B_n - B)P_n = M_n[(a_n I + \mathrm{i}S)V^* - (aI + \mathrm{i}S)V^*]P_n = M_n(a_n - a)V^*P_n,$$

it suffices to show that

$$\|M_n(a_n - a)v_n\|_\sigma \leqslant \gamma_n \|v_n\|_\sigma \quad \text{for all } v_n \in \operatorname{im} P_{n-1} \text{ with } \lim_{n\to\infty} \gamma_n = 0.$$

For this we use the Gaussian rule

$$\int_{-1}^{1} (L_n^\varphi g)(x)\varphi(x)\,\mathrm{d}x = \sum_{j=1}^{n} \lambda_{jn}^\varphi g(x_{jn}^\varphi) =: Q_n^\varphi(g), \quad \text{where } \lambda_{jn}^\varphi = \frac{\pi[1 - (x_{jn}^\varphi)^2]}{n+1}$$

and prove that

$$\|M_n b\, v_n\|_\sigma \leqslant \|b\|_\infty \|v_n\|_\sigma \tag{3.8}$$

for all bounded functions $b : [-1,1] \to \mathbb{C}$ and all $v_n \in \operatorname{im} P_n$. Remember that

$$Q_n^\varphi(p) = \int_{-1}^{1} p(x)\varphi(x)\,\mathrm{d}x \quad \text{for all } p \in \boldsymbol{\Pi}_{2n}.$$

It follows

$$\begin{aligned}
\|M_n b\, v_n\|_\sigma^2 &= \|\varphi L_n^\varphi \varphi^{-1} b\, v_n\|_\sigma^2 = \|L_n^\varphi \varphi^{-1} b\, v_n\|_\varphi^2 \\
&= \sum_{j=1}^{n} \lambda_{jn}^\varphi |\varphi^{-1}(x_{jn}^\varphi) b(x_{jn}^\varphi) v_n(x_{jn}^\varphi)|^2 \\
&\leqslant \|b\|_\infty^2 \int_{-1}^{1} |v_n(x)\varphi^{-1}(x)|^2 \varphi(x)\,\mathrm{d}x = \|b\|_\infty^2 \|v_n\|_\sigma^2,
\end{aligned}$$

which proves (3.8). Consequently,

$$\|M_n(a_n - a)v_n\|_\sigma \leqslant \|a_n - a\|_\infty \|v_n\|_\sigma$$

with $\lim_{n\to\infty} \|a_n - a\|_\infty = 0$ in view of Remark 3.2 and the continuity of the function $F_u : [-1,1] \times \mathbb{R} \to \mathbb{R}$. Thus, relation (3.7) is proved, which together with Theorem 2.7 leads to the assertion. □

Corollary 3.4. *Under the assumptions of Lemma* 3.3 *there exists an* n_0 *such that*

$$\| [H_n'(u_n^{(0)}, d_n^{(0)})]^{-1} \|_{\operatorname{im} P_n \to \boldsymbol{X}_n} \leqslant M_0$$

for all $n \geqslant n_0$.

Proof. Since, for all $(u,d) \in \boldsymbol{X}$, $(a_n I + \mathrm{i}S)u + d = (B_n + K)J_1(u,d)$, from Lemma 3.3 we conclude, for all $(u_n, d_n) \in \operatorname{im} P_{n-1} \times \mathbb{C}$,

$$\begin{aligned} \| [H_n'(u_n^{(0)}, d_n^{(0)})](u_n, d_n) \|_\sigma &= \| M_n(B_n + K)P_n J_1(u_n, d_n) \|_\sigma \\ &\geqslant M_0^{-1} \| J_1(u_n, d_n) \|_\sigma \\ &= M_0^{-1} \| (u_n, d_n) \|_{\boldsymbol{X}}, \end{aligned}$$

and the corollary is proved. □

For our further considerations we need the following concept of weighted Besov spaces. Let ρ and τ be nonnegative real numbers and use the notation $v^{\rho,\tau}(x) = (1-x)^\rho (1+x)^\tau$. By $\boldsymbol{C}_{\rho,\tau}$ we denote the Banach space of all continuous functions $u : (-1,1) \to \mathbb{C}$ for which the norm

$$\| u \|_{\infty,\rho,\tau} = \| v^{\rho,\tau} u \|_\infty = \sup\{ v^{\rho,\tau}(x) \, |u(x)| : x \in (-1,1) \}$$

is finite. Let $E_n^{\rho,\tau}(u)$ be the best weighted uniform approximation of $u \in \boldsymbol{C}_{\rho,\tau}$ by algebraic polynomials of degree less than n, i.e.

$$E_n^{\rho,\tau}(u) = \inf\{ \| p_n - u \|_{\infty,\rho,\tau} : p_n \in \boldsymbol{\Pi}_n \}, \quad E_0^{\rho,\tau}(u) := \| u \|_{\infty,\rho,\tau}.$$

For real numbers $\gamma > 0$ and $q \geqslant 0$, by $\boldsymbol{C}_{\rho,\tau}^{\gamma,q}$ we denote the closed subspace of $\boldsymbol{C}_{\rho,\tau}$ of those functions $u \in \boldsymbol{C}_{\rho,\tau}$ for which

$$\| u \|_{\rho,\tau,\gamma,q} = \sup \left\{ \frac{(n+1)^\gamma E_n^{\rho,\tau}}{1 + \log^q(n+1)} : n = 0, 1, 2, \ldots \right\}$$

is finite (cf. for example, [7]).

Lemma 3.5. *Assume that* $\varphi^{-1} u^*$ *belongs to* $\boldsymbol{C}_{1/2,\,1/2}^{\gamma,q}$ *and that*

$$\| \varphi p_n^* - u^* \|_\infty \leqslant \text{const}\, E_{n-1}^{1/2,\,1/2}(\varphi^{-1} u^*). \tag{3.9}$$

Let $0 < \varepsilon < \gamma$. *If the function* $F_u : [-1,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous then there exists a positive constant* $\tilde{N}_0$, *such that the estimate*

$$\| H_n(u_n^{(0)}, d^*) \|_\sigma \leqslant \tilde{N}_0 n^{\varepsilon - \gamma} \log^q(n+1)$$

holds true.

Proof. Obviously, $H_n(u^*, d^*) = 0$, and, consequently,

$$\| H_n(u_n^{(0)}, d^*) \|_\sigma \leqslant \| M_n [F(.\,, \varphi p_n^*) - F(.\,, u^*)] \|_\sigma + \| M_n S(\varphi p_n^* - u^*) \|_\sigma. \tag{3.10}$$

At first we remark that in view of Lemma 2.1 we have

$$\sup\{ \| M_n \|_{\mathscr{L}(\boldsymbol{R}_0, \boldsymbol{L}_\sigma^2)} : n = 1, 2, \ldots \} < \infty. \tag{3.11}$$

Since $\|\varphi p_n^*\|_\infty$ is bounded, say by c_0, and

$$c_1 := \sup\{|F_u(x,u)|\colon (x,u) \in [-1,1] \times [-c_0, c_0]\} < \infty$$

because of the continuity of F_u, we get

$$\|F(.,\varphi p_n^*) - F(.,u^*)\|_\infty \leqslant c_1 \|\varphi p_n^* - u^*\|_\infty. \tag{3.12}$$

To estimate the second term on the right-hand side of (3.10) we use the fact that $S\varphi I : \boldsymbol{C}^{\varepsilon,q}_{1/2,1/2} \to \boldsymbol{C}^{\varepsilon,q+1}_{0,0}$ is a continuous operator ([7, Remark 4.9]). It follows:

$$\|S(\varphi p_n^* - u^*)\|_\infty \leqslant \|S(\varphi p_n^* - u^*)\|_{0,0,\varepsilon,q+1} \leqslant c_2 \|p_n^* - \varphi^{-1}u^*\|_{1/2,1/2,\varepsilon,q}.$$

Taking into account

$$\|p_n^* - \varphi^{-1}u^*\|_{1/2,1/2,\varepsilon,q} \leqslant c_3 n^{\varepsilon-\gamma} \log^q(n+1) \|\varphi^{-1}u^*\|_{1/2,1/2,\gamma,q},$$

where the constant c_3 does not depend on ε, u^*, and $n = 1,2,\ldots$ ([7, Lemma 6.4]), we get

$$\|S(\varphi p_n^* - u^*)\|_\infty \leqslant c_2 c_3 n^{\varepsilon-\gamma} \log^q(n+1) \|\varphi^{-1}u^*\|_{1/2,1/2,\gamma,q}.$$

This, together with estimates (3.11), (3.10), and (3.12), yields the assertion. □

As an immediate consequence of Corollary 3.4 and Lemma 3.5 we get:

Corollary 3.6. *Under the assumptions of Lemmata* 3.3 *and* 3.5 *there exists an* n_0 *such that, for* $n \geqslant n_0$,

$$\|[H_n'(u_n^{(0)}, d^*)]^{-1} H_n(u_n^{(0)}, d^*)\|_X \leqslant N_0,$$

where $N_0 = M_0 \tilde{N}_0 n^{\varepsilon-\gamma} \log^q(n+1)$.

Now, let us make the following observation. Since $\|\tilde{u}_n\|_\infty \leqslant \sqrt{2/\pi}$, for $u_n \in \operatorname{im} P_n$ and $x \in [-1,1]$, we can estimate

$$|u_n(x)| \leqslant \sqrt{\sum_{k=0}^{n-1} |\langle u_n, \tilde{u}_n\rangle_\sigma|^2} \sqrt{\sum_{k=0}^{n-1} |\tilde{u}_n(x)|^2} \leqslant \|u_n\|_\sigma \sqrt{\frac{2n}{\pi}}.$$

Consequently,

$$\|u_n\|_\infty \leqslant \sqrt{\frac{2n}{\pi}} \|u_n\|_\sigma \quad \text{for all } u_n \in \operatorname{im} P_n. \tag{3.13}$$

With the help of this estimate we can prove the following lemma.

Lemma 3.7. *Assume that* F_u *satisfies the Hölder condition*

$$|F_u(x,u^1) - F_u(x,u^2)| \leqslant c_H |u^1 - u^2|^\eta, \quad x \in [-1,1],\ u^1,\ u^2 \in \mathbb{R} \tag{3.14}$$

for some constants $c_H > 0$ and $\eta \in (0,1]$. *Then there exists a constant* $\tilde{C}_0$ *such that for all* $u_n^1, u_n^2, u_n \in \operatorname{im} P_{n-1}$ *and* $d_1, d_2, d \in \mathbb{C}$,

$$\|[H_n'(u_n^1, d_1) - H_n'(u_n^2, d_2)](u_n, d)\|_\sigma \leqslant \tilde{C}_0 n^{\eta/2} \|u_n^1 - u_n^2\|_\sigma^\eta \|u_n\|_\sigma.$$

Proof. Using (3.8), (3.14) and (3.13) we can estimate

$$\begin{aligned}\|[H_n'(u_n^1,d_1)-H_n'(u_n^2,d_2)](u_n,d)\|_\sigma &= \|M_n[F_u(.,u_n^1)-F_u(.,u_n^2)]u_n\|_\sigma\\ &\leqslant c_H\|u_n^1-u_n^2\|_\infty^\eta\|u_n\|_\sigma\\ &\leqslant c_H\left(\sqrt{\frac{2n}{\pi}}\right)^\eta\|u_n^1-u_n^2\|_\sigma^\eta\|u_n\|_\sigma\end{aligned}$$

and the lemma is proved. □

It follows immediately:

Corollary 3.8. *Under the assumptions of Lemma* 3.7 *the estimate*

$$\|H_n'(u_n,d_n)-H_n'(u_n^{(0)},d_n^{(0)})\|_{X_n\to L_\sigma^2}\leqslant C_0\|(u_n,d_n)-(u_n^{(0)},d_n^{(0)})\|_X^\eta$$

holds true for all $(u_n,d_n),(u_n^{(0)},d_n^{(0)})\in X_n$ *and* $C_0=\tilde{C}_0n^{1/2(1+\eta)}$.

Theorem 3.9. *Assume that the functions* $a(x):=F_u(x,u^*(x))$ *and* $b(x)\equiv \mathrm{i}$ *satisfy all conditions of Theorem* 2.7 *for* $m=1$ *and* $\tilde{v}_0(x)\equiv 1$, *that the function* $F_u:[-1,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous and satisfies the Hölder condition* (3.14), *and that* $\varphi^{-1}u^*$ *belongs to* $C_{1/2,1/2}^{\gamma,q}$ *for some* $\gamma>\frac{1}{2}$. *Then, for all sufficiently large n, there exists an element* $(u_n^{(0)},d_n^{(0)})\in X_n$ *such that Eq.* (3.5) *possesses a unique solution* $(\Delta u_n^{(m)},d_n^{(m+1)})$ *for all* $m=0,1,2\ldots$ *The sequence* $\{(u_n^{(m)},d_n^{(m)})\}_{m=0}^\infty$ *converges in the norm of* X *to a solution* (u_n^*,d_n^*) *of Eq.* (3.2) *and, for* $0<\varepsilon<\gamma-\frac{1}{2}$,

$$\|(u_n^*,d_n^*)-(u^*,d^*)\|_X\leqslant\text{const}\ n^{\varepsilon-\gamma}\log^q(n+1), \tag{3.15}$$

where the constant does not depend on n.

Proof. Choose $u_n^{(0)}=\varphi p_n^*$ satisfying (3.9), $d_n^{(0)}=d^*$, and apply Corollary 3.4, 3.6 and 3.8 as well as Theorem 3.1 with $U=X_n$, $V=\operatorname{im}P_n$, and $P=H_n$. We get $h_0=\text{const}\,n^{\eta(1/2+\varepsilon-\gamma)}\log^{q\eta}(n+1)$ which is small enough if n is sufficiently large. Besides the existence of a solution (u_n^*,d_n^*) of Eq. (3.2) this gives the estimate

$$\|(u_n^*,d_n^*)-(u_n^{(0)},d_n^{(0)})\|_X\leqslant 2N_0=2M_0\tilde{N}_0n^{\varepsilon-\gamma}\log^q(n+1),$$

from which the estimate (3.15) follows taking into account

$$\|u_n^{(0)}-u^*\|_\sigma\leqslant\text{const}\,\|u_n^{(0)}-u^*\|_{\infty,1/2,1/2}.$$

The theorem is proved. □

Remark 3.10. In [4] there Newton collocation methods are also studied for the numerical solution of equations of the form

$$u(x)+\frac{1}{\pi}\int_{-1}^{1}\frac{G(y,u(y))}{y-x}\,\mathrm{d}y+f(x)+d=0,$$

which are based on the classical collocation method. The ideas presented in this section to study the collocation method (3.2), which are essentially based on Theorems 2.7 and 3.1, one can also apply to this kind of equation and to the more general case

$$F(x,u(x))+\frac{1}{\pi}\int_{-1}^{1}\frac{G(y,u(y))}{y-x}\,\mathrm{d}y+d=0.$$

But the generalization of the results of Theorem 3.9 is not straightforward and involves more technical difficulties. Thus, in the present paper we restrict our considerations to equations of the form (1.1) to introduce the main ideas.

4. Numerical results

Before considering concrete examples of Cauchy singular integral equations we state a criterion for the Fredholmness (in $\boldsymbol{L}^2_\sigma$) of an operator $aI+bS$ with $a,b\in\boldsymbol{PC}$ (comp. [3, Theorem IX.4.1]). At first we remark that the Fredholmness of $aI+bS$ implies $a(x\pm0)+b(x\pm0)\neq0$ and $a(x\pm0)-b(x\pm0)\neq0$ for all $x\in[-1,1]$. Let

$$c(x)=\frac{a(x)+b(x)}{a(x)-b(x)}$$

and

$$\boldsymbol{c}(x,\mu):=\begin{cases}c(x-0)\mu+c(x+0)(1-\mu), & \mu\in[0,1],\ x\in(-1,1),\\ c(1)[1-r(\mu)]+r(\mu), & \mu\in[0,1],\ x=1,\\ 1-r(\mu)+c(-1)r(\mu), & \mu\in[0,1],\ x=-1,\end{cases}$$

where

$$r(\mu)=\sin\frac{\pi\mu}{2}\exp\left(\frac{\mathrm{i}\pi(\mu-1)}{2}\right).$$

Note that $z_1[1-r(\mu)]+z_2r(\mu)$, $\mu\in[0,1]$, describes the half-circle line from z_1 to z_2 that lies to the right of the straight line from z_1 to z_2. Thus, the image of $\boldsymbol{c}(x,\mu)$ is a closed curve in the complex plane, which possesses a natural orientation. By $\operatorname{wind}\boldsymbol{c}(x,\mu)$ we denote the winding number of this curve with respect to the origin 0. Then, $A=aI+bS$ is Fredholm in $\boldsymbol{L}^2_\sigma$ if and only if $\boldsymbol{c}(x,\mu)\neq0$ for all $x\in[-1,1]$ and all $\mu\in[0,1]$, where $\operatorname{ind}A=-\operatorname{wind}\boldsymbol{c}(x,\mu)$ and A is one-sided invertible.

Now let us consider the linear equation

$$a(x)u(x)+\frac{1}{\pi}\int_{-1}^{1}\frac{u(y)}{y-x}\,\mathrm{d}y+d=f(x) \tag{4.1}$$

with $a(x)=-2x$. In this case we have

$$c(x)=\frac{2x-\mathrm{i}}{2x+\mathrm{i}}=\frac{4x^2-1-4x\mathrm{i}}{4x^2+1}$$

and $\operatorname{wind}\boldsymbol{c}(x,\mu)=1$. Thus, the operator $A=aI+\mathrm{i}S$ has index -1, which implies $\operatorname{ind}(AV^*+K)=0$. Consequently, to show the invertibility of AV^*+K it remains to prove, that in case of $f\equiv0$ Eq. (4.1) possesses in $\boldsymbol{L}^2_\sigma\times\mathbb{C}$ only the trivial solution. For this end define continuous functions

Table 1
Example (L1)

n	ε_n	$n^{0.5}\varepsilon_n$
20	0.1621197306	0.725
40	0.1187411068	0.751
60	0.0981454447	0.760
80	0.0855274278	0.765
100	0.0767872102	0.768
500	0.0347636544	0.777
1000	0.0246197138	0.779
3000	0.0142288940	0.779
5000	0.0110239315	0.780

$g:[-1,1] \to \mathbb{R}$ and $h:[-1,1] \to (0,\infty)$ by $h(x)\,\mathrm{e}^{\mathrm{i}\pi g(x)} = -2x - \mathrm{i}$ and $-\frac{1}{2} < g(-1) < 0$ and set $\alpha = 1 + g(1)$, $\beta = -g(-1)$. Then $\alpha, \beta \in (0, \frac{1}{2})$ and, in view of [13, Theorem 9.9, Remark 9.10], the homogeneous equation (4.1) has in $\boldsymbol{L}^2_\chi \times \mathbb{C}$ with $\chi(x) = (1-x)^{-\alpha}(1+x)^{-\beta}$ only the trivial solution. Since $\boldsymbol{L}^2_\sigma \subset \boldsymbol{L}^2_\chi$, the same holds true in $\boldsymbol{L}^2_\sigma \times \mathbb{C}$. Analogously, for the operator $aI - \mathrm{i}\,S$ we get $\operatorname{ind}(aI - \mathrm{i}\,S) = 1$ and, consequently, the invertibility of $(aI - \mathrm{i}\,S)V = a_1 I - b_1 S$ (comp. Lemma 2.6). All this together means that the conditions of Theorem 2.7 are satisfied for Eq. (4.1). We consider two examples:

(L1) $u^*(x) = |x|$ and $d^* = 1$, such that

$$f(x) = \frac{x}{\pi} \ln \frac{1-x^2}{x^2} - 2x|x| + 1$$

and

(L2) $u^*(x) = 1 - x^2$ and $d^* = 1$, such that

$$f(x) = \frac{1-x^2}{\pi} \ln \frac{1-x}{1+x} - \frac{2x}{\pi} - 2x(1-x^2) + 1.$$

To observe convergence and stability we use the fact that the solutions u^* in the examples belong to some weighted Sobolev space (cf. for example, [12, Section 8]). For $s \geqslant 0$, define

$$\tilde{\boldsymbol{L}}^2_{\sigma,s} := \left\{ u \in \boldsymbol{L}^2_\sigma : \sum_{n=0}^{\infty} (1+n)^{2s} |\langle u, \tilde{u}_n \rangle_\sigma|^2 < \infty \right\}.$$

Then $u^* \in \tilde{\boldsymbol{L}}^2_{\sigma,s}$ for $s < \gamma$, where $\gamma = \frac{1}{2}$ in example (L1) and $\gamma = \frac{5}{2}$ in example (L2). In Tables 1 and 2 we present the values of $\varepsilon_n = \sqrt{\|P_{n-1}u^* - u_n^*\|^2_\sigma + |d^* - d_n^*|^2}$ and of $n^\gamma \varepsilon_n$ for various n.

As a nonlinear equation we consider

$$-[u(x)]^2 \operatorname{sgn}(x) - f(x) + \frac{1}{\pi} \int_{-1}^{1} \frac{u(y)}{y-x}\,\mathrm{d}y + d = 0, \tag{4.2}$$

such that $F(x,u) = -u^2 \operatorname{sgn}(x) - f(x)$. For $u^*(x) = |x|$ and $d^* = 1$ we have

$$f(x) = -x^2 \operatorname{sgn}(x) + \frac{x}{\pi} \ln \frac{1-x^2}{x^2} + 1$$

Table 2
Example (L2)

n	ε_n	$n^{2.5}\varepsilon_n$
20	0.0006708469	1.200
40	0.0001275452	1.291
60	0.0000475502	1.326
80	0.0000234917	1.345
100	0.0000135641	1.356
500	0.0000002497	1.396
1000	0.0000000443	1.401
3000	0.0000000028	1.405
5000	0.0000000008	1.409

and

$$a(x) = F_u(x, u^*(x)) = -2x,$$

such that $a(x)$ is the same as in the linear examples above. As a last example we take

$$x[1-u(x)]^2 - f(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{u(y)}{y-x}\,\mathrm{d}y + d = 0. \tag{4.3}$$

Here we have $F(x,u) = x(1-u)^2 - f(x)$, such that for $u^*(x) = 1 - x^2$ and $d^* = 1$

$$f(x) = x^5 + \frac{1-x^2}{\pi}\ln\frac{1-x}{1+x} - \frac{2x}{\pi} + 1$$

and

$$a(x) = F_u(x, u^*(x)) = -2x^3.$$

The invertibility of $(aI + \mathrm{i}S)V^* + K$ and $(aI - \mathrm{i}S)V$ in $\mathscr{L}(\boldsymbol{L}^2_\sigma)$ can be shown in the same way as for the linear example above. In Tables 3 and 4 we present the results obtained by applying the usual Newton iteration method (3.3) and the modified Newton iteration method (3.4) to the discrete equations (3.2). In case of the usual Newton iteration method the initial approximation is chosen as $u_n^{(0)}(x) \equiv 0$. The error ε_n is defined as before. The numbers z and z_{mod} denote the number of Newton steps and modified Newton steps, respectively. The number z is the smallest one for which

$$\sqrt{\|u_n^{(z)} - u_n^{(z-1)}\|_\sigma^2 + |d_n^{(z)} - d_n^{(z-1)}|^2} < 10^{-8}. \tag{4.4}$$

In the case of the modified Newton iteration method the initial approximation has to be chosen more carefully in order to get convergence of the Newton iterations. Thus, in both examples we have chosen $u_n^{(0)}(x) \equiv 0.5$. In the case of example (4.2) the number z_{mod} is determined in the same way as z (comp. (4.4)). But, in the case of example (4.3) and for great n it was necessary to take a smaller error bound in order to guarentee the convergence of $(u_n^{(z_{\mathrm{mod}})}, d_n^{(z_{\mathrm{mod}})})$ to the solution (u^*, d^*) of the original equation. So, in this example we replaced the right-hand side of (4.4) by 10^{-10} for $n = 500$, by 10^{-12} $n = 1000$, and by 10^{-13} for $n = 3000$ and 5000. Remark that, in case of Eq. (4.2) $\varphi^{-1}u^*$ does not belong to any space $\boldsymbol{C}^{\gamma,q}_{1/2,\,1/2}$ in contrast to one condition of Theorem 3.9. The numerical results seem to show that the convergence rate of $\|(u_n^*, d_n^*) - (u^*, d^*)\|_X$ is determined by the Sobolev space to which the solution u^* belongs (comp. [12, Theorem 8.3]).

Table 3
Eq. (4.2)

n	ε_n	$n^{0.5}\varepsilon_n$	z	z_{mod}
20	0.1637606725	0.732	7	26
40	0.1198879383	0.758	7	28
60	0.0990778324	0.767	7	28
80	0.0863332032	0.772	7	28
100	0.0775070222	0.775	7	28
500	0.0350843043	0.785	7	27
1000	0.0248463339	0.786	7	26
3000	0.0143596885	0.787	7	25
5000	0.0111252374	0.787	7	24

Table 4
Eq. (4.3)

n	ε_n	$n^{2.5}\varepsilon_n$	z	z_{mod}
20	0.0006907186	1.236	5	22
40	0.0001287444	1.303	5	26
60	0.0000477740	1.332	5	26
80	0.0000235589	1.349	5	26
100	0.0000135904	1.359	5	26
500	0.0000002497	1.396	5	35
1000	0.0000000443	1.401	5	46
3000	0.0000000028	1.405	5	51
5000	0.0000000008	1.407	5	50

5. Final remarks

The (necessary and sufficient) conditions for the stability of the sequence

$$\{M_n[(aI + bS)(V^*)^m + K]P_n\}$$

given in Theorem 2.7 seem to be very restrictive. So, for example, in case of $a \equiv 0$, $b \equiv 1$, and $m = 1$ we get instability for the sequence $\{M_n(S V^* + K)P_n\}$. Thus, the question arises if the choice of other collocation points leads to other stability conditions. Moreover, there is another possibility for the description of the collocation method (2.6), which also allows the application of Banach algebra techniques to study stability. For this define the continuous isomorphism $J_\sigma : \boldsymbol{L}^2_\sigma \to \boldsymbol{L}^2_\sigma$ by

$$J_\sigma u = \sum_{n=1}^{\infty} \langle u, \tilde{u}_n \rangle_\sigma T_n + \sqrt{2}\langle u, \tilde{u}_0 \rangle_\sigma T_0.$$

We remark that $J_\sigma = \varphi I - i\psi S$, $\psi(x) = x$. Then (2.6) is equivalent to

$$J_\sigma^{-1} L_n^\varphi (Au_n + v_n) = J_\sigma^{-1} L_n^\varphi f$$

and one can investigate the stability of the sequence $\{J_\sigma^{-1} L_n^\varphi [A(V^*)^m + K]P_n\}$. These problems of finding the stability conditions in case of other collocation points and in case of another description of the collocation method will be discussed in a forthcoming paper.

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