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Spectral methods for hyperbolic problems ☆

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Abstract

We review the current state of Fourier and Chebyshev collocation methods for the solution of hyperbolic problems with an eye to basic questions of accuracy and stability of the numerical approximations. Throughout the discussion we emphasize recent developments in the area such as spectral penalty methods, the use of filters, the resolution of the Gibbs phenomenon, and issues related to the solution of nonlinear conservations laws such as conservation and convergence. We also include a brief discussion on the formulation of multi-domain methods for hyperbolic problems, and conclude with a few examples of the application of pseudospectral/collocation methods for solving nontrivial systems of conservation laws.

Keywords: Spectral; Pseudospectral; Collocation; Penalty methods; Discontinuous solutions; Gibbs phenomenon; Stability; Filtering; Vanishing viscosity; Multi-domain methods

1. Introduction

The theory, implementation and application of spectral and pseudospectral methods for the solution of partial differential equations has traditionally been centered around problems with a certain amount of inherent regularity of the solutions, e.g., elliptic/parabolic problems. Among many examples, the application that is perhaps most responsible for the widespread use of spectral methods is the accurate and efficient solution of the incompressible Navier–Stokes equations [10].

On the other hand, the application of spectral and pseudospectral methods for the solution of hyperbolic problems, and in particular nonlinear conservation laws which are prone to develop discontinuous solutions in finite time, has traditionally been viewed as problematic. Indeed, with a few noticeable exceptions, very little work was done to adapt spectral and pseudospectral methods to the

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solution of such important classes of problems as the equations of gas-dynamics and electromagnetics until the late 1980s and early 1990s.

The reasons for the perceived difficulty are several. Contrary to parabolic and elliptic problems, there is no physical dissipation inherent in the hyperbolic problem. This again implies that even minor errors and under resolved phenomena can cause the scheme to become unstable, i.e., the question of stability of the spectral approximations tends to be even more critical than for other types of problems. Perhaps the most important reason, however, for the slow acceptance of the use of high-order methods in general and pseudospectral methods in particular for solving hyperbolic conservation laws can be found in the appearance of the Gibbs phenomenon as finite-time discontinuities develop in the solution. Left alone, the nonlinear mixing of the Gibbs oscillations with the approximate solution will eventually cause the scheme to become unstable. Moreover, even if stability is maintained sufficiently long, the computed solution appears to be only first-order accurate in which case the use of a high-order method is questionable. More fundamental issues of conservation and the ability of the scheme to compute the correct entropy solution to conservation laws have also caused considerable concern among practitioners and theoreticians alike.

While many of these issues are genuine and requires careful attention they are not causing the pseudospectral approach to fail if applied correctly. This was hinted to in early work around 1980 [74,66,36] where the first numerical solution of problems with discontinuous solutions and general nonlinear conservation laws were presented. It is, however, mainly within the last decade that many of the most significant advances has been made to establish the soundness of the pseudospectral approach for such problems, often confirming that the superior behavior of these methods for smooth problems carries over to problems involving nonsmooth solutions.

It is the central components of these more recent developments that we shall review in the following. We do not attempt to be complete in the treatment and discussion. We do hope, however, that the review will offer enough information to allow the reader to venture deeper into the fascinating theory and the application of spectral and pseudospectral methods to the solution of hyperbolic conservations laws. It is a topic that has challenged researchers in the last decades of this century and is certain to continue to do so in the next decades.

What remains of this review is organized as follows. In Section 2 we introduce the spectral and pseudospectral approximation of spatial derivatives using Fourier and Chebyshev series approximations of the function. We highlight the duality between the modal representation, exploiting quadratures to approximate inner products, and the nodal formulation, using Lagrange interpolation polynomials, and emphasize how this duality suggests two computationally different but mathematically equivalent formulations of spectral/pseudospectral methods. Section 3 offers a brief overview of the relevant approximation results for spectral and pseudospectral Fourier and Chebyshev expansions of smooth functions. We subsequently discuss the behavior of such expansions for nonsmooth functions, the introduction of the Gibbs phenomenon and recent results on how to improve on the convergence of the approximation away from the discontinuity by the use of filters. We also discuss the complete resolution of the Gibbs phenomenon through the reconstruction of an exponentially convergent approximation to a piecewise smooth function using the information in the global Fourier or Chebyshev expansions only. This sets the stage for Section 4 in which we introduce collocation approximations to hyperbolic problems in general. While we briefly discuss the issues related to Fourier approximations, the emphasis is on the formulation of Chebyshev approximations and techniques to enforce the prescribed boundary conditions. Strongly as well as weakly enforced boundary

conditions are addressed and it is shown, among other things, that the discontinuous Galerkin method is a special case of a much larger class of methods. The flexibility of this general approach, known as spectral penalty methods, is illustrated through a few examples. The critical question of stability is considered in Section 5 where we review the stability of Fourier and Chebyshev pseudospectral approximations of linear and variable coefficient hyperbolic problems. We return to the formulation of the penalty methods in more detail and address the question of stability for linear systems of equations, before we briefly attend to the stability of linear problems with nonsmooth initial conditions and conclude with a brief discussion on issues related to fully discrete stability. The introduction of nonlinearities introduces a number of new complications, discussed in Section 6, such as the impact of the Gibbs phenomenon on the stability of the approximation, the use of stabilization by filtering and conservation properties of the pseudospectral approximations. We briefly review recent results on spectrally vanishing viscosity and its relation to filtering techniques. Section 7 is devoted to a brief overview of multi-domain techniques that allow for solving hyperbolic problems in geometrically complex domains or allows the use of a spatially varying resolution. This last development is critical in enabling the use of pseudospectral methods for the solution of hyperbolic conservations laws for realistic problems as we shall illustrate through a few examples in the concluding Section 8.

2. Modes and nodes

Embedded in all numerical schemes for solving partial differential equations (PDEs) lies an assumption about the behavior of the solution to the PDE as reflected in a choice of how to represent the approximate solution.

For spectral methods the traditional approach has been to assume that the solution, $u(x,t)$, can be expressed as a series of smooth basis functions of the form

$$\mathscr{P}_N u(x,t) = \sum_{n=0}^{N} \hat{u}_n(t)\phi_n(x), \tag{1}$$

where the projection, $\mathscr{P}_N u(x,t)$, of the solution is assumed to approximate $u(x,t)$ well in some appropriate norm as N approaches infinity. Hence, the approximate solution, $\mathscr{P}_N u(x,t) \in \mathsf{P}_N$, at all times where the space, $\mathsf{P}_N \in \{\phi_n(x)\}_{n=0}^{N}$, is spanned by smooth basis-functions, $\phi_n(x)$, which we assume form an L^2-complete basis. For the sake of computational efficiency this basis is typically chosen to be orthogonal in a weighted inner-product although this is not a requirement. However, the actual choice of the basis, $\phi_n(x)$, and the way in which the expansion coefficients, $\hat{u}_n(t)$, are computed offers a number of alternative methods which we shall discuss further in the following.

Leaving these details open for a minute it is clear that, under the assumption of Eq. (1), the approximation of a spatial derivative can be expressed in two different ways:

$$\frac{\partial u(x,t)}{\partial x} \simeq \frac{\partial \mathscr{P}_N u(x,t)}{\partial x} = \sum_{n=0}^{N} \hat{u}_n(t)\frac{\mathrm{d}\phi_n(x)}{\mathrm{d}x} = \sum_{n=0}^{N} \hat{u}_n'(t)\phi_n(x), \tag{2}$$

where the first formulation simply involves derivatives of the smooth basis function, $\phi_n(x)$, while the second expression involves the direct expansion of the spatial derivative itself.

Rather than expressing the unknown solution in terms of a basis as in Eq. (1), one could choose to introduce a grid and assume that the solution can be expressed as a global interpolation polynomial

$$\mathscr{I}_N u(x,t) = \sum_{j=0}^{N} u(x_j,t) l_j(x), \tag{3}$$

where $l_j(x)$ represents the Lagrange interpolation polynomial based on the grid points, x_j. Here we have introduced the notation $\mathscr{I}_N u(x,t)$ to reflect the interpolation property, i.e., $\mathscr{I}_N u(x_j,t) = u(x_j,t)$. In this setting spatial derivatives are approximated by

$$\frac{\partial u(x,t)}{\partial x} \simeq \frac{\partial \mathscr{I}_N u(x,t)}{\partial x} = \sum_{j=0}^{N} u(x_j,t) \frac{\mathrm{d} l_j(x)}{\mathrm{d} x}. \tag{4}$$

As we shall see shortly, the approximation, $\mathscr{P}_N u$, in Eqs. (1)–(2) and the interpolation, $\mathscr{I}_N u$, in Eqs. (3)–(4) are closely related and provides a duality which is pivotal in the formulation, analysis and implementation of efficient spectral methods for the solution of hyperbolic problems.

2.1. Modal expansions and derivatives

If one considers the solution of periodic problems it is natural to express the unknown solution as a Fourier series

$$\mathscr{P}_N u(x) = \sum_{n=-N}^{N} \hat{u}_n \exp(\mathrm{i}nx) \tag{5}$$

with $\phi_n(x) = \exp(\mathrm{i}nx)$ in Eq. (1). Here and in the following we suppress the explicit time dependency of $u(x,t)$ for simplicity.

The expansion coefficients are obtained directly as

$$\hat{u}_n = \frac{1}{2\pi}(u, \exp(\mathrm{i}nx))_{L^2[0,2\pi]} = \frac{1}{2\pi}\int_0^{2\pi} u(x)\exp(-\mathrm{i}nx)\,\mathrm{d}x, \tag{6}$$

through the orthogonality of the basis in the inner product

$$(f,g)_{L^2[0,2\pi]} = \int_0^{2\pi} f\bar{g}\,\mathrm{d}x, \quad \|f\|^2_{L^2[0,2\pi]} = \int_0^{2\pi} |f|^2\,\mathrm{d}x$$

with the associated norm $\|\cdot\|_{L^2[0,2\pi]}$.

The simplicity of the Fourier series makes it straightforward to approximate the spatial derivative

$$\frac{\mathrm{d}^p u(x)}{\mathrm{d}x^p} \simeq \frac{\mathrm{d}^p \mathscr{P}_N u(x)}{\mathrm{d}x^p} = \sum_{n=-N}^{N} (\mathrm{i}n)^p \hat{u}_n \phi_n(x) = \sum_{n=-N}^{N} \hat{u}_n^{(p)} \phi_n(x),$$

i.e., $\hat{u}_n^{(p)} = (\mathrm{i}n)^p \hat{u}_n$, for the approximation of an arbitrary derivative of a function given by its Fourier coefficients.

The computation of the Fourier coefficients, $\hat{u}_n$, poses a problem as one cannot, in general, evaluate the integrals in Eq. (6). The natural solution is to introduce a quadrature approximation to Eq. (6) of the form

$$\tilde{u}_n = \frac{1}{2N\tilde{c}_n} \sum_{j=0}^{2N-1} u(x_j)\exp(-\mathrm{i}nx_j), \tag{7}$$

where $\tilde{c}_N = \tilde{c}_{-N} = 2$ and $\tilde{c}_n = 1$ otherwise. We recognize this as the trapezoidal rule with the equidistant grid

$$x_j = \frac{2\pi}{2N} j, \quad j \in [0, 2N-1]$$

As N increases one hopes that $\tilde{u}_n$ is a good representation of $\hat{u}_n$. To quantify this, we can express $\tilde{u}_n$ using $\hat{u}_n$ as

$$\tilde{c}_n \tilde{u}_n = \hat{u}_n + \sum_{\substack{m=-\infty \\ m\neq 0}}^{m=\infty} \hat{u}_{n+2Nm},$$

where the second term is termed the aliasing error. In particular, if $u(x)$ is bandlimited such that $\hat{u}_{n+2Nm} = 0$ for $|m| > 0$, Eq. (7) is exact.

While the use of trigonometric polynomials is natural for the approximation of periodic problems, an alternative has to be sought when nonperiodic problems are being considered.

A natural basis for the approximation of functions on a finite interval, normalized for convenience to $x \in [-1, 1]$, employs the Chebyshev polynomials, $T_n(x)$, and an approximating expansion of the form

$$\mathscr{P}_N u(x) = \sum_{n=0}^{N} \hat{u}_n \cos(n \arccos x) = \sum_{n=0}^{N} \hat{u}_n T_n(x), \tag{8}$$

i.e., $\phi_n(x) = T_n(x) = \cos(n \arccos x)$ in Eq. (1). The continuous expansion coefficients, $\hat{u}_n$, are found by exploiting the weighted L^2-orthogonality of $T_n(x)$ in the inner product

$$(f, g)_{L^2_w[-1,1]} = \int_{-1}^{1} fg \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x, \quad \|f\|^2_{L^2_w[-1,1]} = \int_{-1}^{1} |f|^2 \frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x$$

with the associated norm, $\|\cdot\|_{L^2_w[-1,1]}$. With this, one immediately recovers

$$\hat{u}_n = \frac{2}{\pi c_n} (u, T_n)_{L^2_w[-1,1]}, \tag{9}$$

where $c_0 = 2$ and $c_n = 1$ otherwise.

Given the series for $u(x)$ in Eq. (8) we need, as for the Fourier series, to recover an approximation to the spatial derivative of $u(x)$. This involves the expression of the derivative of the basis in terms of the basis itself. Utilizing the recursion

$$T_n(x) = -\frac{1}{2(n-1)} T'_{n-1}(x) + \frac{1}{2(n+1)} T'_{n+1}(x),$$

we recover

$$T'_n(x) = 2n \sum_{\substack{p=0 \\ p+n \text{ odd}}}^{n-1} \frac{T_p(x)}{c_p}.$$

Hence, the expansion coefficients for the spatial derivative is recovered by matching terms in Eq. (2) to obtain

$$c_n \hat{u}'_n = 2 \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{N} p\hat{u}_p.$$

Similar types of expressions can be derived to express higher derivatives. Contrary to the Fourier series, however, the computation of $\hat{u}'_n$ involves global spectral information which makes the straightforward formulation computationally inefficient. The resolution lies in realizing that for all finite expansions the coefficients can be recovered through a backward recursion of the form

$$c_{n-1}\hat{u}'_{n-1} = 2n\hat{u}_n + \hat{u}'_{n+1},$$

keeping in mind that $\hat{u}'_{N+1} = \hat{u}'_N = 0$ due to the nature of the finite approximation. This reduces the computation of a derivative to a linear process.

As for the Fourier expansion the evaluation of the continuous inner product, Eq. (9), is a source of considerable problems. The classical solution lies in the introduction of a Gauss quadrature of the form

$$\tilde{u}_n = \frac{2}{\tilde{c}_n \pi} \sum_{j=0}^{N} u(x_j) T_n(x_j) w_j,$$

as an approximation to the inner product with $N+1$ being the number of grid points in the Gauss quadrature. Among several possible choices, the Chebyshev–Gauss–Lobatto quadrature with

$$x_j = -\cos\left(\frac{\pi}{N} j\right), \quad w_j = \frac{\pi}{\tilde{c}_j N}, \quad \tilde{c}_j = \begin{cases} 2, & j = 0, N, \\ 1, & j = 1, \ldots, N-1 \end{cases} \tag{10}$$

is the most popular as it includes the endpoints of the finite interval among the grid points. This is clearly an advantage if one needs to impose boundary conditions. With this approximation, which suffers from aliasing similar to the Fourier series, approximations to derivatives can be recovered as if the continuous expansion coefficients were being used.

2.2. Nodal methods and differentiation matrices

As the use of the modal expansions for all practical purposes requires the introduction of a finite grid one may question the need to consider special basis functions at all. Indeed, given a specific nodal set, x_j, we can construct a global interpolation

$$\mathcal{I}_N u(x) = \sum_{j=0}^{N} u(x_j) l_j(x),$$

where the Lagrange interpolating polynomials, $l_j(x)$, takes the form

$$l_j(x) = \frac{q(x)}{(x - x_j) q'(x_j)}, \quad q(x) = \prod_{j=0}^{N} (x - x_j).$$

Clearly, if the x_j's are distinct, $l_j(x)$ is uniquely determined as an Nth-order polynomial specified at $N+1$ points and we can approximate derivatives of $u(x)$ directly as in Eq. (4). In particular, if we restrict our attention to the approximation of the derivative of $u(x)$ at the grid points, x_j, we have

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x_i} \simeq \left.\frac{\mathrm{d}\mathscr{I}_N u}{\mathrm{d}x}\right|_{x_i} = \sum_{j=0}^{N} u(x_j) \left.\frac{\mathrm{d}l_j}{\mathrm{d}x}\right|_{x_i} = \sum_{j=0}^{N} u(x_j) D_{ij},$$

where D_{ij} is recognized as a differentiation matrix similar in spirit to a traditional finite difference approximation to spatial derivatives. One should keep in mind, however, that the global nature of the interpolation implies that the differentiation matrix is full.

The specification of the nodes uniquely defines the interpolation polynomials and, hence, the differentiation matrices. Indeed, if we simply take an equidistant grid for $x \in [0, 2\pi[$ of the form

$$x_j = \frac{2\pi}{2N} j, \quad j \in [0, 2N-1]$$

and assume that the interpolating polynomial itself it 2π-periodic we recover

$$\mathscr{I}_N u(x) = \sum_{j=0}^{2N-1} u(x_j) g_j(x) = \sum_{j=0}^{2N-1} u(x_j) \frac{1}{2N} \sin[N(x-x_j)] \cot\left[\frac{1}{2}(x-x_j)\right] \tag{11}$$

with the entries of the differentiation matrix being

$$D_{ij} = \left.\frac{\mathrm{d}g_j}{\mathrm{d}x}\right|_{x_i} = \begin{cases} \dfrac{1}{2}(-1)^{i+j} \cot\left[\dfrac{x_i - x_j}{2}\right], & i \neq j, \\ 0, & i = j. \end{cases} \tag{12}$$

Among other things, we see that the differentiation matrix is an anti-symmetric, circulant Toeplitz matrix. It is interesting to note that Eq. (12) essentially can be obtained as the limit of an infinite order central finite difference stencil under the assumption of periodicity [25].

Turning to the interpolation of nonperiodic functions on finite intervals it is well known [56,26] that one must abandon the equidistant grid to avoid the Runge-phenomenon and the divergence of the Lagrange polynomials and choose a grid that clusters quadratically as

$$x_j \sim -1 + c\left(\frac{j}{N}\right)^2,$$

close to the endpoints. Such nodal sets are plentiful, among them all the zeros of the classical orthogonal polynomials and their derivatives. Indeed, if we consider the set of nodes given as

$$x_j = -\cos\left(\frac{\pi}{N} j\right),$$

which are the roots of the polynomial, $(1-x^2)T'_N(x)$, and recognized as the Chebyshev–Gauss–Lobatto quadrature nodes, Eq. (10), we recover

$$\mathscr{I}_N u(x) = \sum_{j=0}^{N} u(x_j) h_j(x) = \sum_{j=0}^{N} u(x_j) \frac{(-1)^{N+1+j}(1-x^2)T'_N(x)}{\tilde{c}_j N^2 (x - x_j)}, \tag{13}$$

where $\tilde{c}_0 = \tilde{c}_N = 2$ and $\tilde{c}_j = 1$ otherwise. The corresponding differentiation matrix has the entries

$$D_{ij} = \left.\frac{\mathrm{d}h_j}{\mathrm{d}x}\right|_{x_i} = \begin{cases} -\dfrac{2N^2+1}{6}, & i = j = 0, \\ \dfrac{\tilde{c}_i}{\tilde{c}_j}\dfrac{(-1)^{i+j+N}}{x_i - x_j}, & i \neq j, \\ -\dfrac{x_i}{2(1-x_i^2)}, & 0 < i = j < N, \\ \dfrac{2N^2+1}{6}, & i = j = N. \end{cases} \tag{14}$$

It is easy to see that $D_{ij} = -D_{N-i,N-j}$, i.e., the differentiation matrix is centro-antisymmetric as a consequence of the reflection symmetry of the nodal set. Moreover, one can show that D is nilpotent.

2.3. The duality between modes and nodes

While there is a great deal of flexibility in the choice of the quadrature rules used to compute the discrete expansion coefficients in the modal expansions, and similar freedom in choosing a nodal set on which to base the Lagrange interpolation polynomials, particular choices are awarded by a deeper insight.

Consider, as an example, the modal expansion, Eq. (5), with the expansion coefficients approximated as in Eq. (7). Inserting the latter directly into the former yields

$$\begin{aligned} \mathscr{P}_N u(x) &= \sum_{n=-N}^{N} \left[\frac{1}{2N\tilde{c}_n} \sum_{j=0}^{2N-1} u(x_j)\exp(-\mathrm{i}nx_j) \right] \exp(\mathrm{i}nx) \\ &= \sum_{j=0}^{2N-1} u(x_j) \left[\frac{1}{2N} \sum_{n=-N}^{N} \frac{1}{\tilde{c}_n} \exp(\mathrm{i}n(x-x_j)) \right] \\ &= \sum_{j=0}^{2N-1} u(x_j) \frac{1}{2N} \sin[N(x-x_j)] \cot\left[\frac{1}{2}(x-x_j)\right], \end{aligned}$$

which we recognize as the periodic interpolation polynomial based on the equidistant grid, Eq. (11). In other words, we have that

$$\mathscr{I}_N u(x) = \sum_{n=-N}^{N} \tilde{u}_n \exp(\mathrm{i}nx) = \sum_{j=0}^{2N-1} u(x_j) g_j(x),$$

provided the expansion coefficients are approximated by the trapezoidal rule as in Eq. (7). This particular combination of grid points and quadrature rules results in two mathematically equivalent, but computationally very different, ways of expressing the interpolation and hence the computation of spatial derivatives.

In a similar fashion one can show that as a consequence of the Christoffel–Darboux identity for orthogonal polynomials [81] we have

$$\mathscr{I}_N u(x) = \sum_{n=0}^{N} \tilde{u}_n T_n(x) = \sum_{j=0}^{N} u(x_j) h_j(x)$$

with $h_j(x)$ being given in Eq. (13), provided only that the Chebyshev–Gauss–Lobatto quadrature is used to approximate the expansion coefficients, $\tilde{u}_n$, in the modal expansion, Eq. (8). Similar results can be obtained if one chooses a different Gauss quadrature and the corresponding nodes as the grid for the interpolation polynomials.

It is important to appreciate that one need not choose the special quadratures used here or the special nodal sets for the interpolation polynomials to obtain robust and stable spectral schemes. Doing so, however, provides a duality in the formulation that have major advantages in the analysis as well as in the practical implementation of methods for solving hyperbolic problems.

3. Approximation results

In attempting to understand the quality of the computed approximate solutions we need to consider the behavior of the finite-order expansions given in Eqs. (1) and (3) as N increases.

While we shall focus the attention on the behavior of the polynomial interpolation, Eq. (3), we shall also find it useful understand the purely modal expansion, Eq. (1), as the difference between the two is a measure of the aliasing error.

We have chosen to split the subsequent discussion of the approximation results into that of problems possessing a minimum amount of smoothness and the approximation of truly discontinuous functions. Many more results on the approximation of functions using spectral expansions can be found in [10,27,4,6] and references therein.

3.1. Smooth problems

Although the approximations based on Fourier and Chebyshev series are closely related it is instructive to split the discussion of the two as the latter provides an example of results for the much broader class of approximations based on orthogonal polynomials.

3.1.1. Fourier expansions

We begin by considering the truncated continuous Fourier expansion

$$\mathscr{P}_N u(x) = \sum_{n=-N}^{N} \hat{u}_n \exp(\mathrm{i}nx)\,\mathrm{d}x, \quad \hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)\exp(-\mathrm{i}nx)\,\mathrm{d}x \tag{15}$$

for which it is clear that

$$\|u - \mathscr{P}_N u\|^2_{L^2[0,2\pi]} = 2\pi \sum_{|n|>N} |\hat{u}_n|^2,$$

as a direct consequence of Parsevals identity for Fourier series. The truncation error depends solely on the decay of $\hat{u}_n$ which behaves as

$$|\hat{u}_n| = \frac{1}{2\pi n^q}\left|\int_0^{2\pi} u^{(q)}(x)\exp(-\mathrm{i}nx)\,\mathrm{d}x\right|,$$

provided $u(x)\in C^{(q-1)}[0,2\pi]$, i.e., $u^{(q)}\in L^2[0,2\pi]$ and $u(x)$ as well as its first $(q-1)$-derivatives are 2π-periodic. Hence, the truncation error is directly related to the smoothness of $u(x)$ as

$$\|u-\mathscr{P}_N u\|_{L^2[0,2\pi]} \leqslant C(q)N^{-q}\|u^{(q)}\|_{L^2[0,2\pi]}. \tag{16}$$

In the event that $u(x)$ is analytic one recovers the remarkable property that [83]

$$\begin{aligned}\|u-\mathscr{P}_N u\|_{L^2[0,2\pi]} &\leqslant C(q)N^{-q}\|u^{(q)}\|_{L^2[0,2\pi]}\\ &\sim C(q)\frac{q!}{N^q}\|u\|_{L^2[0,2\pi]} \sim C(q)e^{-cN}\|u\|_{L^2[0,2\pi]},\end{aligned}$$

known as spectral accuracy or spectral convergence. This is indeed the basic property that has given name to spectral methods.

The diagonality of the modal differentiation operator implies that truncation and differentiation commutes, i.e.,

$$\mathscr{P}_N\frac{\mathrm{d}^q u}{\mathrm{d}x^q} = \frac{\mathrm{d}^q}{\mathrm{d}x^q}\mathscr{P}_N u$$

for $u(x)\in C^{(q-1)}[0,2\pi]$, and by repeatedly applying Eq. (16) we obtain

$$\|u-\mathscr{P}_N u\|_{W^p[0,2\pi]} \leqslant C(p,q)N^{p-q}\|u\|_{W^q[0,2\pi]},$$

provided only that $0\leqslant p\leqslant q$. Here, we have introduced the Sobolev norm

$$\|u\|^2_{W^q[0,2\pi]} = \sum_{s=0}^{q}\|u^{(s)}\|^2_{L^2[0,2\pi]}$$

to measure the error on the spatial derivative. Clearly, as long as u is sufficiently smooth and periodic, the error decays rapidly for increasing number of terms, N, and we recover spectral convergence for all spatial derivatives if $u(x)\in C^\infty[0,2\pi]$.

These results address the mean convergence, while the pointwise convergence is a harder but often more useful measure. For the truncated Fourier series one recovers [10]

$$\|u-\mathscr{P}_N u\|_{L^\infty[0,2\pi]} \leqslant C(q)(1+\log N)N^{-q}\|u^{(q)}\|_{L^\infty[0,2\pi]},$$

where we have introduced the familiar $L^\infty[0,2\pi]$ norm to measure the maximum pointwise error. This result provides an early indication that we may experience problems if $u(x)$ is only piecewise smooth.

Before addressing this, however, let us consider the behavior of the discrete expansion, Eq. (3), as N increases. As discussed in Section 2.1, the continuous and the discrete expansion coefficients are related as

$$\tilde{c}_n\tilde{u}_n = \hat{u}_n + \sum_{\substack{m=-\infty\\ m\neq 0}}^{m=\infty}\hat{u}_{n+2Nm},$$

provided $u(x) \in W^{1/2}[0, 2\pi]$. The aliasing error, however, has been shown to be of the same order as the truncation error [66], obtained directly from Eq. (16). Thus, as for the continuous expansion, we recover

$$\|u - \mathscr{I}_N u\|_{L^2[0,2\pi]} \leqslant C(q) N^{-q} \|u^{(q)}\|_{L^2[0,2\pi]}$$

and exponential convergence in cases where $u(x) \in C^\infty[0, 2\pi]$ [83]. Numerical evidence for this was first given in [75,25].

Contrary to the continuous expansion, however, truncation and differentiation does not commute

$$\mathscr{I}_N \frac{\mathrm{d}u}{\mathrm{d}x} \neq \frac{\mathrm{d}}{\mathrm{d}x} \mathscr{I}_N u$$

as a consequence of the aliasing error. Nevertheless, if $u(x)$ is at least continuous, the difference is bounded as [76,83]

$$\left\| \mathscr{I}_N \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x} \mathscr{I}_N u \right\|_{L^2[0,2\pi]} \leqslant C(q) N^{1-q} \|u^{(q)}\|_{L^2[0,2\pi]}, \tag{17}$$

provided $u(x) \in W^q[0, 2\pi]$, $q > \frac{1}{2}$. This suggests that the estimate

$$\|u - \mathscr{I}_N u\|_{W^p[0,2\pi]} \leqslant C(p, q) N^{p-q} \|u\|_{W^q[0,2\pi]}$$

for $0 \leqslant p \leqslant q$ provides a good bound on the expected accuracy of the approximation of spatial derivatives using the discrete expansions.

An estimate of the pointwise error is made difficult by the influence of the aliasing error. A bound on the pointwise error is given as [57]

$$\|u - \mathscr{I}_N u\|_{L^\infty[0,2\pi]} \leqslant C(q) \log(N) N^{-q} \|u^{(q)}\|_{L^\infty[0,2\pi]},$$

provided $u(x) \in C^q[0, 2\pi]$, $q > 0$.

3.1.2. Chebyshev expansions

Turning to the truncated Chebyshev expansion we have

$$\mathscr{P}_N u(x) = \sum_{n=0}^{N} \hat{u}_n T_n(x), \quad \hat{u}_n = \frac{2}{\pi c_n} \int_{-1}^{1} u(x) T_n(x) \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x. \tag{18}$$

A direct consequence of Bessels equality for orthogonal expansions is

$$\|u - \mathscr{P}_N u\|^2_{L^2_w[-1,1]} = \frac{2}{\pi} \sum_{|n|>N} |\hat{u}_n|^2.$$

Hence, as for the Fourier series, the truncation error depends solely on the decay of $\hat{u}_n$ as

$$|\hat{u}_n| = \frac{2}{\pi c_n n^{2q}} \left| \int_{-1}^{-1} \left[\sqrt{1-x^2} \frac{\mathrm{d}}{\mathrm{d}x} \right]^{2q} u(x) T_n(x) \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \right|,$$

provided $u(x) \in C^{2q-1}[-1, 1]$. This implies that the truncation error is directly related to the smoothness of $u(x)$ as

$$\|u - \mathscr{P}_N u\|_{L^2_w[-1,1]} \leqslant C(q) N^{-q} \left\| \left[\sqrt{1-x^2} \frac{\mathrm{d}}{\mathrm{d}x} \right]^q u \right\|_{L^2_w[-1,1]} \leqslant C N^{-q} \|u\|_{W^q_w[-1,1]}, \tag{19}$$

where we, as for the Fourier series, have introduced the weighted Sobolev norm

$$\|u\|^2_{W^q_w[-1,1]} = \sum_{s=0}^{q} \|u^{(s)}\|^2_{L^2_w[-1,1]}$$

as a measure of the regularity of $u(x)$. As for the Fourier approximation, the truncation error decays faster than any algebraic order of N if $u(x) \in C^\infty[-1,1]$ [83]. This exponential convergence, however, is achieved with no conditions on $u(x)$ at the boundaries, emphasizing the usefulness of Chebyshev expansions for the approximation of problems defined on finite, nonperiodic domains.

Contrary to the Fourier series, however, even the continuous expansions do not permit commutation of truncation and differentiation without introducing an error as [12]

$$\left\| \mathscr{P}_N \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x} \mathscr{P}_N u \right\|_{L^2_w[-1,1]} \leqslant C(q) N^{3/2-q} \|u\|_{W^q_w[-1,1]}$$

for $q > 1$. Thus, the estimate

$$\|u - \mathscr{P}_N u\|_{W^p_w[-1,1]} \leqslant C(p,q) N^{2p-q-1/2} \|u\|_{W^q_w[-1,1]}$$

and $u(x) \in C^{(q-1)}[-1,1]$, $q \geqslant 1$, provides a bound on the accuracy we can expect for the Chebyshev approximation of spatial derivatives of smooth functions.

Let us finally consider the properties of the truncated discrete expansion, or interpolation, of the form

$$\mathscr{I}_N u(x) = \sum_{n=0}^{N} \tilde{u}_n T_n(x), \quad \tilde{u}_n = \frac{2}{\tilde{c}_n \pi} \sum_{j=0}^{N} u(x_j) T_n(x_j) w_j,$$

where the grid points, x_j, and the weights, w_j, of the Chebyshev–Gauss–Lobatto quadrature rule are given in Eq. (10). As for the Fourier interpolation, the grid introduces aliasing errors as reflected in the connection between the discrete and the continuous expansion coefficients

$$\tilde{c}_n \tilde{u}_n = \hat{u}_n + \sum_{\substack{m=-\infty \\ m \neq 0}}^{m=\infty} \hat{u}_{n+2Nm},$$

provided $u(x) \in W^{1/2}_w[-1,1]$, i.e., the aliasing term takes the exact same form as for the Fourier series. Recalling that the Chebyshev series can be written as a cosine series, this is only natural and one recovers a bound on the interpolation error [10,5]

$$\|u - \mathscr{I}_N u\|_{L^2_w[-1,1]} \leqslant C(q) N^{-q} \|u\|_{W^q_w[-1,1]}$$

for $q \geqslant 1$. Hence, the aliasing error is of the same order as the truncation error and the interpolation maintains exponential convergence if $u(x) \in C^\infty[-1,1]$.

Exploiting the close connection between the Fourier basis and the Chebyshev basis one can show that [10]

$$\left\| \mathscr{I}_N \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x} \mathscr{I}_N u \right\|_{L^2_w[-1,1]} \leqslant C(q) N^{2-q} \|u\|_{W^q_w[-1,1]}$$

for $q \geqslant 1$ being a special case of the general estimate [11]

$$\|u - \mathscr{I}_N u\|_{W_w^p[-1,1]} \leqslant C(p,q) N^{2p-q} \|u\|_{W_w^q[-1,1]}$$

for $0 \leqslant p \leqslant q$. This provides a bound on the accuracy we can expect for a discrete Chebyshev approximation of a spatial derivative.

The pointwise error is given as [11]

$$\|u - \mathscr{I}_N u\|_{L^\infty[-1,1]} \leqslant C(q) N^{1/2-q} \|u\|_{W_w^q[-1,1]}$$

when $q \geqslant 1$.

3.2. Nonsmooth problems

The main conclusion to draw from the previous section is that if the solution possesses significant regularity we can expect the spectral expansion to be highly efficient for the representation of the solution and its spatial derivatives. In other words, only relatively few terms are needed in the expansion to produce a very accurate approximation.

Considering problems with only limited regularity, however, the picture is considerably more complex and the results given above tell us little about the accuracy of the approximation of such solutions. In particular, if the solution is only piecewise smooth the results discussed so far ensure mean convergence only while the question of pointwise convergence remains open.

It is by now a classical result that the Fourier series, Eq. (15), in the neighborhood of a point of discontinuity, x_0, behaves as [39]

$$\mathscr{P}_N u\left(x_0 + \frac{2z}{2N+1}\right) \sim \frac{1}{2}[u(x_0^+) + u(x_0^-)] + \frac{1}{\pi}[u(x_0^+) - u(x_0^-)]\ \mathrm{Si}(z),$$

where z is a constant and $\mathrm{Si}(z)$ signifies the sine integral. Away from the point of discontinuity, x_0, we recover linear pointwise convergence as $\mathrm{Si}(z) \simeq \pi/2$ for z large. Close to the point of discontinuity, however, we observe that for any fixed value of z, pointwise convergence is lost regardless of the value of N. We recognize this nonuniform convergence and complete loss of pointwise convergence as the celebrated Gibbs phenomenon. Moreover, the oscillatory behavior of the sine integral is recognized as the familiar Gibbs oscillations which are high frequency and appear as being noise.

While the use of the Chebyshev expansion, Eq. (18), eliminates the Gibbs phenomenon at the boundaries of the domain, the problem remains in the interior of the domain on the form [39]

$$\mathscr{P}_N u\left(x_0 + \frac{2z}{\sqrt{1-x_0^2}(2N+1)}\right) \sim \frac{1}{2}[u(x_0^+) + u(x_0^-)] + \frac{1}{\pi}[u(x_0^+) - u(x_0^-)]\ \mathrm{Si}(z),$$

i.e., a nonuniform convergence of the expansion close to the point of discontinuity.

With the situation being even more complex and the details partially unresolved for the discrete expansions, the Gibbs phenomenon and the loss of fast global convergence is often perceived as an argument against the use of spectral expansions for the representation of piecewise smooth functions and, ultimately, against the use of spectral methods for the solution of problems with discontinuous solutions.

As we shall discuss in the following, however, recent results allow us to dramatically improve on this situation and even completely overcome the Gibbs oscillations to recover an exponential accurate approximation to a piecewise analytic function based on the information contained in the global expansion coefficients.

3.2.1. Enhanced convergence by the use of filters

One manifestation of the slow and nonuniform convergence of $\mathscr{I}_N u$ for a piecewise smooth functions is found in the linear decay of the global expansion coefficients, $\tilde{u}_n$. This realization also suggests that one could attempt to modify the global expansion coefficients to enhance the convergence rate of the spectral approximation. The critical question to address naturally is exactly how one should modify the expansion to ensure enhanced convergence to the correct solution.

Let us consider the filtered approximation, $\mathscr{F}_N u_N(x)$, of the form

$$\mathscr{F}_N u_N(x) = \sum_{n=-N}^{N} \sigma\left(\frac{n}{N}\right) \tilde{u}_n \exp(\mathrm{i}nx), \tag{20}$$

where $\tilde{u}_n$ signifies the discrete expansion coefficients of $u_N(x,t)$ and $\sigma(\eta)$ is a real filter function with the following properties [88]:

$$\sigma(\eta) = \begin{cases} \sigma(-\eta), \\ \sigma(0) = 1, \\ \sigma^{(q)}(0) = 0, \quad 1 \leqslant q \leqslant 2p-1, \\ \sigma(\eta) = 0, \qquad |\eta| \geqslant 1. \end{cases} \tag{21}$$

If $\sigma(\eta)$ has at least $2p-1$ continuous derivatives, $\sigma(\eta)$ is termed a filter of order $2p$.

As the filter is nothing more than a lowpass filter, it is not surprising that the filtered function converges faster than the unfiltered filtered original expansion. To understand exactly how the filtering modifies the convergence rate, let us assume that $u(x)$ is piecewise $C^{2p}[0,2\pi]$ with one discontinuity located at $x = \xi$. Let us furthermore assume that the filter is of order $2p$. Then the pointwise error of the filtered approximation is given as [88,42]

$$|u(x) - \mathscr{F}_N u_N(x)| \leqslant C \frac{1}{N^{2p-1} d(x,\xi)^{2p-1}} K(u) + C \frac{\sqrt{N}}{N^{2p}} \|u^{(2p)}\|_{L^2_B[0,2\pi]},$$

where $d(x,\xi)$ measures the distance from x to the point of discontinuity, ξ, $K(u)$ is uniformly bounded away from the discontinuity and a function of $u(x)$ only. Also, $\|\cdot\|_{L^2_B[0,2\pi]}$ signifies the broken $L^2[0,2\pi]$-norm.

While the details of the proof of this result are quite technical and can be found in [88,42], the interpretation of the result is simple, and perhaps somewhat surprising. It states that the convergence rate of the filtered approximation is determined solely by the order, $2p$, of the filter, $\sigma(\eta)$, and the regularity of the function, $u(x)$, away from the point of discontinuity. In particular, if the function, $u(x)$, is piecewise analytic and the order of the filter increases with N, one recovers an exponentially accurate approximation to the unfiltered function everywhere except very close to the discontinuity [88,42].

The actual choice of the filter function, $\sigma(\eta)$, is one of great variety and numerous alternatives are discussed in [10,42]. A particularly popular one is the exponential filter [74]

$$\sigma(\eta) = \exp(-\alpha\eta^{2p}),$$

which satisfies all the conditions in Eq. (21) except that of being zero for $|\eta| \geqslant 1$. However, by choosing $\alpha = -\ln \varepsilon_M$, with ε_M representing the machine accuracy, $\sigma(\eta)$ vanishes for all practical purposes when $|\eta|$ exceeds one and the exponential filter allows for the recovery of a piecewise exponentially accurate representation of a piecewise analytic function away from the point of discontinuity.

When applying the filter it is worth noticing that, as for differentiation, the spectral filtering, Eq. (20), has a dual formulation, expressed as a matrix operation, of the form

$$\mathscr{F}_N u_N(x_i) = \sum_{j=0}^{2N-1} F_{ij} u_N(x_j), \quad F_{ij} = \frac{1}{N} \sum_{n=0}^{N} \frac{1}{c_n} \sigma\left(\frac{n}{N}\right) \cos(n(x_i - x_j)),$$

where $c_0 = c_N = 2$ and $c_n = 1$ otherwise.

The use of a filter in a Chebyshev collocation method is similar to that of the Fourier approximation, i.e., it can be expressed as

$$\mathscr{F}_N u_N(x) = \sum_{n=0}^{N} \sigma\left(\frac{n}{N}\right) \tilde{u}_n T_n(x),$$

or on its dual form

$$\mathscr{F}_N u_N(x_i) = \sum_{j=0}^{N} F_{ij} u_N(x_j), \quad F_{ij} = \frac{2}{N c_j} \sum_{n=0}^{N} \frac{1}{c_n} \sigma\left(\frac{n}{N}\right) T_n(x_i) T_n(x_j).$$

While the use of spectral filtering as discussed in the above remains the most popular way of enhancing the convergence rate, an alternative approach can be realized by observing that the Gibbs phenomenon is also a manifestation of the global nature of the interpolating polynomial. In other words, one could attempt to improve on the quality of the approximation by localizing the approximation close to the point of the discontinuity.

This approach, known as physical space filtering, operates directly on the interpolating polynomials rather than the expansion coefficients. To illustrate the basic idea consider the filtered Fourier approximation

$$\mathscr{F}_N u_N(x) = \sum_{j=0}^{2N-1} u(x_j) l_j^{\sigma}(x),$$

where x_j are the usual equidistant grid and the filtered Lagrange interpolation polynomial takes the form

$$l_j^{\sigma}(x) = \frac{1}{2N} \sum_{n=-N}^{N} \sigma\left(\frac{n}{N}\right) \frac{1}{\tilde{c}_n} \exp(\mathrm{i}n(x - x_j)).$$

Clearly, if $\sigma(\eta) = 1$ we recover the Fourier interpolation polynomial given in Eq. (11). However, we do not need to maintain the close connection to the trigonometric interpolation polynomials, as expressed in $l_j^{\sigma}(x)$, but can choose to use any reasonable kernel, $\psi(x, x_j)$, that approximates a delta function as $x - x_j$ approaches zero. Following [44,42] we can exemplify this idea by considering

$$\psi(x, x_j) = \rho(\xi(x, x_j)) \frac{1}{2\varepsilon} \sum_{n=-N}^{N} \exp(\mathrm{i}n\xi(x, x_j)),$$

where

$$\xi(x,x_j)=\frac{x_j-x}{\varepsilon}.$$

We assume that $u(x)\in C^{2p}[x_j-\varepsilon,x_j+\varepsilon]$ and $\rho(\xi(x,x_j))$ controls the amounts of localization as

$$\rho(0)=1,\quad \rho(\xi(x,x_j))=0 \quad \text{for } |\xi(x,x_j)|\geqslant 1,$$

i.e., the kernel vanished outside of the symmetric interval $[x_j-\varepsilon,x_j+\varepsilon]$. Note also that $(1/2\varepsilon)\sum_{n=-N}^{N}\exp(\mathrm{i}n\xi(x,x_j))$ is an approximation to a x_j-centered delta function.

In this setting it can be shown [44,42] that the order of the filter and the regularity of the function away from the point of discontinuity solely determines the convergence rate of the filtered approximation. Exponential convergence can be recovered everywhere except very close to the point of discontinuity as measured through ε. The need to specify the size, ε, of the symmetric interval remains a practical obstacle to the use of the physical space filters.

While the use of filters can have a dramatic impact on the quality of the convergence of a global approximation to a discontinuous function, such techniques are unable to improve on the quality of the approximation as one approaches the point of discontinuity. Moreover, filtering generally treats the Gibbs oscillations as a source of noise and attempts to remove it. However, as has been speculated in the past [67], and recently shown rigorously, the Gibbs oscillations are not noise but rather contain sufficient information to reconstruct an exponentially convergent approximation everywhere provided only that the location of the discontinuity is known, i.e., the Gibbs phenomenon can be overcome completely.

3.2.2. Resolving the Gibbs phenomenon

In the following, we outline the key elements of a general theory that establishes the possibility of recovering a piecewise exponentially convergent series to a piecewise analytic function, $u(x)\in L_w^2[-1,1]$, having knowledge of the global expansion coefficients and the position of the discontinuities only.

The basic element of this new approach is the identification of a new basis with very special properties and, subsequently, the expansion of the slowly convergent truncated global expansion in this new basis. Provided this new basis satisfies certain conditions, the new expansion has the remarkable property that it is exponentially convergent to the original piecewise analytic function even though it uses information from the slowly convergent global expansion.

As previously we assume that

$$\mathscr{P}_N u(x)=\sum_{n=0}^{N}\hat{u}_n\phi_n(x),\quad \hat{u}_n=\frac{1}{\gamma_n}(u,\phi_n)_w$$

with $\gamma_n=\|\phi_n\|_w^2$. Note in particular that this also contains the Fourier case provided

$$\phi_n(x)=\exp\left[\mathrm{i}\left(n-\frac{N}{2}\right)\pi x\right].$$

Let us also assume that there exists an interval $[a,b]\subset[-1,1]$ in which $u(x)$ is analytic and, furthermore, that the original truncated expansion is pointwise convergent in all of $[-1,1]$ with the

exception of a finite number of points. We shall introduce the scaled variable

$$\xi(x) = -1 + 2\frac{x-a}{b-a}.$$

Clearly, $\xi : [a,b] \to [-1,1]$.

We define a new basis, $\psi_n^\lambda(\xi)$, which is orthogonal in the weighted inner product, $(\cdot,\cdot)_w^\lambda$ where λ signifies that the weight, $w(x)$, may depend on λ, i.e.,

$$(\psi_k^\lambda, \psi_n^\lambda)_w^\lambda = \|\psi_n^\lambda\|^2_{L^2_w[-1,1]}\delta_{kn} = \gamma_n^\lambda \delta_{kn}.$$

Furthermore, we require that if $v(\xi)$ is analytic then

$$\mathscr{P}_\lambda v(\xi) = \sum_{n=0}^{\lambda} \frac{1}{\gamma_n^\lambda}(v, \psi_n^\lambda)_w^\lambda \psi_n^\lambda(\xi)$$

is pointwise exponentially convergent as λ increases, i.e.,

$$\|v - \mathscr{P}_\lambda v\|_{L^\infty[-1,1]} \leqslant C\mathrm{e}^{-c\lambda}$$

with $c > 0$. This is similar to the case of classical expansions discussed in Section 3.1.

A final condition, however, sets this basis apart and is central in order to overcome the Gibbs phenomenon. We shall require that there exists a number $\beta < 1$, such that for $\lambda = \beta N$ we have

$$\left|\frac{1}{\gamma_n^\lambda}(\phi_k(x(\xi)), \psi_n^\lambda(\xi))_w^\lambda\right| \|\psi_n^\lambda\|_{L^\infty[-1,1]} \leqslant \left(\frac{\alpha N}{k}\right)^\lambda \tag{22}$$

for $k > N$, $n \leqslant \lambda$ and $\alpha < 1$. The interpretation of this condition is that the projection of the high modes of ϕ_k onto the basis, ψ_n^λ, is exponentially small in the interval, $\xi \in [-1,1]$. In other words, by reexpanding the slowly decaying ϕ_k-based global expansion in the local ψ_n^λ-basis an exponentially accurate local approximation is recovered. We shall term this latter, and crucial, condition on ψ_n^λ the Gibbs condition to emphasize its close connection to the resolution of the Gibbs phenomenon.

Provided only that the ψ_n^λ-basis, which we term the Gibbs complementary basis, is complete we recover the key result

$$\left\| u(x) - \sum_{n=0}^{\lambda} \frac{1}{\gamma_n^\lambda}(\mathscr{P}_N u, \psi_n^\lambda)_w^\lambda \psi_n^\lambda(\xi(x)) \right\|_{L^\infty[a,b]} \leqslant C\exp(-cN),$$

where $\lambda = \beta N$ and $u(x)$ is analytic in the interval $[a,b]$.

In other words, if a Gibbs complementary basis exists it is possible to reconstruct a piecewise exponentially convergent approximation to a piecewise analytic function from the information contained in the original very slowly converging global approximation using only knowledge about the location of the points of discontinuity. Hence, the impact of the Gibbs phenomenon can be overcome.

A constructive approach to the identification of the complementary basis is currently unknown. The existence of such a basis, however, has been established by carefully examining the properties of the basis

$$\psi_n^\lambda(\xi) = C_n^\lambda(\xi),$$

where $C_n^\lambda(\xi)$ represent the Gegenbauer polynomials, also known as the symmetric Jacobi polynomials or ultraspherical polynomials [81]. It is well known that the polynomials are orthogonal in the inner product

$$(f,g)_w^\lambda = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\lambda-1/2}\,\mathrm{d}x$$

and that

$$\gamma_n^\lambda = (\psi_n^\lambda, \psi_n^\lambda)_w^\lambda = \sqrt{\pi}\frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}\frac{\Gamma(\lambda+\frac{1}{2})}{(n+\lambda)\Gamma(\lambda)}.$$

The spectral convergence of Gegenbauer expansions of analytic functions is natural as the polynomials appears as eigensolutions to a singular Sturm–Liouville problem. Hence, to establish that the Gegenbauer polynomials provide an example of a Gibbs complementary basis we need to verify the Gibbs condition, Eq. (22).

If we consider the Fourier basis

$$\phi_n(x) = \exp(\mathrm{i}n\pi x),$$

it must be established that

$$\left|\frac{1}{\gamma_n^\lambda}(\phi_k, \psi_n^\lambda)_w^\lambda\right| \leqslant \left(\frac{\alpha N}{k}\right)^\lambda$$

for $k > N$, $0 < \alpha < 1$, and $n \leqslant \beta N = \lambda$.

For the Fourier basis the inner product allows an exact evaluation

$$\frac{1}{\gamma_n^\lambda}(\phi_k, \psi_n^\lambda)_w^\lambda = \mathrm{i}^n\Gamma(\lambda)\left(\frac{2}{\pi k\varepsilon}\right)^\lambda (n+\lambda)J_{n+\lambda}(\pi\varepsilon k)$$

with $J_\nu(x)$ being the Bessel function and $\varepsilon = b - a$ measures the width of the interval. Using the properties of the Bessel function and the Stirling formula for the asymptotic of the Γ-function, the Gibbs condition is satisfied if [42]

$$\beta = \frac{2\pi\varepsilon}{27}.$$

This establishes the existence of a Gibbs complementary basis to the Fourier basis.

For the Chebyshev case with

$$\phi_n(x) = T_n(x),$$

it can again be shown that the Gegenbauer polynomials provides an example of a Gibbs complementary basis although the proof is considerably more complicated as the means by which the inner product is bounded is recursive [42,43]. Nevertheless, the existence of a Gibbs complementary basis for the Chebyshev basis has been established. This paves the way for postprocessing of global approximations of piecewise smooth problems to recover a piecewise exponentially accurate approximation. Whether the Gegenbauer basis is the optimal choice as the Gibbs complementary basis for the Fourier and Chebyshev basis remains an open questions.

We have focused on the resolution of the Gibbs phenomenon for continuous spectral expansions which, unfortunately, have little practical importance. A similar discussion and analysis for the discrete expansion is complicated by the introduction of the aliasing error.

If one constructs the special interpolating polynomial

$$v(\xi) = \mathcal{I}_N[(1-\xi^2)^{\lambda-1/2}u(x(\xi))],$$

the proof that the Gegenbauer polynomials remain a Gibbs complementary basis for the Fourier and Chebyshev polynomials has be completed [41]. However, experience shows that the straightforward approach in which one considers the expansion

$$\mathcal{I}_\lambda v(\xi) = \sum_{n=0}^{\lambda} \tilde{v}_n^\lambda C_n^\lambda(\xi)$$

with the discrete expansion coefficients being approximated by a Gaussian quadrature rule as

$$\tilde{v}_n^\lambda = \frac{1}{\tilde{\gamma}_n^\lambda}\sum_{j=0}^{\lambda} v(\xi_j)w_j C_n^\lambda(\xi_j) = \frac{1}{\tilde{\gamma}_n^\lambda}\sum_{j=0}^{\lambda} \mathcal{I}_N u(x(\xi_j))w_i C_n^\lambda(\xi_j),$$

works well although a proof for this remains unknown.

The reconstruction of piecewise smooth solutions to conservation laws as a postprocessing technique has been exploited in [18,20,31]. Other applications of the reconstruction technique can be found in [89,90] and a two-dimensional version is discussed in [30].

It is worth mentioning in passing that alternatives to the identification of the Gibbs complementary basis for the reconstruction of piecewise analytic functions are known. These techniques all exploit the idea, originally proposed in [36], that by knowing or computing the location and size of the discontinuities, one can subtract these to recover a function with enhanced regularity. This was originally used as a postprocessing technique only [36] but later used as part of a time-dependent solution [8]. A high-order version of this approach, accounting also for discontinuities in the derivatives of the solution, has recently been developed [23,3] and tested on linear hyperbolic problems [22].

4. Collocation approximations of hyperbolic problems

Let us now turn the attention towards the actual solution of hyperbolic problems. Prominent examples of such problems include Maxwells equations from electromagnetics, the Euler equations from gas dynamics and the equations of elasticity. However, for the sake of simplicity we shall concentrate on methods for the scalar conservation law of the type

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

subject to appropriate boundary and initial conditions. As a prominent special case we shall devote much attention to the variable coefficient linear wave problem

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0, \tag{23}$$

where $a > 0$ implies a rightward propagating wave and $a < 0$ corresponds to a leftward propagating wave.

4.1. Fourier methods

Restricting the attention to problems of a purely periodic character, i.e.,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{24}$$
$$u(0,t) = u(2\pi,t),$$
$$u(x,0) = g(t),$$

it is only natural to seek numerical solutions, $u_N(x,t)$, expressed in terms of the Fourier series. If we require that the solution, $u_N(x,t)$, satisfies Eq. (24) in a collocation sense, $u_N(x,t)$ is a trigonometric polynomial that satisfies the equation

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + \left.\frac{\partial \mathscr{I}_N f(u_N)}{\partial x}\right|_{x_j} = 0$$

at the grid points, x_j. It is worth while emphasizing that in solving the conservation law, we encounter three types of solutions. The exact solution, $u(x,t)$, will generally not be available. However, when solving the partial differential equation we conjecture that the computable numerical solution, $u_N(x,t)$, is very close to the interpolation of the exact solution, $\mathscr{I}_N u(x,t)$. Due to aliasing errors and effects of nonlinearities the two solutions are generally not equivalent although for well resolved problems it is a reasonable assumption. A complete analysis of these aspects involves the derivation of the error equation which is a complex task, even for simple equations.

Consider the simple wave equation, Eq. (23), for which the Fourier collocation scheme is given as

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + a(x_j)\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = 0,$$

where we have left the derivative on symbolic form to emphasize the two mathematically equivalent, but computationally different ways of computing this operation.

In the same simple fashion, the Fourier collocation approximation to Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

is obtained by seeking the approximate solution, $u_N(x,t)$, such that

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + \left.\frac{1}{2}\frac{\partial}{\partial x}\mathscr{I}_N u_N^2\right|_{x_j} = 0. \tag{25}$$

Note that while the partial differential equation has the equivalent formulation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

the corresponding nonconservative Fourier approximation

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + u_N(x_j)\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = 0$$

is not equivalent to Eq. (25) and will in general yield a different results due to the aliasing errors and the mixing of these through the nonlinear term.

4.2. Chebyshev methods

Let us now consider the more general initial–boundary value problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{26}$$
$$u(x,0) = g(t)$$

posed on a finite domain which we take to be $[-1,1]$ without loss of generality. For the problem to be wellposed, we must specify boundary conditions of the form

$$\alpha u(-1,t) = f^-(t), \quad \beta u(1,t) = f^+(t),$$

where the specification of α and β is closely related to the flux function, e.g., if

$$x\frac{\partial f}{\partial u} < 0,$$

at the boundary, information is incoming and a boundary condition must be given. For a system of equations, the equivalent condition is posed through the characteristic variables, i.e., characteristic waves entering the computational domain must be specified and, hence, require a boundary condition to ensure wellposedness of the problem.

What separates the polynomial collocation approximation from the trigonometric schemes discussed in Section 4.1 is the need to impose the boundary conditions in such a way that we restrict the numerical solutions, $u_N(x,t)$, to those obeying the boundary conditions. The details of how this is done leads to different schemes.

4.2.1. Strongly imposed boundary conditions

In the classic approach one requires that the boundary conditions are imposed strongly, i.e., exactly. Hence, we shall seek a polynomial, $u_N(x,t)$, that satisfies Eq. (26) in a collocation sense at all the interior grid points, x_j, as

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + \left.\frac{\partial \mathscr{I}_N f(u_N)}{\partial x}\right|_{x_j} = 0,$$

while the boundary conditions are imposed exactly

$$\alpha u_N(-1,t) = f^-(t), \quad \beta u_N(1,t) = f^+(t).$$

If we again consider the wave equation, Eq. (23), the Chebyshev collocation scheme becomes

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + a(x_j)\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = 0$$

at all interior grid points, i.e., for $a > 0$, $j \in [1,N]$, while $u_N(x_0,t) = f^-(t)$.

In the same spirit the conservative Chebyshev collocation approximation to Burgers equation becomes

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + \frac{1}{2}\left.\frac{\partial}{\partial x}\mathscr{I}_N u_N^2\right|_{x_j} = 0,$$

which, if subjected to pure Dirichlet boundary conditions, are computed under the constraint

$$u_N(-1,t) = f^-(t), \quad u_N(1,t) = f^+(t).$$

4.2.2. Weakly imposed boundary conditions

The conceptual leap that leads one to consider other ways of imposing boundary conditions is the observation that it is sufficient to impose the boundary conditions to the order of the scheme, i.e., weakly, such that only in the limit of infinite order is the boundary condition is enforced exactly.

This simple idea, put forward in the context of spectral methods in [11] in a weak formulation and in [28,29] for the strong formulation considered here, has recently been developed further into a flexible and very general technique to impose boundary conditions in pseudospectral approximations to a variety of problems [19,52,13,49,53,50,55].

In this setting, one seeks a polynomial solution, $u_N(x,t)$, to Eq. (26) satisfying

$$\frac{\partial u_N}{\partial t} + \mathscr{I}_N \frac{\partial \mathscr{I}_N f(u_N)}{\partial x} = -\tau^- \alpha Q^-(x)[u_N(-1,t) - \tilde{f}^-(t)] + \tau^+ \beta Q^+(x)[u_N(1,t) - \tilde{f}^+(t)], \tag{27}$$

where we have introduced the polynomials, $Q^{\pm}(x) \in \mathsf{P}_N$, and the scalars, $\tau^{\pm}$.

To complete the scheme we must specify how the equation is to be satisfied which in most cases amounts to a choice between a Galerkin or a collocation approach. Moreover, we must choose $Q^{\pm}(x)$ and an approach by which to specify the scalar parameters, $\tau^{\pm}$. While the latter choice usually is dictated by requiring semi-discrete stability, the former choice of $Q^{\pm}(x)$ is associated with a great deal of freedom.

Before we discuss this in more detail, let us briefly introduce the discrete expansions based on Legendre polynomials, $P_n(x)$, as

$$\mathscr{I}_N u(x) = \sum_{n=0}^{N} \tilde{u}_n P_n(x), \quad \tilde{u}_n = \frac{1}{\tilde{\gamma}_n} \sum_{j=0}^{N} u(x_j) P_n(x_j) w_j,$$

where the quadrature to compute $\tilde{u}_n$ is based on the Legendre–Gauss–Lobatto points, i.e., the zeros of $(1-x^2)P_N'(x)$, or the Gauss points, i.e., the zeros of $P_{N+1}(x)$. The weights, w_i, of the quadrature naturally depend on the choice of the grid points [27]. We recall that the summation is exact provided $f(x)$ is a polynomial of degree at most $2N+1$ if the Gauss quadrature is used and exact for a polynomial of order at most $2N-1$ if the Gauss–Lobatto quadrature is used.

Consider the approximation to the constant coefficient wave equation, Eq. (23),

$$\frac{\partial u_N}{\partial t} + a\frac{\partial u_N}{\partial x} = -\tau^- a Q^-(x)[u_N(-1,t) - f(t)],$$

where $u_N(x,t)$ is based on the Legendre–Gauss–Lobatto points. A viable choice of $Q^-(x)$ is

$$Q^-(x) = \frac{(1-x)P_N'(x)}{2P_N'(-1)} = \begin{cases} 1, & x = -1, \\ 0, & x = x_j \neq -1, \end{cases}$$

where x_j refers to the Legendre–Gauss–Lobatto points. By requesting that the equation be satisfied in a collocation sense, we recover the scheme

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + a\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = -a\frac{N(N+1)}{4}\frac{(1-x_j)P_N'(x_j)}{2P_N'(-1)}[u_N(-1,t) - f(t)],$$

which we shall show in Section 5.2 to be asymptotically stable. Although the boundary condition is imposed only weakly, the approximation is clearly consistent, i.e., if $u_N(x,t) = u(x,t)$ the penalty term vanishes identically.

To illustrate the flexibility of the weakly imposed boundary conditions, let us again consider

$$\frac{\partial u_N}{\partial t} + a\frac{\partial u_N}{\partial x} = -\tau^- aQ^-(x)[u_N(-1,t) - f(t)],$$

where $Q^-(x)$ is as above, but $u_N(x,t)$ is based on the Chebyshev–Gauss–Lobatto grid, Eq. (10). In this case, $Q^-(x)$ is different from zero at all the interior Chebyshev–Gauss–Lobatto grid points and the boundary term reflects a global correction. Nevertheless, if we require that the wave equation be satisfied at the Legendre–Gauss–Lobatto nodes we recover the scheme

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + a\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = -a\frac{N(N+1)}{4}\frac{(1-x_j)P'_N(x_j)}{2P'_N(-1)}[u_N(-1,t) - f(t)],$$

where x_j are the Chebyshev–Gauss–Lobatto nodes. In other words, we have constructed an asymptotically stable Legendre–Gauss–Lobatto collocation method using the Chebyshev–Gauss–Lobatto grid for the approximation. This method, known as the Chebyshev–Legendre method [19], provides an example of a scheme where the equation is satisfied at points different from those on which the approximation is based. This example also shows that there is nothing special about the quadrature points in terms of stability. Indeed, we can construct a stable scheme on any set of grid points [13]. In terms of accuracy, however, the use of very special families of grid points is crucial.

As a final example, let us consider the general conservation law, Eq. (24), subject to boundary conditions at $x = \pm 1$, and assume that the polynomial solution, $u_N(x,t)$, is based on the Legendre–Gauss nodes, i.e.,

$$u_N(x,t) = \sum_{n=0}^{N} \tilde{u}_n P_n(x) = \sum_{j=0}^{N} u(x_j) l_j(x) = \sum_{j=0}^{N} u(x_j)\frac{P_{N+1}(x)}{(x-x_j)P'_{N+1}(x_j)},$$

where x_j signifies the Legendre–Gauss nodes.

We now request that Eq. (27) be satisfied in the following Galerkin-like way [53]:

$$\begin{aligned}\int_{-1}^{1}\left(\frac{\partial u_N}{\partial t} + \frac{\partial f_N}{\partial x}\right) l_i(x)\,\mathrm{d}x &= \oint_{-1}^{1} \tau(x) l_i(x)[f_N(x,t) - g(x,t)]\,\mathrm{d}s \\ &= -\tau^- l_i(-1)[f_N(-1,t) - g^-(t)] \\ &\quad +\tau^+ l_i(1)[f_N(1,t) - g^+(t)],\end{aligned}$$

where we have abused the notation a bit to make the multi-dimensional generalization more obvious. Here the boundary integral enforces the boundary conditions and we have introduced

$$f_N(x) = \mathscr{I}_N f(u_N) = \sum_{j=0}^{N} f_j l_j(x)$$

with $f_j = \mathscr{I}_N f(u_N(x_j))$. After integration by parts once, we recover

$$\begin{aligned}&\sum_{j=0}^{N} M_{ij}\left.\frac{\mathrm{d}u_N}{\mathrm{d}t}\right|_{x_j} - \sum_{j=0}^{N} S_{ij} f_j + f_N(1) l_i(1) - f_N(-1) l_i(-1) \\ &\quad = -\tau^- l_i(-1)[f_N(-1) - g^-(t)] + \tau^+ l_i(1)[f_N(1) - g^+(t)]\end{aligned}$$

with the mass-matrix, M, and the stiffness matrix, S, having the entries

$$M_{ij} = (l_i, l_j)_{L^2_w[-1,1]}, \quad S_{ij} = \left(\frac{\mathrm{d}l_i}{\mathrm{d}x}, l_j\right)_{L^2_w[-1,1]}$$

and the inner product is the usual unweighted inner product, i.e., $w(x) = 1$.

Exploiting the exactness of the Gauss quadrature and the fact that $l_i(x)$ are based on the Gauss-nodes we recover

$$M_{ij} = \begin{cases} w_i, & i = j, \\ 0, & i \neq j, \end{cases} \qquad S_{ij} = \left.\frac{\mathrm{d}l_i}{\mathrm{d}x}\right|_{x_j} w_j = D_{ji} w_j,$$

where w_j are the Gauss–Legendre quadrature weights [27] and D_{ji} represents, in the spirit of Section 2.2, the entries of the differentiation matrix based on the Gauss–Legendre grid point. This results in the collocation scheme

$$\left.\frac{\mathrm{d}u_N}{\mathrm{d}t}\right|_{x_i} - \sum_{j=0}^{N} D_{ji} f_j \frac{w_j}{w_i} + f_N(1)\frac{l_i(1)}{w_i} - f_N(-1)\frac{l_i(-1)}{w_i}$$

$$= -\tau^- \frac{l_i(-1)}{w_i}[f_N(-1) - g^-(t)] + \tau^+ \frac{l_i(1)}{w_i}[f_N(1) - g^+(t)] \tag{28}$$

at the grid points, x_i. Taking $\tau^\pm = 1$ we recover

$$\left.\frac{\mathrm{d}u_N}{\mathrm{d}t}\right|_{x_j} - \sum_{j=0}^{N} D_{ji} f_j \frac{w_j}{w_i} + g^+(t)\frac{l_i(1)}{w_i} - g^-(t)\frac{l_i(-1)}{w_i} = 0,$$

which one recognizes as the collocation form of the discontinuous Galerkin method [16,63], i.e., by taking $\tau^\pm$ to unity, one ensures that the scheme is conservative. Nevertheless, the discontinuous Galerkin method is only a special case of a much larger family of schemes with weakly imposed boundary conditions. The advantage in this realization lies in the flexibility of choosing $\tau^\pm$. In particular, if conservation is unnecessary, as for linear or smooth nonlinear problems, one can simply require asymptotic stability of Eq. (28) as [53]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2_{L^2_w[-1,1]} \leqslant 0 \rightarrow \tau^\pm \geqslant \frac{1}{2},$$

i.e., by sacrificing conservation we may lower $\tau^\pm$ while maintaining stability. As discussed in [52], a lower value of $\tau^\pm$ typically allows for increasing the discretely stable time-step when using explicit time-stepping.

5. Stability results for hyperbolic problems

To establish stability for the collocation schemes one traditionally either exploits the structure in the differentiation matrices, e.g., the Fourier differentiation matrices are antisymmetric, or utilizes the exactness of the quadrature rules to go from the semi-discrete formulation to the continuous formulation.

These techniques all lead to semi-discrete energy-estimates of the form

$$\|u_N(t)\| \leqslant K \mathrm{e}^{\alpha t}\|u_N(0)\|,$$

assuming homogeneous boundary conditions without loss of generality. Clearly, $\alpha \leqslant 0$ implies asymptotic stability.

5.1. Stability of the Fourier collocation method

Consider the variable coefficient linear wave problem, Eq. (23), subject to periodic boundary conditions for which the Fourier collocation approximation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} + AD\boldsymbol{u} = 0, \tag{29}$$

where $\boldsymbol{u} = [u_N(x_0), \ldots, u_N(x_{2N-1})]^{\mathrm{T}}$ represents the solution vector, D is the Fourier differentiation matrix, Eq. (12), and $A_{jj} = a(x_j)$ is diagonal.

Let us define the discrete inner product and L^2-equivalent norm as

$$[f, g]_N = \frac{\pi}{N} \sum_{j=0}^{2N-1} f(x_j) g(x_j), \quad \|f\|_N^2 = [f, f]_N.$$

If we initially assume that $|a(x)| > 0$ [75,65,25,39,76], it is easy to see that for $\boldsymbol{v} = A^{-1/2}\boldsymbol{u}$, we recover

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} + A^{1/2} D A^{1/2} \boldsymbol{v} = 0,$$

such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_N\|_N^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^{\mathrm{T}} A^{-1}\boldsymbol{u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}^{\mathrm{T}} H \boldsymbol{u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|_H^2 = 0,$$

since $A^{1/2} D A^{1/2}$ is antisymmetric. Here we have introduced the usual notation for the L^2-equivalent H-norm, $\|\cdot\|_H$ [25], also known as the elliptic norm.

For the general case where $a(x)$ changes sign within the computational domain, the situation is more complex. The straightforward way to guarantee stability is to consider the skew-symmetric form [65]

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial a(x) u}{\partial x} + \frac{1}{2} a(x)\frac{\partial u}{\partial x} - \frac{1}{2} a_x(x) u(x) = 0 \tag{30}$$

with the discrete form

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + \frac{1}{2}\left.\frac{\partial \mathscr{I}_N a(x) u_N}{\partial x}\right|_{x_j} + \frac{1}{2} a(x_j)\left.\frac{\partial u_N}{\partial x}\right|_{x_j} - \frac{1}{2} a_x(x_j) u_N(x_j) = 0.$$

Stability follows since

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|_N^2 \leqslant \frac{1}{2}\max_{x\in[0,2\pi]} |a_x(x)|\, \|u_N\|_N^2.$$

The disadvantage of the skew-symmetric formulation, however, clearly lies in a doubling of the computational work.

The question of stability of the simple formulation, Eq. (29), for general $a(x)$ has remained an open question until very recently, although partial results has been known for a while [40,86]. The difficulty in resolving this issue is associated with the development of very steep spatial gradients

which, for a fixed resolution, eventually introduce significant aliasing that affect the stability. A testament to this observation is the trivial stability of the Fourier–Galerkin approximation [39,84]. By carefully examining the interplay between aliasing, resolution, and stability, it has recently been shown [34] that the Fourier approximation is only algebraically stable [39], i.e.,

$$\|u_N(t)\|_N \leqslant C(t)N\|u_N(0)\|_N, \tag{31}$$

or weakly unstable. However, the weak instability spreads from the high modes through aliasing and results in at most an $\mathcal{O}(N)$ amplification of the Fourier components of the solution. In other words, for well-resolved computations where these aliasing components are very small the computation will appear stable for all practical purposes.

As an example of one of the few nonlinear cases for which stability can be established, recall Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0. \tag{32}$$

A stable approximation is obtained by considering the skew-symmetric form [82]

$$\frac{\partial u}{\partial t} + \frac{1}{3}\frac{\partial u^2}{\partial x} + \frac{1}{3}u\frac{\partial u}{\partial x} = 0,$$

from which stability of the collocation approximation

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + \frac{1}{3}\left.\frac{\partial}{\partial x}\mathcal{I}_N u_N^2\right|_{x_j} + \frac{1}{3}u_N(x_j)\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = 0$$

follows directly from the exactness of the quadrature.

5.2. Stability of Chebyshev collocation method

Establishing stability of the Chebyshev collocation approximations is considerably more challenging than for the Fourier collocation approximation. To expose the sources of this difficulty, let us consider the simple wave equation, Eq. (23), with $a(x) = 1$ and subject to the conditions

$$u(x,0) = g(x), \quad u(-1,t) = 0.$$

A Chebyshev collocation scheme based on the Gauss–Lobatto nodes yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = -\tilde{D}\boldsymbol{u}. \tag{33}$$

Here $\boldsymbol{u}(t) = [u_N(-1,t),\ldots,u_N(x_j,t),\ldots,u_N(1,t)]^{\mathrm{T}}$ represents the grid vector at the Gauss–Lobatto nodes, x_j, and the matrix $\tilde{D}$ represents the Chebyshev differentiation matrix, Eq. (14), modified to enforce the boundary condition strongly, i.e., by introducing zeros in the first row and column.

The strongly enforced boundary condition introduces the first main obstacle as the delicate structure of the differentiation matrix is destroyed by this modification, leaving us with the quadrature formula in trying to establish stability. The straightforward quadrature formula, however, is closely related to the weighted inner product, $(f,g)_{L^2_w[-1,1]}$, in which the Chebyshev polynomials are orthogonal. The norm associated with this inner product is, unfortunately, not uniformly equivalent to the usual L^2-norm [39]. This loss of equivalence eliminates the straightforward use of the quadrature rules

in the quest to establish stability as the corresponding norm is too weak. Thus, the two central techniques utilized for the Fourier methods are not directly applicable to the case of the Chebyshev collocation methods. It is worth mentioning that the situation for the Legendre approximations is considerably better as the Legendre polynomials are orthogonal in the unweighted $L^2[-1,1]$ inner product.

It seems natural to attempt to construct a new inner product and associated norm, uniformly equivalent to L^2, and subsequently establish stability in this norm. This is exactly the approach that was taken in [35,37,46] where Eq. (23) is considered and the following inner product was introduced [39]:

$$(f,g)_{L^2_{\tilde{w}}[-1,1]} = \int_{-1}^{1} f(x)g(x)\frac{1-x}{\sqrt{1-x^2}}\,\mathrm{d}x \tag{34}$$

with the associated quadrature rule

$$\int_{-1}^{1} f(x)\frac{1-x}{\sqrt{1-x^2}}\,\mathrm{d}x = \sum_{j=0}^{N-1} f(x_j)\tilde{w}_j,$$

which is exact for $f(x)\in\mathsf{P}_{2N-2}$. The weights, $\tilde{w}_j$, are given in [35,46] where it is also shown that $L^2_{\tilde{w}[-1,1]}$ is equivalent to $L^2_{w[-1,1]}$.

We note that the quadrature sum does not include the outflow boundary at $x=1$. To utilize the above quadrature we introduce the grid vector, $\boldsymbol{v}(t) = [u_N(-1,t),\ldots,u_N(x_j,t),\ldots,u_N(x_{N-1},t)]^{\mathrm{T}}$, containing the first N components of u_N. The evolution of $\boldsymbol{v}$ is simply described

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = -\hat{D}\boldsymbol{v}, \tag{35}$$

where $\hat{D}$ signifies $\tilde{D}$, Eq. (33), with the last row and column removed.

Stability of $\boldsymbol{v}$ in Eq. (35) follows from the exactness of the quadrature as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=0}^{N-1} v_N^2(x_j)\tilde{w}_j \leqslant 0.$$

The exact relation between this result and the stability of $\boldsymbol{u}$ in Eq. (33) is nontrivial and we refer to [35] where stability is established by directly relating $\boldsymbol{u}$ and $\boldsymbol{v}$.

The more general variable coefficient problem, Eq. (23), with $a(x)$ being smooth can be addressed using a similar approach. In particular, if $a(x)$ is smooth and uniformly bounded away from zero stability is established in the elliptic norm [35]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=0}^{N-1} v_N^2(x_j)\frac{\tilde{w}_j}{a(x_j)} \leqslant 0.$$

For the more general case of $a(x)$ changing sign the only known results are based on the skew-symmetric form [11], Eq. (30), although numerical experiments suggest that the straightforward Chebyshev collocation approximation of the wave equation with a variable coefficient behaves much as the Fourier approximation discussed above, i.e., if the solution is well resolved, the approximation is stable [39,35].

The extension of these results to a hyperbolic system of equations with constant coefficients is discussed in [37,38]. Stability of the scalar problem in combination with a dissipative boundary

operator is shown to be sufficient to guarantee algebraic stability of the Chebyshev approximation to a hyperbolic system of equations.

What we have discussed so far can be viewed as the traditional approach to stability, i.e., one formulates a meaningful approximation to the hyperbolic problem and subsequently attempts to establish stability of the scheme. As we have experienced this approach may well lead to very significant technical difficulties and it is worth while looking for an alternative approach.

Rather than first proposing an approximation and then attempting to establish stability it would seem natural to ensure stability as part of the construction. If we recall the idea of enforcing the boundary conditions only weakly, discussed in some detail in Section 4.2.2, we realize that this approach provides an example of just such a constructive approach to stability.

Let us again consider the general variable coefficient problem, Eq. (23), subject to appropriate boundary conditions. If we first assume that a is a constant, and recall the Chebyshev–Legendre approximation [19] discussed in Section 4.2.2 we have

$$\left.\frac{\partial u_N}{\partial t}\right|_{x_j} + a\left.\frac{\partial u_N}{\partial x}\right|_{x_j} = -a\tau\frac{(1-x_j)P_N'(x_j)}{2P_N'(-1)}[u_N(-1,t) - f(t)].$$

To establish stability, we exploit that both sides can be represented as an Nth-order polynomial specified at $N+1$ grid points, i.e., it is unique. We can thus read it at the Legendre–Gauss–Lobatto point, y_j, multiply from the left with $u_N(y_j)w_j$ with w_j being the weight associated with the Legendre–Gauss–Lobatto quadrature [27], and sum over all the nodes to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=0}^{N} u_N^2(y_j,t)w_j = -\frac{1}{2}a[u_N^2(1,t) - u_N^2(-1,t)] - \tau\omega_0 a u_N^2(-1,t),$$

from which we recover L^2 stability provided only that

$$\tau \geqslant \frac{1}{2\omega_0} = \frac{N(N+1)}{4}$$

as mentioned in Section 4.2.2. A direct proof of stability for the Chebyshev approximation using a Chebyshev penalty term is given in [17]. These results generalize directly to the case of a variable coefficient with a constant sign by introducing an elliptic norm and the general case can be addressed by writing the problem on skew-symmetric form, Eq. (30).

Let us finally consider the strictly hyperbolic system

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0,$$

where $U = [u_1(x,t),\ldots,u_M(x,t)]^{\mathrm{T}}$ represent the statevector and A is an $M\times M$ matrix which we without loss of generality take to be diagonal. If we split A into A^- and A^+, corresponding to the the negative and positive entries of A, respectively, the scheme is given as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U} + (A\otimes D)\boldsymbol{U} = &-\tau^-(A^+\otimes Q^-)(I_M\otimes I_N[\boldsymbol{U}(-1,t) - \boldsymbol{g}^-(t)])\\ &+\tau^+(A^-\otimes Q^+)(I_M\otimes I_N[\boldsymbol{U}(+1,t) - \boldsymbol{g}^+(t)]),\end{aligned}$$

where $\otimes$ signifies the Kronecker product, the grid vector $\boldsymbol{U}$ is ordered by the component as $\boldsymbol{U}=[u_1(-1,t),\ldots,u_1(1,t),u_2(-1,t),\ldots,u_M(1,t)]^{\mathrm{T}}$ and we have introduced the two diagonal matrices

$$Q^-_{jj}=\frac{(1-x_j)P'_N(x_j)}{2P'_N(-1)},\quad Q^+_{jj}=\frac{(1+x_j)P'_N(x_j)}{2P'_N(1)}, \tag{36}$$

while I_L signifies the order L identity matrix. The boundary conditions are represented by the vectors $\boldsymbol{g}^-(t)$ and $\boldsymbol{g}^+(t)$ accounting for the left and right boundaries, respectively.

The stability of this approximation follows directly by choosing $\tau^{\pm}$ as for the scalar case. This illustrates well how this approach lends it self to the formulation of stable spectral approximations of even very complex systems of equations [52,54,49,50].

For cases with variable coefficients, the situation is unchanged as long as the coefficients vary smoothly and the frozen coefficient problem is stable [64]. The same is true for the nonlinear case with smooth solutions. For problems with discontinuities, however, the general question of stability remains a significant challenge that we shall discuss further in the following section.

5.3. Stability of problems with nonsmooth initial conditions

Prior to that, however, let us briefly consider a situation where one solves Eq. (23) with discontinuous initial conditions. The question to raise is whether we can expect anything meaningful from such a solution due to the appearance of the Gibbs phenomenon and its potential impact on the time-dependent solution.

The simplest case of a constant coefficient problem clearly poses no problem as there is no means by which the aliasing errors can be redistributed. Let us therefore focus the attention on the linear variable coefficient problem [1]

$$\frac{\partial u}{\partial t}+\frac{1}{2}a(x)\frac{\partial u}{\partial x}+\frac{1}{2}\frac{\partial au}{\partial x}-\frac{1}{2}a_x(x)u=\frac{\partial u}{\partial t}+\mathscr{L}u=0. \tag{37}$$

Note that we have written $\mathscr{L}$ on skew-symmetric form to avoid the instabilities discussed above. We wish to solve this problem subject to periodic boundary conditions and with a discontinuous initial condition

$$u(x,t)=u_0(x).$$

Approximating the initial condition introduces the Gibbs phenomenon and the variable coefficient enables the mixing of the aliasing error with the solution itself. One could speculate that this process eventually could destroy the accuracy of the computed solution.

To understand this scenario let us introduce the dual problem

$$\frac{\partial v}{\partial t}+\frac{1}{2}a(x)\frac{\partial v}{\partial x}+\frac{1}{2}\frac{\partial av}{\partial x}-\frac{1}{2}a_x(x)v=\frac{\partial v}{\partial t}-\mathscr{L}^*v=0, \tag{38}$$

where $(\mathscr{L}u,v)_{L^2[0,2\pi]}=(u,\mathscr{L}^*v)_{L^2[0,2\pi]}$. Contrary to Eq. (37), we assume that Eq. (38) has the smooth initial condition

$$v(x,0)=v_0(x).$$

An immediate consequence of the structure of Eqs. (37)–(38) is that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,v)_{L^2[0,2\pi]}=(\mathscr{L}u,v)_{L^2[0,2\pi]}-(u,\mathscr{L}^*v)_{L^2[0,2\pi]}=0$$

implying that

$$(u(t), v(t))_{L^2[0,2\pi]} = (u(0), v(0))_{L^2[0,2\pi]}. \tag{39}$$

If we consider the pseudospectral Fourier approximations of Eqs. (37)–(38)

$$\frac{\partial u_N}{\partial t} + \mathscr{L}_N u_N = 0, \quad \frac{\partial v_N}{\partial t} - \mathscr{L}_N^* v_N = 0,$$

where $u_N \in \mathsf{P}_N$ and $v_N \in \mathsf{P}_N$ represent the polynomial solutions and $\mathscr{L}_N = \mathscr{I}_N \mathscr{L} \mathscr{I}_N$ and $\mathscr{L}_N^* = \mathscr{I}_N \mathscr{L}^* \mathscr{I}_N$, respectively, we recover

$$\frac{\pi}{N} \sum_{j=0}^{2N-1} u_N(x_j, t) v_N(x_j, t) = \frac{\pi}{N} \sum_{j=0}^{2N-1} u_N(x_j, 0) v_N(x_j, 0) \tag{40}$$

as a consequence of the skew-symmetry of $\mathscr{L}_N$ and $\mathscr{L}_N^*$.

A deceptive element of the pseudospectral approximation of the initial conditions is that it hides the oscillations, i.e., if we look at $u_N(x_j, 0)$ it appears perfectly smooth. To reinforce the oscillatory behavior of $u_N(0)$ we preprocess the initial conditions, u_0 and v_0, such that they are the pseudospectral representation of the Galerkin representation, i.e., the truncated continuous expansion, of u_0 and v_0. For the latter, this will have little impact as v_0 is smooth. However, the Galerkin representation of u_0 is oscillatory and this will be reflected in $u_N(x_j, 0)$. Since $u_N(0)$ and $v_N(0)$ are both Nth-order trigonometric polynomials Eq. (40) implies

$$\frac{\pi}{N} \sum_{j=0}^{2N-1} u_N(x_j, t) v_N(x_j, t) = (u_N(0), v_N(0))_{L^2[0,2\pi]},$$

by the quadrature. Exploiting the smoothness of v_0 it is straightforward to show that [1]

$$(u_0, v_0)_{L^2[0,2\pi]} = (u_N(0), v_N(0))_{L^2[0,2\pi]} + CN^{-q} \|v_0^{(q)}\|_{L^2[0,2\pi]}.$$

In combination with Eqs. (39) and (40) this yields

$$\frac{\pi}{N} \sum_{j=0}^{2N-1} u_N(x_j, t) v_N(x_j, t) = (u(t), v(t))_{L^2[0,2\pi]} + CN^{-q} \|v_0^{(q)}\|_{L^2[0,2\pi]}.$$

Further assuming that the approximation of the dual problem is stable, which is supported by the discussion in Section 5.1 and the assumption that v_0 is smooth, we have convergence as

$$\|v(t) - v_N(t)\|_{L^2[0,2\pi]} \leqslant CN^{1-q} \|v^{(q)}\|_{L^2[0,2\pi]}.$$

We can therefore replace v_N with v, thereby introducing an error of the order of the scheme, to obtain

$$\frac{\pi}{N} \sum_{j=0}^{2N-1} u_N(x_j, t) v(x_j, t) = (u(t), v(t))_{L^2[0,2\pi]} + \varepsilon,$$

where ε is exponentially small if $v(x,t)$ is analytic. In other words, $u_N(x,t)$ approximates $u(x,t)$ weakly to within spectral accuracy and the stability of the problem with smooth initial conditions is sufficient to guarantee stability of the problem with nonsmooth initial conditions.

Moreover, there exists a smooth function, $v(x,t)$, that allows one to extract highly accurate information about $u(x,t)$ from $u_N(x,t)$ even after propagation in time and the accumulated effects of the aliasing error. This justifies the use of the Gibbs reconstruction techniques discussed in Section 3.2 as a postprocessing technique after propagating the oscillatory initial conditions.

5.4. Aspects of fully discrete stability

The results summarized in the past sections on semi-discrete stability provide a necessary foundation for understanding the behavior of the fully discrete approximation, i.e., an approximation in which also the temporal dimension is discretized. This last, yet essential step, in constructing a fully discrete scheme introduces a number of additional complications.

A thorough discussion of temporal integration techniques and their properties is well beyond the scope of this review and we shall focus the attention on the widely used Runge–Kutta methods. Numerous alternative techniques, explicit as well as implicit, are discussed in [39,10].

Let us consider the problem, Eq. (23), on the generic form, Eq. (37), and assume that $a(x)$ is uniformly bounded away from zero to avoid unnecessary complications. In this case, the fully discrete s-stage Runge–Kutta scheme

$$u_N^{n+1} = \sum_{k=0}^{s} \frac{(\Delta t \mathscr{L}_N)^k}{k!} = \mathscr{P}(\Delta t \mathscr{L}_N) u_N^n,$$

advances the solution, u_N^n, from $t = n\Delta t$ to $t = (n+1)\Delta t$. The central issue to address is which value of Δt ensures that this is a stable process in the sense that $\|\mathscr{P}(\Delta t \mathscr{L}_N)^n\|$ remains bounded for all n.

If we first consider the Fourier collocation approximation to Eq. (23) we have

$$\mathscr{L}_N = AD,$$

in the notation of Section 5.1. Recall that D, Eq. (12), is skew-symmetric which immediately implies that $D = S\Lambda S^{\mathrm{T}}$ with $\|S\| = \|S^{\mathrm{T}}\| = 1$ and [65]

$$\Lambda = \mathrm{diag}[-\mathrm{i}(N-1), -\mathrm{i}(N-2), \ldots, -\mathrm{i}, 0, 0, \mathrm{i}, \ldots, \mathrm{i}(N-2), \mathrm{i}(N-1)],$$

such that $\|D\| = (N-1)$. We note the double zero eigenvalue which appears as a result of having $2N+1$ modes but only $2N$ nodes in the expansion. The double eigenvalues is not degenerate and is thus not introducing any problems.

Given the purely imaginary spectrum, this yields a necessary and sufficient condition for fully discrete stability on the form

$$\Delta t \leqslant \frac{C_{\Delta t}}{a_\infty (N-1)}, \quad a_\infty = \max_{x \in [0,2\pi]} |a(x)|, \tag{41}$$

and $C_{\Delta t} = \sqrt{3}$ for the third-order third-stage Runge–Kutta method while it is $C_{\Delta t} = \sqrt{8}$ for the fourth-order fourth-stage Runge–Kutta method.

While the von Neumann stability analysis suffices to establish both necessary and sufficient conditions for discrete stability of the Fourier method it fails to provide more than necessary conditions for the Chebyshev case. This is caused by $\hat{D}$, Eq. (14) modified to account for boundary conditions, being nonnormal, i.e., even if $\hat{D}$ can be diagonalized as $\hat{D} = S\Lambda S^{-1}$ we cannot in general ensure that $\|S\|\|S^{-1}\|$ remains bounded for $N \to \infty$.

As for the semi-discrete case discussed in Section 5.2 the difficulty in establishing rigorous stability results lies in the need to identify the right norm. If we restrict the attention to Eq. (23) in the simplest case of $a(x) = a > 0$ it is natural to employ the norm used in Eq. (34) and consider the stability of the slightly changed problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} + aD\boldsymbol{v} = 0, \tag{42}$$

where $\boldsymbol{v}(t) = [u_N(-1,t),\ldots,u_N(x_j,t),\ldots,u_N(x_{N-1},t)]^{\mathrm{T}}$, contains the first N components of u_N and $\hat{D}$ is discussed in relation with Eq. (35).

Using a first-order one-stage Runge–Kutta scheme, also known as the forward Euler method, a necessary and sufficient condition for fully discrete stability of the Chebyshev collocation approximation of Eq. (42) has been obtained for Eq. (35) of the form [45]

$$\Delta t a \left(N^2 + \frac{2}{\Delta x_{\min}} \right) \leqslant \frac{1}{4},$$

where the first term, N^2, is associated with the Chebyshev basis itself, while

$$\Delta x_{\min} = \min(1 + x_0, 1 - x_N),$$

reflects a dependency on the minimum grid size. Although one can only conjecture that the discrete stability remains valid for the pseudospectral Chebyshev–Gauss–Lobatto approximation, Eq. (29), of $\boldsymbol{u}$ rather than $\boldsymbol{v}$ it is interesting to note that

$$\Delta x_{\min} \simeq \frac{\pi^2}{2} N^{-2}.$$

Hence, we recover the well-known empirical stability condition [39]

$$\Delta t \leqslant \frac{C}{aN^2},$$

where C is of order one. Similar results have been established for higher-order Runge–Kutta schemes in [68].

To associate this limit with the clustering of the grid, however, is a deceptive path. To see this consider Eq. (23) with $a(x) > 0$, in which case the stability condition becomes [45]

$$\Delta t \left(a_\infty N^2 + 2 \max_{x_j} \frac{a(x_j)}{1 - x_j} \right) \leqslant \frac{1}{4}.$$

Clearly, in the event where $a(x_j)$ approaches zero faster than N^{-2} it is the first term, $a_\infty N^2$, rather than the minimum grid size that controls the time step.

The effect of the nonnormality on the performance of the pseudospectral approximations is discussed in detail in [87] for both Legendre and Chebyshev approximations. Attempts to extend the applicability of the von Neumann analysis to problems with nonnormal operators has been discussed in the context of spectral methods in [78,79] where it is advocated that one considers the pseudo-spectrum rather than the simpler eigenspectrum of D to properly understand the fully discrete stability of the approximation.

6. Convergence results for nonlinear hyperbolic problems

The return to the general nonlinear problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{43}$$

introduces additional issues and new problems which are not relevant or of less importance for the variable coefficient problem discussed in Section 5.

One of the central difficulties in using spectral methods for the solution of nonlinear conservation laws lies in the potential development of nonsmooth solutions in finite time even for problems with very smooth initial conditions. As we have discussed previously, this introduces the Gibbs phenomenon which, through the nonlinearity, interacts with the solution. What we shall discuss in the following is the impact this has on the performance of the numerical approximation and techniques that allow us to recover accurate and physically meaningful solutions to the conservation laws even when the Gibbs oscillations are apparent.

6.1. Stability by the use of filters

Maintaining stability of the numerical approximation becomes increasingly hard as the discontinuity develops and generates energy with higher and higher frequency content. This process, amplified by the nonlinear mixing of the Gibbs oscillations and the numerical solution, eventually renders the scheme unstable.

Understanding the source of the stability problem, i.e., accumulation of high-frequency energy, also suggests a possible solution by introducing a dissipative mechanism that continuously remove these high-frequency components.

A classical way to accomplish this is to modify the original problem by adding artificial dissipation as

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon(-1)^{p+1}\frac{\partial^{2p} u}{\partial x^{2p}}.$$

A direct implementation of this, however, may be costly and could introduce additional stiffness which would limit the stable time-step. We shall hence seek a different approach to achieve a similar effect.

In Section 3.2.1 we discussed the use of low pass filtering to improve on the convergence rate of the global approximation away from the point of discontinuity. This was achieved by modifying the numerical solution, $u_N(x,t)$, through the use of a spectral filter as

$$\mathscr{F}_N u_N(x,t) = \sum_{n=-N}^{N} \sigma\left(\frac{n}{N}\right)\tilde{u}_n(t)\exp(\mathrm{i}nx). \tag{44}$$

To understand the impact of using the filter at regular intervals as a stabilizing mechanism, a procedure first proposed in [74,66], let us consider the exponential filter

$$\sigma(\eta) = \exp(-\alpha\eta^{2p}).$$

As discussed in Section 3.2.1 this filter allows for a dramatic improvement in the accuracy of the approximation away from points of discontinuity.

To appreciate its impact on stability, consider the generic initial value problem

$$\frac{\partial u}{\partial t} = \mathscr{L} u$$

with the pseudospectral Fourier approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = \mathscr{L}_N \boldsymbol{u}.$$

Advancing the solution from $t = 0$ to Δt followed by the filtering is conveniently expressed as

$$\boldsymbol{u}(\Delta t) = \mathscr{F}_N \exp(\mathscr{L}_N \Delta t)\boldsymbol{u}(0).$$

If we first assume that $\mathscr{L}_N$ represents the constant coefficient hyperbolic problem, $\mathscr{L} = a(\partial/\partial x)$, we recover that

$$\tilde{u}_n(\Delta t) = \exp(-\alpha\eta^{2p} + a(\mathrm{i}k)\Delta t)\tilde{u}_n(0), \tag{45}$$

i.e., we are in fact computing the solution to the modified problem

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} - \alpha\frac{(-1)^p}{\Delta t N^{2p}}\frac{\partial^{2p} u}{\partial x^{2p}}.$$

The effect of the filter is thus equivalent to the classical approach of adding a small dissipative term to the original equation, but the process of adding the dissipation is very simple as discussed in Section 3.2.1. Note in particular that $\Delta t N$ essentially represents the *CFL* condition, Eq. (41), and hence is of order one.

For a general $\mathscr{L}$, e.g., with a variable coefficient or of a nonlinear form, in which case $\mathscr{F}_N$ and $\mathscr{L}_N$ no longer commute, the modified equation being solved takes the form

$$\frac{\partial u}{\partial t} = \mathscr{L}u - \alpha\frac{(-1)^p}{\Delta t N^{2p}}\frac{\partial^{2p} u}{\partial x^{2p}} + \mathscr{O}(\Delta t^2),$$

by viewing the application of the filter as an operator splitting problem [7,20].

With this in mind it is not surprising that using a filter has a stabilizing effect. Moreover, we observe that if p increases with N the modification caused by the filter vanishes spectrally as N increases. These loose arguments for the stabilizing effect of filtering have been put on firm ground for problem with smooth and nonsmooth initial data [74,66,83] for the Fourier approximation to the general variable coefficient problem, Eq. (23). These results, however, are typically derived under the assumption that $\sigma(\eta)$ is of polynomial form. While such filtering indeed stabilizes the approximation it also reduces the global accuracy of the scheme [74,88,42]. Let us therefore briefly consider the stabilizing effect of the exponential filter in the pseudospectral Fourier approximation of Eq. (23), known to be weakly unstable as discussed in Section 5.1, Eq. (31).

Consider the filtered approximation of the form

$$\frac{\partial u_N}{\partial t} + \mathscr{I}_N\left(a(x)\frac{\partial u_N}{\partial x}\right) = \varepsilon_N(-1)^{p+1}\frac{\partial^{2p} u_N}{\partial x^{2p}}, \tag{46}$$

where the superviscosity term on the right can be implemented through a filter and

$$\varepsilon_N = \frac{\alpha}{\Delta t N^{2p}}.$$

To establish stability, let us rewrite Eq. (46) as

$$\frac{\partial u_N}{\partial t} + \mathscr{N}_1 u_N + \mathscr{N}_2 u_N + \mathscr{N}_3 u_N = \varepsilon_N(-1)^{p+1}\frac{\partial^{2p} u_N}{\partial x^{2p}},$$

where

$$\mathscr{N}_1 u_N = \frac{1}{2}\frac{\partial}{\partial x}\mathscr{I}_N a(x)u_N + \frac{1}{2}\mathscr{I}_N\left(a(x)\frac{\partial u_N}{\partial x}\right)$$

is the skew-symmetric form of the operator, Eq. (30),

$$\mathscr{N}_2 u_N = \frac{1}{2}\mathscr{I}_N\left(a(x)\frac{\partial u_N}{\partial x}\right) - \frac{1}{2}\mathscr{I}_N\frac{\partial a(x)u_N}{\partial x}$$

and

$$\mathscr{N}_3 u_N = \frac{1}{2}\mathscr{I}_N\frac{\partial a(x)u_N}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\mathscr{I}_N a(x)u_N.$$

To establish stability, consider

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|_N^2 = -[u_N, \mathscr{N}_1 u_N]_N - [u_N, \mathscr{N}_2 u_N]_N$$

$$-[u_N, \mathscr{N}_3 u_N]_N + \left[u_N, \varepsilon_N(-1)^{p+1}\frac{\partial^{2p}u_N}{\partial x^{2p}}\right]_N.$$

Clearly $[u_N, \mathscr{N}_1 u_N]_N = 0$ due to the skew-symmetry of $\mathscr{N}_1 u_N$ and by inspection we can bound

$$[u_N, \mathscr{N}_2 u_N]_N \leqslant \tfrac{1}{2}\max_{x\in[0,\,2\pi]}|a_x(x)|\|u_N\|_N^2.$$

It is indeed the term associated with $\mathscr{N}_3 u_N$ that is the troublemaker. To appreciate this, simply note that if $\mathscr{P}_N$ was used rather than $\mathscr{I}_N$ such that differentiation and truncation commute, the term would vanishes identically and the scheme would be stable. To bound this term, we can use that

$$[u_N, \mathscr{N}_3 u_N]_N \leqslant C(\|u_N\|_N^2 + \|\mathscr{N}_3 u_N\|_N^2).$$

Noting that $\|\mathscr{N}_3 u_N\|_{L^2[0,2\pi]}^2$ is nothing more than the commutation error and that $\|\cdot\|_N$ is L^2-equivalent, we can borrow the result of Eq. (17) to obtain

$$[u_N, \mathscr{N}_3 u_N]_N \leqslant C(\|u_N\|_{L^2[0,2\pi]}^2 + N^{2-2p}\|u_N^{(p)}\|_{L^2[0,2\pi]}^2),$$

where C depends on $a(x)$ and its first p derivatives. If we finally note that

$$\left[u_N, \varepsilon_N(-1)^{p+1}\frac{\partial^{2p}u_N}{\partial x^{2p}}\right]_N = -\varepsilon_N\|u_N^{(p)}\|_N^2,$$

it is clear that we can always choose $\varepsilon_N = AN^{2-2p}$ and A sufficiently large to ensure stability. In other words, using an exponential filter is sufficient to stabilize the Fourier approximation of Eq. (23).

There is one central difference in the effect of using the filter in the Fourier and the Chebyshev approximation. In the latter, the modified equation takes the form

$$\frac{\partial u}{\partial t} = \mathscr{L}u - \alpha\frac{(-1)^p}{\Delta t N^{2p}}\left[\sqrt{1-x^2}\frac{\partial}{\partial x}\right]^{2p} u + \mathscr{O}(\Delta t^2). \tag{47}$$

Hence, while the filtering continues to introduce dissipation, it is spatially varying. In particular, it vanishes as one approaches the boundaries of the domain. In computations with moving discontinuities this may be a source of problems since the stabilization decreases as the discontinuity approaches the boundaries of the computational domain.

6.2. Spectrally vanishing viscosity and entropy solutions

The foundation of a convergence theory for spectral approximations to conservation laws has been laid in [85,72,15] for the periodic case and subsequently extended in [73] to the Legendre approximation and recently to the Chebyshev–Legendre scheme in [70,71].

To appreciate the basic elements of this convergence theory let us first restrict ourselves to the periodic case. For the discrete approximation to Eq. (43) we must add a dissipative term that is strong enough to stabilize the approximation, yet small enough not to ruin the spectral accuracy of the scheme. In [85,72] the following spectral viscosity method was considered

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}\mathscr{P}_N(f(u_N)) = \varepsilon_N(-1)^{p+1}\frac{\partial^p}{\partial x^p}\left[Q_m(x,t)\frac{\partial^p u_N}{\partial x^p}\right], \tag{48}$$

where

$$\frac{\partial^p}{\partial x^p}\left[Q_m(x,t)\frac{\partial^p u_N}{\partial x^p}\right] = \sum_{m<|n|\leqslant N}(\mathrm{i}k)^{2p}\hat{Q}_n\hat{u}_n\exp(\mathrm{i}nx).$$

To ensure that stability is maintained m should not be taken too big. On the other hand, taking m too small will impact the accuracy in a negative way. An acceptable compromise seems to be

$$m \sim N^\theta, \quad \theta < \frac{2p-1}{2p}.$$

Moreover, the smoothing factors, $\hat{Q}_n$, should only be activated for high modes as

$$\hat{Q}_n = 1 - \left(\frac{m}{|n|}\right)^{(2p-1)/\theta}$$

for $|n| > m$ and $\hat{Q}_n = 1$ otherwise. Finally, we shall assume that the amplitude of the viscosity is small as

$$\varepsilon_N \sim \frac{C}{N^{2p-1}}.$$

Under these assumptions, one can prove for $p = 1$ that the solution is bounded in $L^\infty[0,2\pi]$ and obtain the estimate

$$\|u_N\|_{L^2[0,2\pi]} + \sqrt{\varepsilon_N}\left\|\frac{\partial u_N}{\partial x}\right\|_{L^2_{loc}} \leqslant C.$$

Convergence to the correct entropy solution then follows from compensated compactness arguments [85,72].

To realize the close connection between the spectral viscosity method and the use of filters discussed in Section 6.1, consider the simple case where $f(u) = au$. In this case, the solution to Eq. (48) is given as

$$\hat{u}_n(t) = \exp(\mathrm{i}nat - \varepsilon_N n^2\hat{Q}_n)\hat{u}_n(0), \quad |n| > m,$$

which is equivalent to the effect of the filtering discussed in Section 6.1. Note that the direct application of the vanishing viscosity term in Eq. (48) amounts to $2p$ spatial derivatives while filtering as discussed in Section 3.2.1 can be done at little or no additional cost.

For $p \neq 1$ a bound on the $L^\infty[0,2\pi]$ is no longer known. However, experience suggests that it is better to filter from the first mode but to employ a slower decay of the expansion coefficients, corresponding to taking $p > 1$. This yields the superviscosity method in which one solves

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}\mathscr{P}_N f(u_N) = \varepsilon_N(-1)^{p+1}\frac{\partial^{2p} u_N}{\partial x^{2p}},$$

which we recognize from Eq. (46) as being equivalent to that obtained when using a high-order exponential filter.

The vanishing viscosity approximation to Eq. (43) using a Chebyshev collocation approach takes the form

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}\mathscr{I}_N f(u_N) = \varepsilon_N(-1)^{p+1}\left[\sqrt{1-x^2}\frac{\partial}{\partial x}\right]^{2p} u_N + \mathscr{B}u_N,$$

where again

$$\varepsilon_N \sim \frac{C}{N^{2p-1}},$$

and p grows with N [73]. Here the boundary operator, $\mathscr{B}u_N$, may vanish or it may take values as

$$\mathscr{B}u_N = -\tau^-\frac{(1-x)T_N'(x)}{2T_N'(-1)}(u_N(-1,t)-g^-) + \tau^+\frac{(1+x)T_N'(x)}{2T_N'(1)}(u_N(1,t)-g^+),$$

which we recognize as the weakly imposed penalty terms discussed in Sections 4.2 and 5.2. Note again that the vanishing viscosity term is equivalent to that obtained from the analysis of the effect of spectral space filtering, Eq. (47). Similar results can be obtained for the Legendre approximation and for the Chebyshev–Legendre method for which convergence has been proven [70,71], using arguments similar to those in [85,72], for $p = 1$ as well as for $p > 1$.

6.3. Conservation

It is natural to question whether the introduction of the artificial Gibbs oscillations has any impact on the basic physical properties described by the conservation law, e.g., mass conservation and the speed by which discontinuities propagate.

To come to an understanding of this, assume a spatially periodic problem and consider the pseudospectral Fourier scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} + D\boldsymbol{f} = 0,$$

where $\boldsymbol{u} = [u_N(0,t),\ldots,u_N(x_{2N-1},t)]^{\mathrm{T}}$ represent the grid vector and the interpolation of the flux is given as $\boldsymbol{f} = [\mathscr{I}_N f(u_N(0,t),t),\ldots,\mathscr{I}_N f(u_N(x_{2N-1},t),t)]^{\mathrm{T}}$.

The first thing to note is that

$$\int_0^{2\pi} u_N(x,t)\,\mathrm{d}x = \int_0^{2\pi} u_N(x,0)\,\mathrm{d}x$$

as an immediate consequence of the accuracy of the trapezoidal rule and the assumption of periodicity. Hence, the approximation conserves the 'mass' of the interpolation of the initial conditions.

Let us introduce a smooth periodic test function, $\psi(x,t)$, with the corresponding grid vector, $\boldsymbol{\psi} = [\psi_N(x_0,t),\ldots,\psi_N(x_{2N-1},t)]^{\mathrm{T}}$. The test function, $\psi(x,t)$, is assumed to vanish at large t. If we consider [36]

$$\boldsymbol{\psi}^{\mathrm{T}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} + D\boldsymbol{f}\right) = 0$$

and utilize the accuracy of the trapezoidal rule we recover

$$\int_0^{2\pi}\left[\psi_N(x,t)\frac{\partial u_N(x,t)}{\partial t} - \frac{\partial \psi_N(x,t)}{\partial x}\mathscr{I}_N f(u_N(x,t),t)\right]\mathrm{d}x = 0,$$

after integration by parts which is permitted if the solution, $u_N(x,t)$, is bounded. This implicitly assumes that the numerical approximation itself is stable which generally implies that a vanishing viscosity term is to be added, potentially through the use of a filter.

Integrating over time and by parts once more, we recover the result

$$\int_0^{\infty}\int_0^{2\pi}\left[u_N(x,t)\frac{\partial \psi_N(x,t)}{\partial t} + \frac{\partial \psi_N(x,t)}{\partial x}\mathscr{I}_N f(u_N(x,t),t)\right]\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^{2\pi}\psi_N(x,0)u_N(x,0)\,\mathrm{d}x = 0.$$

Thus, for $N\to\infty$ the solution, $u_N(x,t)$, is a weak solution to the conservation law. This again implies that the limit solution satisfies the Rankine–Hugoniot conditions which guarantees that shocks propagate at the right speed to within the order of the scheme. Results similar to these have also been obtained for the Chebyshev approximation to the conservation law [36].

To appreciate that the addition of the vanishing viscosity has no impact on the conservation of the scheme, consider the Legendre superviscosity case [73] and let $\psi(x,t)$ be a test function in $C^3[-1,1]$ that vanishes at the endpoints. Taking $\psi_{N-1}(x,t) = \mathscr{I}_{N-1}\psi(x,t)$, then clearly $\psi_{N-1}(x,t)\to\psi(x,t)$, $(\psi_{N-1})_x(x,t)\to\psi_x(x,t)$, and $(\psi_{N-1})_t(x,t)\to\psi_t(x,t)$ uniformly in N.

Since $\psi_{N-1}(x)$ is a polynomial that vanishes at the boundaries we have

$$\int_{-1}^{1}(1+x)P_N'(x)\psi_{N-1}(x,t)\,\mathrm{d}x = 0, \quad \int_{-1}^{1}(1-x)P_N'(x)\psi_{N-1}(x,t)\,\mathrm{d}x = 0.$$

Moreover, integration by parts yields that

$$\lim_{N\to\infty}\frac{\varepsilon_N(-1)^p}{N^{2p-1}}\int_{-1}^{1}\psi_{N-1}(x,t)\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\right]^{2p}u_N(x,t)\,\mathrm{d}x = 0.$$

Hence, the superviscosity term does not cause any problems and one can show that

$$-\int_0^{\mathrm{T}}\int_{-1}^{1}\left(u_N(x,t)\frac{\partial \psi_{N-1}(x,t)}{\partial t} + \mathscr{I}_N f(u_N(x,t))\frac{\partial \psi_{N-1}(x,t)}{\partial x}\right)\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{-1}^{1}u_N(x,0)\psi_{N-1}(x,0)\,\mathrm{d}x = 0.$$

The main conclusion of this is that if $u_N(x,t)$ is a solution to the Legendre collocation approximation at the Gauss–Lobatto points and if $u_N(x,t)$ converges almost everywhere to a function $u(x,t)$, then

$u(x,t)$ is a weak solution to Eq. (43). The technical details of this proof can be found in [14] where also a similar result for the Chebyshev superviscosity approximation is given.

The theory of convergence of spectral methods equipped with spectral viscosity or superviscosity is limited to the scalar case as discussed in Section 6.2. For the system case a more limited result can be obtained, stating that if the solution converges to a bounded solution, it converges to the correct weak solution.

7. Multi-domain methods

As a final technique, playing a pivotal role in making many of the techniques discussed previously amenable to the solution of problems of interest to scientists and engineers, let us briefly discuss multi-domain methods for hyperbolic problems.

The original motivation for the introduction of multi-domain methods can be found in the restrictions that the fixed grids, required to ensure the high spatial accuracy, impose. This fixed grid makes it difficult to utilize adaptivity and, for multi-dimensional problems, to address problems in complex geometries. Moreover, the use of global spectral expansions makes it difficult to achieve a high parallel efficiency on contemporary parallel computers.

Many of these concerns can be overcome if one splits the computational domain into a number of geometrically simple building blocks, e.g., squares and cubes, and then employs tensor-product forms of the simple one-dimensional approximations as the basis of an element by element approximation. While this technique opens up for the use of a highly nonuniform resolution and the ability to model problems in geometrically complex domains, it also introduces the need to connect the many local solutions in an accurate, stable, and efficient manner to reconstruct the global solution.

7.1. Patching techniques

The patching of the local solutions in a way consistent with the nature of the hyperbolic problem can be performed in at least two different yet related ways. Borrowing the terminology introduced in [59], we shall refer to these two different methods as the differential and the correctional method, respectively.

To expose the differences between the two methods, let us consider the two domain scalar problem

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in [-1, 0], \tag{49}$$

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0, \quad x \in [0, 1].$$

To recover the global solution $U = [u, v]$ under the constraint that $u(0,t) = v(0,t)$, the central issue is how one decides which of the multiple solutions at $x = 0$ takes preference and hence determines the evolution of $u(0,t)$ and $v(0,t)$.

Provided that the initial conditions are consistent with the continuity condition it will clearly remain continuous if we ensure that $u_t(0,t) = v_t(0,t)$. This approach, known as the differential method, involves the exchange of information between the two domains to ensure that the flux of

$u(0,t)$ and $v(0,t)$ are identical throughout the computation. There are, however, several ways to do so.

In the original work [58], the solution is assumed to be smooth and one introduces the flux derivative

$$\lambda = \left.\frac{\partial f}{\partial u}\right|_{u(0,t)} = \left.\frac{\partial f}{\partial v}\right|_{v(0,t)}$$

and requires that u and v be updated at $x = 0$ as

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = \left.\frac{\partial v}{\partial t}\right|_{x=0} = -\frac{1}{2}(\lambda + |\lambda|)\left.\frac{\partial u}{\partial x}\right|_{x=0} - \frac{1}{2}(\lambda - |\lambda|)\left.\frac{\partial v}{\partial x}\right|_{x=0}.$$

This can be recognized as nothing else than pure upwinding. The extension to systems of equations employs the characteristic form of the system and the multi-dimensional case is treated by dimensional splitting.

An alternative formulation, based on the weakly imposed boundary conditions discussed in Section 4.2.2 and introduced in [48,19,50], takes the form

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} + \left.\frac{\partial f(u)}{\partial x}\right|_{x=0} = -\tau\frac{|\lambda - |\lambda||}{2}(u(0,t) - v(0,t)),$$

$$\left.\frac{\partial v}{\partial t}\right|_{x=0} + \left.\frac{\partial f(v)}{\partial x}\right|_{x=0} = -\tau\frac{|\lambda + |\lambda||}{2}(v(0,t) - u(0,t)),$$

which again amounts to upwinding, although on a weak form. The advantage of this latter formulation is that it allows for establishing stability and it makes the enforcement of very complex interface conditions simple. The extension to systems employs the characteristic variables and is discussed in detail in [48,50] while the multi-dimensional case is treated in [49]. Similar developments for methods employing multi-variate polynomials [53,55] or a purely modal basis [69,91,93] defined on triangles and tetrahedra has recently been developed, pawing the way for the formulation of stable spectral methods for the solution of hyperbolic conservation laws using a fully unstructured grid.

Rather than correcting the local temporal derivative to ensure continuity of the flux across the interface one could choose to modify the solution itself. This observation provides the basic foundation for correctional methods in which both u and v is advanced everywhere within the each domain, leading to a multiplicity of solutions at $x = 0$. For the specific case discussed here, the correctional approach amounts to

$$u(0,t) = v(0,t) = \begin{cases} u(0,t) & \text{if } \lambda \geqslant 0, \\ v(0,t) & \text{if } \lambda < 0, \end{cases}$$

which we again recognize as upwinding. The system case is treated similarly by exploiting the characteristic variables. As for the differential methods, the use of the characteristics implicitly assumes a minimum degree of smoothness of the solution. However, as no information about the spatial derivatives are passed between domains, the correctional method imposes no constraints on the smoothness of the grid.

The main appeal of the correctional method lies in its simplicity and robustness and it has been utilized to formulate very general multi-domain method for problems in gas dynamics [59,60], in acoustics and elasticity [2], and electromagnetic [95,96,51,97].

Note that both methods employ the local flux Jacobian, λ, which implicitly requires a certain amount of smoothness of the solution at the interface. A differential method overcoming this can be realized by borrowing a few ideas from classical finite volume methods.

Consider the cell-averaged formulation

$$\frac{\mathrm{d}\overline{u}_j}{\mathrm{d}t} + \frac{f(u(x_{j+1/2})) - f(u(x_{j-1/2}))}{\Delta x_j} = 0,$$

where

$$\Delta x_j = x_{i+1/2} - x_{i-1/2}, \quad \overline{u}_j = \frac{1}{\Delta x_j}\int_{x_{j-1/2}}^{x_{j+1/2}} u(s)\,\mathrm{d}s.$$

Here $x_{j\pm1/2}$ signifies the Chebyshev–Gauss–Lobatto grid and x_j refers to the interlaced Chebyshev–Gauss grid. No assumptions are made about the smoothness of the flux and since each individual cell requires reconstruction, the patching of the subdomains is achieved by flux-splitting techniques known from finite volume methods. This approach was first proposed in [8] for Fourier methods and subsequently in [9] for the Chebyshev approximation and has the advantage of being conservative by construction. The averaging and reconstruction procedure, which can be done in an essentially nonoscillatory way, is essential for the accuracy and stability of the scheme and several alternatives, exploiting a similar framework, has been proposed in [80,32,33].

The use of a staggered grid, collocating the solution u at the Gauss grid and the fluxes, $f(u)$, at the Gauss–Lobatto grid, has the additional advantage of allowing for the formulation of multi-dimensional multi-domain methods with no grid points at the vertices of the elements. This approach, introduced in [62,61], has been developed for smooth problems and eliminates complications associated with the treatment of vertices in multi-domain methods [60].

Alternative differential patching methods has been discussed in [77,10] where the patching is achieved by the use of compatibility conditions.

7.2. Conservation properties of multi-domain schemes

The important question of the conservation properties of multi-domain schemes is discussed in [14] in which the following polynomial approximation to Eq. (49) is considered:

$$\begin{aligned}\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}\mathcal{I}_N f(u_N) &= \tau_1 Q_{\mathrm{I}}^{+}(x)[f^{+}(u_N(0,t)) - f^{+}(v_N(0,t))] \\ &\quad + \tau_2 Q_{\mathrm{I}}^{+}(x)[f^{-}(u_N(0,t)) - f^{-}(v_N(0,t))] + \mathrm{SV}(u_N),\end{aligned}$$

$$\begin{aligned}\frac{\partial v_N}{\partial t} + \frac{\partial}{\partial x}\mathcal{I}_N f(v_N) &= -\tau_3 Q_{\mathrm{II}}^{-}(x)[f^{+}(v_N(0,t)) - f^{+}(u_N(0,t))] \\ &\quad - \tau_4 Q_{\mathrm{II}}^{-}(x)[f^{-}(v_N(0,t)) - f^{-}(u_N(0,t))] + \mathrm{SV}(v_N),\end{aligned}$$

where $Q^{\pm}(x)$ are polynomials given in Eq. (36), i.e., they vanish at all collocation points except $x=\pm1$. Furthermore, $\mathrm{SV}(u_N)$ and $\mathrm{SV}(v_N)$ represent the vanishing viscosity terms, or filtering, required to stabilize the nonlinear problem as discussed in Sections 6.1 and 6.2, while $f = f^{+} + f^{-}$ signifies a splitting into the upwind and downwind components of the flux.

To establish conservation of the approximation, consider a test function $\psi(x)$ and denote by $\psi_{\rm I}$ and $\psi_{\rm II}$ its restriction to the first and second domain respectively. We can assume that $\psi_{\rm I}$ and $\psi_{\rm II}$ are polynomials of order $N-1$ and that $\psi_{\rm I}$ vanishes at $x=-1$ of the first domain while $\psi^{\rm II}$ vanishes at $x=1$ of the second domain, but not at $x=0$.

Repeated integration by parts using the fact that $\mathrm{SV}(u_N)$ vanishes at the boundaries of each domain yields

$$\int_{-1}^{0}\psi_{\rm I}(x)\,\mathrm{SV}(u_N)\,\mathrm{d}x=(-1)^P\int_{-1}^{0}u_N\,\mathrm{SV}(\psi_{\rm I})\,\mathrm{d}x$$

which tends to zero with increasing N. A similar result can be obtained for the second domain.

Consider now

$$\int_{-1}^{0}\psi^{\rm I}(x)\frac{\partial u_N}{\partial t}\,\mathrm{d}x+\int_{0}^{1}\psi^{\rm II}(x)\frac{\partial v_N}{\partial t}\,\mathrm{d}x.$$

To recover that u_N and v_N are weak solutions to Eq. (49), i.e., the above integral vanishes, we must require

$$\tau_1+\tau_3=1,\quad \tau_2+\tau_4=1. \tag{50}$$

However, for linear stability, as we discussed in Section 5.2, one can show that

$$\tau_1\geqslant\tfrac{1}{2},\quad \tau_2\leqslant\tfrac{1}{2},$$

$$\tau_3\leqslant\tfrac{1}{2},\quad \tau_4\geqslant\tfrac{1}{2},$$

together with Eq. (50), are necessary and sufficient conditions to guarantee stability.

This leaves us with a set of conditions under which to design stable and conservative scheme. In particular, if we choose to do pure upwinding at the interfaces by specifying

$$\tau_1=\tau_4=1,\quad \tau_2=\tau_3=0,$$

we essentially recover the discontinuous Galerkin method discussed in Section 4.2.

An appealing alternative appears by considering

$$\tau_1=\tau_2=\tau_3=\tau_4=\tfrac{1}{2},$$

which yields a marginally stable and conservative scheme of the form

$$\frac{\partial u_N}{\partial t}+\frac{\partial}{\partial x}\mathcal{I}_N f(u_N)=\frac{1}{2}Q_{\rm I}^{+}(x)[f(u_N(0,t))-f(v_N(0,t))]+\mathrm{SV}(u_N),$$

$$\frac{\partial v_N}{\partial t}+\frac{\partial}{\partial x}\mathcal{I}_N f(v_N)=-\frac{1}{2}Q_{\rm II}^{-}(x)[f(v_N(0,t))-f(u_N(0,t))]+\mathrm{SV}(v_N),$$

i.e., the interface boundary conditions are imposed on the fluxes f rather than on the split fluxes f^+ and f^-.

7.3. Computational efficiency

An interesting question pertaining to the use of multi-domain methods is how one decides how many elements and what resolution to use within each element. In pragmatic terms, what we are

interested in is to identify the optimal combination of the order of the polynomial, N, and the number of elements, K, needed to solve a particular problem to within a maximum error using minimum computational resources.

On one hand, it is the high order of the interpolation polynomial that results in the very accurate approximation. On the other hand, the computation of derivatives generally scales as $\mathcal{O}(N^2)$ while the total work scales only linearly with the number of elements. To develop guidelines for choosing the optimal N and K, consider a one-dimensional wave problem with a smooth solution. Assume that the approximation error, $E(N,K)$, scales as

$$E(N,K) \propto \left(\frac{\pi k}{KN}\right)^N,$$

where k is the maximum wavenumber in the solution, i.e., it is proportional to the inverse of the minimum spatial length scale. We shall require that the maximum absolute error, E, is bounded as $E \leqslant \exp(-\gamma)$, and estimate the computational work as

$$W(N,K) = c_1 KN^2 + c_2 KN,$$

where c_1 and c_2 are problem specific constants. Minimizing the work subject to the error constraint yields the optimal values

$$N_{\text{opt}} = \gamma, \quad K_{\text{opt}} = \frac{\pi k}{N_{\text{opt}}} \exp\left(\frac{\gamma}{N_{\text{opt}}}\right).$$

One observes that high accuracy, i.e., γ large, should be achieved by using a large number of modes, N, and not, as one could expect, by employing many subdomains each with a low number of modes. For very smooth and regular functions, where k is small, or if only moderate accuracy is required, the use of many domains may not be the optimal method of choice. On the other hand, if the function exhibits strongly localized phenomena, i.e., k is large, one should introduce many domains to minimize the computational burden. While these arguments are loose, they indicate that an optimal choice of N and K for most problems seems to be a few larger subdomains, each with a reasonable number of modes to maintain an acceptable spatial accuracy.

These results have been confirmed in computational experiments in [48,49,47] indicating that $N=8$–16 is reasonable for two-dimensional problems and $N=4$–8 is reasonable for three-dimensional problems. If this results in insufficient resolution one should generally increase the number of domains rather than the resolution. Similar conclusions have been reached for the analysis of spectral multi-domain methods in a parallel setting [24].

8. A few applications and concluding remarks

Less than a decade ago, the formulation and implementation of robust, accurate and efficient spectral methods for the solution of conservation laws was considered an extremely challenging task. This was partly due to problems of a more theoretical character but partly due also to many practical concerns introduced by the appearance of discontinuous solutions and the need to accurately model the long time behavior of hyperbolic problems in complex geometries.

To illustrate the impact of many of the recent developments discussed in this review, let us conclude by presenting a few contemporary examples of the use of spectral methods for the solution

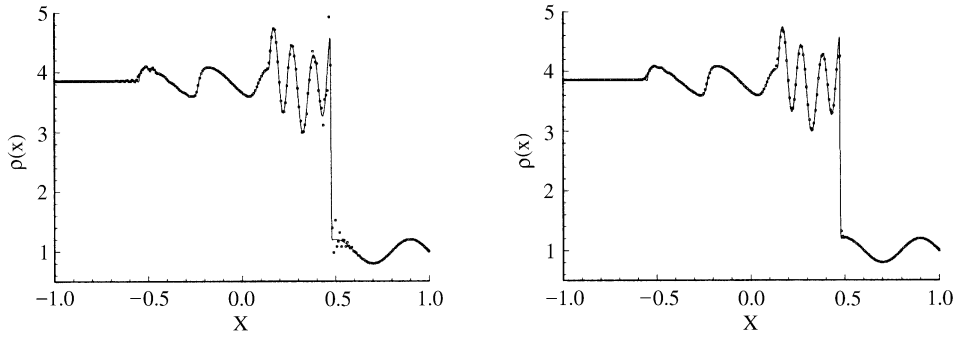

Fig. 1. The solution of the compressible Euler equations for a Mach 3 shock interacting with an entropy wave. On the left is shown the density at $t = 1.8$ computed by using a Chebyshev collocation method with 256 modes and a stabilizing filter. On the right is shown the solution after Gegenbauer reconstruction, removing the effect of the Gibbs phenomenon. The solid line represents the solution computed with a high-order ENO scheme [18,20].

of conservation laws. These few examples all use extensive parts of what we have discussed here and it would indeed have been difficult, if at all possible, to complete the computations without these recent developments.

As a first example, consider the solution of the one-dimensional Euler equations for a compressible gas. We consider the situation where a Mach 3 shock interacts with an entropy wave, producing a strongly oscillatory moving shock [18,20]. Fig. 1 shows the density at $t = 1.8$ computed using a Chebyshev collocation method and compared to the solution obtained using a high-order ENO scheme. A high-order exponential filter is used to stabilize the solution which exhibits strong Gibbs oscillations. Using the Gegenbauer reconstruction technique discussed in Section 3.2, one can recover the nonoscillatory solution with a sharp shock front as is also illustrated in Fig. 1, and a global solution that is essentially identical to that computed using the ENO scheme.

As a second more realistic problem, consider that of shock-induced combustion in which a strong shock, propagating in a oxygen atmosphere, impinges on one or several hydrogen jets, igniting the combustion by compressional heating. This is a problem of great interest to designers of jet engines but also of great difficulty due to the very rich dynamics, the strong shock and the development of very sharp interfaces and flame fronts.

This problem has been studied intensively in [18,20] from which also Fig. 2 is taken. The setting is a strong shock propagating through two aligned hydrogen jets, causing combustion and strong mixing of the fluid. The computation is performed using a two-dimensional Chebyshev collocation method with filtering, a simplified model for the combustion processes and high-order time-stepping. The accuracy as well as the efficiency of the spectral code for combustion problems with strong shocks has been confirmed by detailed comparisons with high-order ENO schemes [21,20].

As a final example, emphasizing geometric complexity rather than shock-induced complications, consider the problem of time-domain electromagnetic scattering by a geometrically very complex metallic scatterer, exemplified in this case by an F-15 fighter (see Fig. 3). The solution of the vectorial Maxwells equations for a problem of such complexity is achieved through the use of novel high-order multi-variate spectral methods defined on tetrahedra [55] with a flexible, stable, and efficient implementation of the penalty method as a discontinuous Galerkin formulation [92,94].

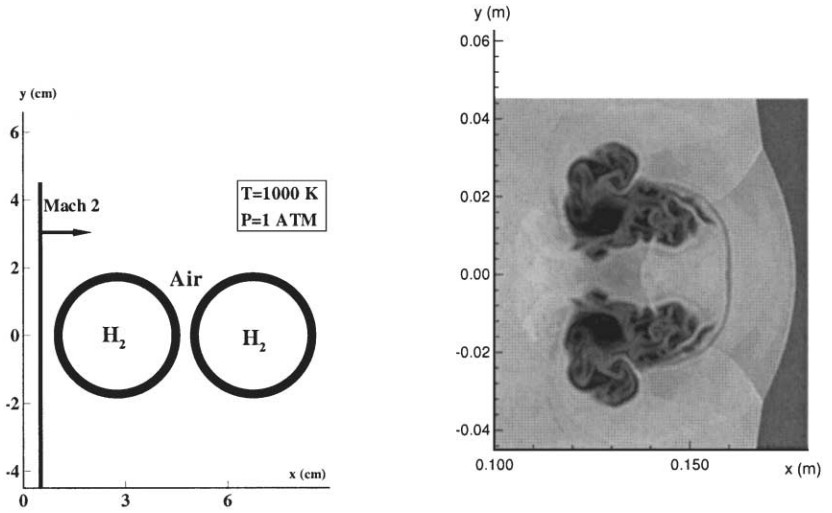

Fig. 2. Example of shock-induced combustion computed using a stabilized Chebyshev collocation method. On the left is shown the initial two-dimensional configuration of two hydrogen jets in an oxygen atmosphere. The left shows a snapshot of the density shortly after the strong shock has passed the jets, showing the very rich dynamics of the mixing process as well as the very complex shock structure [20].

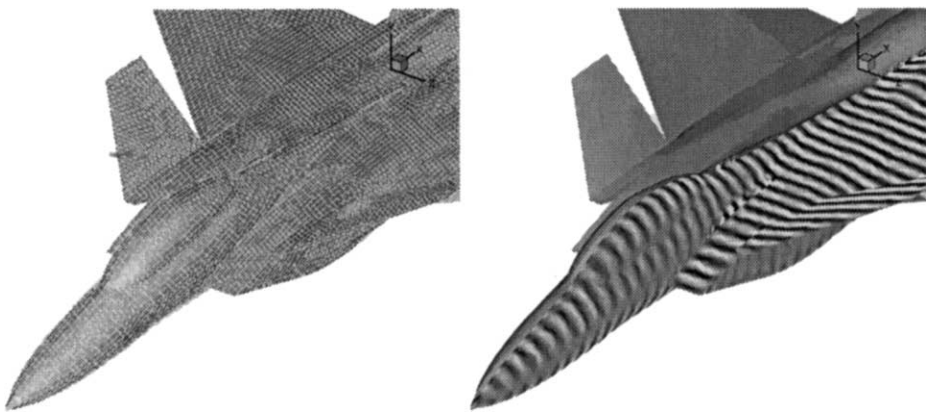

Fig. 3. Application of a fully unstructured spectral multi-domain method to the solution of electromagnetic scattering from an F-15 fighter. The frequency of the incoming plane wave is 600 MHz. On the left is shown a part of the triangulated surface grid and on the right is shown one of the magnetic field components on the surface of the plane. The computation is performed with fourth-order elements and approximately 120.000 tetrahedra to fill the computational volume.

This framework allows for the use of existing unstructured grid technology from finite elements to achieve similar geometric flexibility while it dramatically improves on the accuracy and paves the way for completing reliable simulations of very large scattering and penetration problems. Although the particular example here is for the solution of Maxwells equations, the general computational framework is amenable to the solution of general systems of conservation laws.

While the developments of spectral methods during the last decade have been very significant, a number of critical issues remains open. On the theoretical side, many issues related to stability, even for linear problems, remains open. The results are naturally even more sparse for nonlinear problems. In many instances the experienced user of spectral methods can do much more than can be justified — and often with remarkable success.

Spectral collocation methods have reached a level of sophistication and maturity where it allows for the accurate and efficient solution of nonlinear problems with strong discontinuities using only one domain. For smooth problems, one the other hand, the development of multi-domain formulations has reached a level where it allows for the efficient and accurate solution of problems of almost arbitrary geometric complexity. One of the great challenges of the immediate future lies in understanding how to do strongly nonlinear conservation laws in complex geometries in a stable manner using a spectrally accurate, geometrically flexible, and computationally efficient formulation.

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