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# On hypergeometric functions and function spaces

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## Abstract

The aim of this paper is to discuss the role of hypergeometric functions in function spaces and to prove some new results for these functions. The first part of this paper proves results such as monotone, convexity and concavity properties of sums of products of hypergeometric functions. The second part of our results deals with the space  $\mathcal{A}$  of all normalized analytic functions  $f$ ,  $f(0)=0=f'(0)-1$ , in the unit disk  $\Delta$  and the subspace

$$\mathcal{R}(\beta) = \{f \in \mathcal{A} : \exists \eta \in \mathbb{R} \text{ such that } \operatorname{Re} e^{i\eta}(f'(z) - \beta) > 0, z \in \Delta\}.$$

For  $f \in \mathcal{A}$ , we consider integral transforms of the type

$$V_\lambda(f) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where  $\lambda(t)$  is a real valued nonnegative weight function normalized so that  $\int_0^1 \lambda(t) dt = 1$ . We obtain conditions on  $\beta$  and the function  $\lambda$  such that  $V_\lambda(f)$  takes each member of  $\mathcal{R}(\beta)$  into a starlike function of order  $\beta$ ,  $\beta \in [0, 1/2]$ . These results extend and improve the earlier known results in these directions. We end the paper with an open problem. © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction and main results

The familiar hypergeometric function defined by the series

$${}_2F_1(a, b; c; z) := F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^n, \quad (1.1)$$

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is analytic in the unit disc  $\Delta = \{z: |z| < 1\}$ . It arises naturally in the study of second order linear differential equations with regular singular points. In (1.1),  $(a, 0) = 1$  for  $a \neq 0$  and the rising factorial notation

$$(a, n) = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \geq 1,$$

is used. To avoid division by 0, the parameter  $c$  in (1.1) should be neither 0 nor a negative integer. If  $a$  or  $b$  is 0 or a negative integer, then the power series reduces to a polynomial. The theory of hypergeometric functions has found many applications and generalizations [4,6,7,19,20,31,50] and the study of this theory acquired an independent status, see for more detail the recent book [6]. Until recently, there have been few attempts to look at the interconnectivity of the special functions with geometric function theory. The use of hypergeometric functions in the proof of the Bieberbach conjecture by de Branges has given function theorists a renewed interest to study the role of special functions. We use certain basic facts about hypergeometric functions (see e.g. [6,24,48,51]). These are the Euler representation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt \quad (\operatorname{Re} c > \operatorname{Re} b > 0),$$

the beta function formula

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{b-1} (1-t)^{a-1} dt \quad (\operatorname{Re} a > 0, \operatorname{Re} b > 0)$$

and asymptotic behaviour of  $F(a, b; c; z)$  near  $z = 1$ :

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty, \quad \operatorname{Re} c > \operatorname{Re}(a+b),$$

$$\lim_{z \rightarrow 1^-} \frac{F(a, b; a+b; z)}{\log(1/(1-z))} = \frac{1}{B(a, b)}, \quad \operatorname{Re} c = \operatorname{Re}(a+b),$$

$$\lim_{z \rightarrow 1^-} \frac{F(a, b; a+b; z)}{(1-z)^{c-a-b}} = \frac{B(c, a+b-c)}{B(a, b)} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad \operatorname{Re} c < \operatorname{Re}(a+b).$$

The particular case  $c = a+b$  is called *zero-balanced*. When  $z = x$ ,  $x \in (0, 1)$ , cases for  $\operatorname{Re} c \leq \operatorname{Re}(a+b)$  the above results have been extended and, in fact the results about the asymptotic approximation have been improved in [2,45] (see also [7,12]).

Some of the results which also have impact on other areas such as number theory and algebraic geometry, are related to the work of the Indian genius S. Ramanujan on hypergeometric functions. The unpublished results of S. Ramanujan were edited with reconstructed proofs by Berndt (5 volumes) in 1985–1996 and these results now become widely accessible. In this paper, we prove some inequalities for hypergeometric functions of a real variable and study inclusion relations between various subclasses of univalent functions. Some of our main results are Theorems 1.4, 3.8, 3.15 and 3.16. We now give some necessary preliminaries.

Recall that the complete elliptic integrals of the first and second kind defined by the functions  $\mathcal{K}$  and  $\mathcal{E}$  for  $r \in [0, 1)$  are

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2 \phi}},$$

$$\mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 \phi} d\phi$$

and their complements are defined by  $\mathcal{K}'(r) = \mathcal{K}(r')$  and  $\mathcal{E}'(r) = \mathcal{E}(r')$  where  $r^2 + r'^2 = 1$ . An important property of these integrals is described by the Legendre relation

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}$$

and a generalization of this due to Elliott [22] is

$$\begin{aligned} & F\left(\frac{1}{2} + \lambda, -\frac{1}{2} - \nu; 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \nu + \mu; 1 - r\right) \\ & + F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu; 1 + \lambda + \mu; r\right) F\left(-\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \nu + \mu; 1 - r\right) \\ & - F\left(\frac{1}{2} + \lambda, \frac{1}{2} - \nu; 1 + \lambda + \mu; r\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \nu; 1 + \nu + \mu; 1 - r\right) \\ & = \frac{\Gamma(1 + \lambda + \mu)\Gamma(1 + \nu + \mu)}{\Gamma\left(\lambda + \mu + \nu + \frac{3}{2}\right)\Gamma\left(\frac{1}{2} + \mu\right)}. \end{aligned}$$

Clearly, the choice  $\lambda = \mu = \nu = 0$  gives the Legendre relation. Another natural generalization of the Legendre relation was suggested in [3] by an  $L$ -function which we define as follows:

Let  $a, b, c > 0$ ,  $u(r) = F(a - 1, b; c; r)$ ,  $v(r) = F(a, b; c; r)$ , and

$$\mathcal{L}(a, b, c, r) = u(r)v(1 - r) + u(1 - r)v(r) - v(r)v(1 - r), \quad r \in (0, 1). \quad (1.2)$$

In [3, Conjecture 3.16], Anderson et al. presented the following.

**Conjecture 1.1.** For  $a, b \in (0, 1)$ ,  $a + b \leq 1$  ( $\geq 1$ ),  $\mathcal{L}(a, b, c, r)$  is concave (convex) as a function of  $r$  on  $(0, 1)$ .

Motivated by this conjecture, Karatsuba and Vuorinen [30] proved the following results.

**Theorem 1.2.** Let  $a, b > 0$ . Then

- (i)  $\mathcal{L}(r)$  has only one extremum  $r = 1/2$ ,
- (ii) For  $c > b$  with  $a + b > 1$ , or  $-1 < c \neq 0 < b$  with  $a + b < 1$ , we have  $\mathcal{L}'(r) > 0$  for  $r > 1/2$  and  $\mathcal{L}'(r) < 0$  for  $r < 1/2$ ,
- (iii) For  $-1 < c \neq 0 < b$  with  $a + b > 1$ , or  $c > b$  with  $a + b < 1$ , we have  $\mathcal{L}'(r) > 0$  for  $r < 1/2$  and  $\mathcal{L}'(r) < 0$  for  $r > 1/2$ .

In [30], Theorem 1.2 was stated for  $c > 0$ . However, from the proof of Theorem 1.2 it is clear that the condition  $c > 0$  can be replaced by  $0 \neq c > -1$ . Now, we recall another result from [30].

**Theorem 1.3.** Let  $a, b, c > 0$ .

- (1) For  $c + 1 > ab/(a + b + 1)$ ,
  - (a) if  $c > b, a + b > 1$  or  $c < b, a + b < 1$ , then  $\mathcal{L}(a, b, c, r)$  is strictly convex,
  - (b) if  $c > b, a + b < 1$  or  $c < b, a + b > 1$ , then  $\mathcal{L}(a, b, c, r)$  is strictly concave.
- (2) (a)  $\mathcal{L}(a, b, c, r) > 0$  for  $c > b$ ,  
 (b)  $\mathcal{L}(a, b, c, r) < 0$  for  $c < b$ ,  
 (c)  $\mathcal{L}(a, b, b, r) = \mathcal{L}(a, c, c, r) = 0$ .
- (3)  $\mathcal{L}(a, b, c, r)$  is constant for  $a + b = 1$ .
- (4)  $\mathcal{L}(a, b, c, 1/2)$  is the unique extremum of the function  $\mathcal{L}(a, b, c, r)$ .

We improve part (1) in Theorem 1.4 in the following way:

**Theorem 1.4.** Let  $a, b, c > -1$  and let  $\alpha = \alpha(a, b) \geq 2$  be defined by

$$\alpha(a, b) = \min \left\{ p + \frac{2(a+b)+3}{p-1} + \frac{2ab}{p(p-1)} : p \geq 2 \right\}. \quad (1.3)$$

We have

- (i) if either  $c > b$  with  $a + b > 1$ , or  $6ab/(39 + 12(a + b) + 2ab) - 1 < c < b$  with  $a + b < 1$  then  $\mathcal{L}(r)$  is strictly convex for  $r \in (0, 1)$ .
- (ii) if either  $c > b$  with  $a + b < 1$ , or  $ab/(a + b + 2 + \alpha) - 1 \leq c < b$  with  $a + b > 1$  then  $\mathcal{L}(r)$  is strictly concave for  $r \in (0, 1)$ .

The proof of (2)–(4) in Theorem 1.3 is fairly simple and hence, we do not require details. In Section 2, we give a simple proof of Theorem 1.2 but with the help of the outline given in [30]. We also provide a straightforward proof of Theorem 1.4. Further, it does not seem to be easy to state the precise value for the right-hand side expression of (1.3) because of the fact that the terms involved inside the brackets depend on  $a, b$ . However, for example, for all  $a, b > 0$ , it is easy to see that

$$p + \frac{2(a+b)+3}{p-1} + \frac{2ab}{p(p-1)} \geq p + \frac{2(a+b)+3}{p-1} \geq 1 + 2\sqrt{2(a+b)+3}$$

and therefore  $\alpha(a, b) \geq 1 + 2\sqrt{2(a+b)+3}$  is a lower bound for  $\alpha(a, b)$ . Also, for  $a, b > 0$ , one can obtain a lower bound for  $\alpha(a, b)$  by observing the fact that

$$p + \frac{2(a+b)+3}{p-1} + \frac{2ab}{p(p-1)} \geq p + \frac{2ab}{p(p-1)} \geq 3 \left( \frac{ab}{2} \right)^{1/3} + \frac{1}{2}.$$

## 2. Proofs of generalization of Legendre relation

Throughout this section, we let  $\mathcal{L}(r) = \mathcal{L}(a, b, c, r)$ . Clearly, (1.2) can be rewritten as

$$\mathcal{L}(r) = u(r)u(1-r) - V(r)V(1-r), \quad (2.1)$$

where

$$V(r) = v(r) - u(r) \quad \text{i.e., } v(r) = V(r) + u(r). \quad (2.2)$$

The series expansion of the hypergeometric function gives

$$\begin{aligned} V(r) &= F(a, b; c; r) - F(a - 1, b; c; r) \\ &= \sum_{n=1}^{\infty} \frac{(b, n)}{(c, n)(1, n)} [(a, n - 1)\{(a + n - 1) - (a - 1)\}] r^n \\ &= \frac{br}{c} F(a, b + 1; c + 1; r). \end{aligned}$$

Moreover, the power series expansion of  $u(r)$  and  $v(r)$  imply that [3, Theorem 3.12(1), (2)]

$$r \frac{du(r)}{dr} = (a - 1)[v(r) - u(r)] = (a - 1)V(r) \quad (2.3)$$

and

$$r(1 - r) \frac{dv(r)}{dr} = (a - c)V(r) + brv(r) \quad \text{i.e., } r(1 - r)v'(r) = (a - c)V(r) + brv(r). \quad (2.4)$$

Differentiating (2.1) we find that

$$\mathcal{L}'(r) = u'(r)u(1 - r) - u(r)u'(1 - r) - V'(r)V(1 - r) + V(r)V'(1 - r). \quad (2.5)$$

In view of (2.2) and (2.3), (2.4) yields that

$$r(1 - r)V'(r) = \{1 - c + (a - 1)r\}V(r) + brv(r). \quad (2.6)$$

Using the derivative formulas (2.3) and (2.6), we can rewrite (2.5) as

$$\begin{aligned} \mathcal{L}'(r) &= \left\{ \frac{(a - 1)V(r)}{r} \right\} u(1 - r) - u(r) \left\{ \frac{(a - 1)V(1 - r)}{1 - r} \right\} \\ &\quad - \left\{ \frac{V(r)[1 - c + (a - 1)r] + brv(r)}{r(1 - r)} \right\} V(1 - r) \\ &\quad + V(r) \left\{ \frac{V(1 - r)[1 - c + (a - 1)(1 - r)] + b(1 - r)v(1 - r)}{r(1 - r)} \right\}. \end{aligned}$$

Using the fact that  $u(r) = v(r) - V(r)$ , we can simplify the last equality to obtain

$$r(1 - r)\mathcal{L}'(r) = (a + b - 1)[(1 - r)v(1 - r)V(r) - rv(r)V(1 - r)]$$

or equivalently,

$$\mathcal{L}'(r) = (a + b - 1) \left[ \frac{v(1 - r)V(r)}{r} - \frac{v(r)V(1 - r)}{1 - r} \right].$$

Since  $V(r) = (b/c)rF(a, b+1; c+1; r)$ , the last equation for  $\mathcal{L}'(r)$  can be rewritten in the following convenient form

$$\mathcal{L}'(r) = \frac{(a+b-1)b}{c} [F(a, b+1; c+1; r)F(a, b; c; 1-r) - F(a, b; c; r)F(a, b+1; c+1; 1-r)]. \quad (2.7)$$

This form of representation helps us to get the proof of Theorems 1.2 and 1.4.

**Proof of Theorem 1.2.** Using the power series representation of the hypergeometric functions involved in (2.7) we obtain

$$\mathcal{L}'(r) = (a+b-1) \sum_{k,l \geq 0} [B_{k,l} - B_{l,k}] r^k (1-r)^l,$$

where

$$B_{k,l} = \frac{b(a,k)(b+1,k)(a,l)(b,l)}{c(c+1,k)(1,k)(c,l)(1,l)}$$

and, by the definition of the ascending factorial notation, we see that

$$B_{k,l} - B_{l,k} = (c-b) \left[ \frac{(a,k)(b,k)}{(c,k+1)(1,k)} \frac{(a,l)(b,l)}{(c,l+1)(1,l)} (k-l) \right] := (c-b)A_{k,l}, \quad \text{say.}$$

Therefore, we find that

$$\begin{aligned} \mathcal{L}'(r) &= (a+b-1)(c-b) \sum_{k,l \geq 0} A_{k,l} r^k (1-r)^l \\ &= (a+b-1)(c-b) \sum_{l \geq 0} \left( \sum_{k > l} + \sum_{k < l} \right) A_{k,l} r^k (1-r)^l \\ &= (a+b-1)(c-b) \sum_{l \geq 0} \sum_{k > l} A_{k,l} \{r^k (1-r)^l - r^l (1-r)^k\}. \end{aligned}$$

Now, for  $k > l$ , we obtain

$$\begin{aligned} r^k (1-r)^l - r^l (1-r)^k &= r^l (1-r)^l [r^{k-l} - (1-r)^{k-l}] \\ &= r^l (1-r)^l (2r-1) \sum_{n=1}^{k-l} r^{k-l-n} (1-r)^{n-1}, \end{aligned}$$

which shows

$$\mathcal{L}'(r) \begin{cases} = 0 & \text{for all } r = 1/2, \\ > 0 & \text{for all } 0 < r < 1/2 \text{ and for } (a+b-1)(c-b) < 0, \\ > 0 & \text{for all } 1/2 < r < 1 \text{ and for } (a+b-1)(c-b) > 0, \\ < 0 & \text{for all } 0 < r < 1/2 \text{ and for } (a+b-1)(c-b) > 0, \\ < 0 & \text{for all } 1/2 < r < 1 \text{ and for } (a+b-1)(c-b) < 0, \end{cases}$$

and the proof is complete.  $\square$

**Remark 2.1.** It is easy to see that

$$bF(a, b+1; c+1; r) = cF(a, b; c; r) - (c-b)F(a, b; c+1; r).$$

In fact, the last identity is clear if we compare the coefficients of  $r^n$  on both sides of it. Substituting this value of  $bF(a, b+1; c+1; r)$  and the corresponding value for  $bF(a, b+1; c+1; 1-r)$  in (2.7), we find that (2.7) is equivalent to

$$\mathcal{L}'(r) = \frac{(a+b-1)(c-b)}{c} [F(a, b; c; r)F(a, b; c+1; 1-r) - F(a, b; c+1; r)F(a, b; c; 1-r)].$$

It is interesting to point out that if  $a+b=1$ , then  $\mathcal{L}'(r)=0$  for all  $r \in (0, 1)$  so that  $\mathcal{L}(a, 1-a, c, r)$  is constant. However, it is proved in [3, Corollary 3.13(5)] that

$$\mathcal{L}(a, 1-a, c, r) = \frac{\Gamma^2(c)}{\Gamma(c+a-1)\Gamma(c-a+1)}$$

which is clearly a generalization of the Legendre relation.

**Proof of Theorem 1.4.** Differentiating (2.7), we find that

$$\begin{aligned} \mathcal{L}''(r) &= (a+b-1) \left[ \frac{ab(b+1)}{c(c+1)} \{F(a+1, b+2; c+2; r)F(a, b; c; 1-r) \right. \\ &\quad \left. + F(a, b; c; r)F(a+1, b+2; c+2; 1-r)\} \right. \\ &\quad \left. - \frac{ab^2}{c^2} \{F(a, b+1; c+1; r)F(a+1, b+1; c+1; 1-r) \right. \\ &\quad \left. + F(a+1, b+1; c+1; r)F(a, b+1; c+1; 1-r)\} \right] \\ &= (a+b-1) \sum_{k, l \geq 0} [(C_{k, l} + C_{l, k}) - (D_{k, l} + D_{l, k})] r^k (1-r)^l, \end{aligned}$$

where

$$C_{k, l} = \frac{ab(b+1)}{c(c+1)} \left( \frac{(a+1, k)(b+2, k)}{(c+2, k)(1, k)} \right) \left( \frac{(a, l)(b, l)}{(c, l)(1, l)} \right)$$

and

$$D_{k, l} = \frac{ab^2}{c^2} \left( \frac{(a, k)(b+1, k)}{(c+1, k)(1, k)} \right) \left( \frac{(a+1, l)(b+1, l)}{(c+1, l)(1, l)} \right).$$

Again, by the definition of the ascending factorial notation, it is easy to see that

$$(C_{k, l} + C_{l, k}) - (D_{k, l} + D_{l, k}) = \frac{(a, k)(b, k)}{(c, k+2)(1, k)} \frac{(a, l)(b, l)}{(c, l+2)(1, l)} [E(k, l) + E(l, k)]$$

with

$$\begin{aligned} E(k, l) &= (a+k)(b+k)(b+k+1)(c+l)(c+l+1) \\ &\quad - (a+k)(b+l)(b+k)(c+l+1)(c+k+1) \\ &= (c-b)(a+k)(b+k)(c+l+1)(k-l+1) \end{aligned}$$

so that

$$\begin{aligned} E(k, l) + E(l, k) &= (c-b)[(k-l)\{(a+k)(b+k)(c+l+1) - (a+l)(b+l)(c+k+1)\} \\ &\quad + \{(a+k)(b+k)(c+l+1) + (a+l)(b+l)(c+k+1)\}] \\ &= (c-b)F(k, l), \end{aligned}$$

where

$$\begin{aligned} F(k, l) &= (k-l)^2\{(c+1)(k+l) + (c+1)(a+b) + lk - ab\} \\ &\quad + \{(a+k)(b+k)(c+l+1) + (a+l)(b+l)(c+k+1)\}. \end{aligned}$$

Therefore, we have

$$\mathcal{L}''(r) = \frac{(a+b-1)(c-b)}{c^2(c+1)^2} \sum_{k, l \geq 0} \left( \frac{(a, k)(b, k)}{(c+2, k)(1, k)} \frac{(a, l)(b, l)}{(c+2, l)(1, l)} \right) F(k, l) r^k (1-r)^l.$$

We claim that  $F(k, l) > 0$  for all  $k, l \geq 0$ , under the hypotheses. For this we first observe that, as  $F(k, l)$  is symmetric over  $k$  and  $l$ , it suffices to assume that  $k \geq l$ . Clearly for  $k = l$ , we have  $F(k, k) > 0$  and therefore, we can assume that  $k = l + p$ ,  $p \geq 1$ . Also,  $F(l + p, l)$  is clearly an increasing function of  $l$  so that  $F(l + p, l) \geq F(p, 0)$ , where

$$\begin{aligned} F(p, 0) &= (c+1)p^3 + ((c+1)(a+b+1) - ab)p^2 \\ &\quad + ((c+1)(a+b+1) + ab)p + 2(c+1)ab. \end{aligned}$$

Therefore, since  $F(1, 0) > 0$ , it suffices to show that  $F(p, 0) \geq 0$  for all  $p \geq 2$ .

If  $c > b$ , then it is clear that each of the coefficients of the polynomial is positive and therefore,  $F(p, 0) > 0$  for all  $p \geq 2$ . This observation suggests to look at the remaining cases.

*Case 1:* Let  $a+b < 1$  and  $-1 < c \neq 0$ . First, we observe that the inequality  $F(p, 0) \geq 0$  is equivalent to

$$c+1 \geq ab\phi(p), \tag{2.8}$$

where

$$\phi(p) = \frac{p^2 - p}{p^3 + (a+b+1)p^2 + (a+b+1)p + 2ab}, \quad p \geq 2.$$

It can be easily seen that the function  $\phi$  satisfies the condition  $\phi(p) < \phi(3)$  for every  $p \geq 2$ . It is a simple exercise to verify this inequality for  $p = 2, 4, 5, 6$ . For  $p \geq 7$ , we write the equivalent form



of  $\phi(p) < \phi(3)$  as

$$\psi(p) = 6p^3 + [6 - 6(a+b) - 39 - 2ab]p^2 + [45 + 18(a+b) + 2ab]p + 12ab \geq 0.$$

Clearly, to verify this inequality for  $p \geq 7$ , it suffices to replace  $p^3$  by  $7p^2$  (since  $p^3 \geq 7p^2$ ) and use the fact that  $a+b < 1$  and  $ab < 1$ . Thus, if we replace  $p^3$  by  $7p^2$ , then the resulting equation is clearly positive. Thus,

$$c \geq \phi(3) = \frac{6ab}{39 + 12(a+b) + 2ab} - 1,$$

is a sufficient condition for  $F(p, 0) > 0$  for all  $p \geq 1$ .

Case 2: Let  $a+b > 1$  and  $-1 < c \neq 0$ . Note that

$$\frac{1}{\phi(p)} = a+b+2+p + \frac{2(a+b)+3}{p-1} + \frac{2ab}{p(p-1)}, \quad p \geq 2.$$

By (1.3), we have

$$p + \frac{2(a+b)+3}{p-1} + \frac{2ab}{p(p-1)} \geq \alpha(a, b) \geq 2, \quad p \geq 2$$

so that  $\phi(p) \geq 1/(a+b+2+\alpha)$  and the theorem follows.  $\square$

### 3. Mapping properties

Let  $\mathcal{H}$  denote the space of all analytic functions in the unit disk  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  with the topology of uniform convergence on compact subsets and denote by  $\mathcal{A}$  the subspace of  $\mathcal{H}$  with the usual normalization  $f(0) = f'(0) - 1 = 0$ . The class  $\mathcal{A}$  has been studied extensively, together with its subclass of univalent (schlicht) functions, denoted by  $\mathcal{S}$ . See the books [21, 28, 29, 35, 36] and the bibliography of Bernardi [14]. The remarkable result of de Branges [15] shows that for each  $n \geq 2$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S} \Rightarrow |a_n| \leq n$$

settling a conjecture of L. Bieberbach from 1916 and the proof of this conjecture relies on hypergeometric functions. One of the fundamental questions in the theory of univalent functions is to ask for coefficient univalence criteria, that is, results converse to the de Branges theorem.

**Problem 3.1.** Find conditions on the Maclaurin coefficients  $a_n$  of  $f \in \mathcal{A}$  that ensure the membership of  $f$  in  $\mathcal{S}$  and also  $f$  in some of its interesting subclasses.

We now give a brief outline of the known results on Problem 3.1. Along with the classes  $\mathcal{A}$  and  $\mathcal{S}$  several subclasses of  $\mathcal{S}$  have been widely studied. Two such subclasses are

$$\mathcal{H}(\beta) = \left\{ f \in \mathcal{A} \mid \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, z \in \Delta \right\}, \quad \beta < 1,$$

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} \mid \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta, z \in \Delta \right\}, \quad \beta < 1.$$

These classes are called *convex of order  $\beta$*  and *starlike of order  $\beta$* , respectively. If  $\beta = 0$  these classes are called just convex and starlike and denoted by  $\mathcal{K}$  and  $\mathcal{S}^*$ , respectively. The names of the classes  $\mathcal{K}$  and  $\mathcal{S}^*$  correspond to the geometric properties of the image domains of the functions in these classes (see [21, p. 41, Theorem 2.10 and p. 42, Theorem 2.11] and [28, p. 111, Theorem 1]). Given a convex function  $g \in \mathcal{K}$  and  $\beta < 1$ , set

$$\mathcal{C}_g(\beta) = \left\{ f \in \mathcal{A} \mid \exists \eta \in \mathbb{R} \text{ such that } \operatorname{Re} \left( e^{i\eta} \frac{f'(z)}{g'(z)} \right) > \beta, z \in \Delta \right\}. \quad (3.2)$$

Now,  $\mathcal{C}_g(0) \equiv \mathcal{C}_g$  is the class of functions *close-to-convex* with respect to  $g$ . Let  $\mathcal{C} = \{\mathcal{C}_g; g \in \mathcal{K}\}$  denote the class of all close-to-convex functions. The strict inclusions

$$\mathcal{K} \subsetneq \mathcal{S}^* \subsetneq \mathcal{C} \subsetneq \mathcal{S} \quad (3.3)$$

hold [21,28]. For  $\beta < 1$ , we also introduce the class

$$\mathcal{P}(\beta) = \{ p \in \mathcal{H} : \exists \eta \in \mathbb{R} \text{ such that } p(0) = 1, \operatorname{Re}[e^{i\eta}(p(z) - \beta)] > 0, z \in \Delta \}$$

and define

$$\mathcal{R}(\beta) = \{ f \in \mathcal{A} : f'(z) \in \mathcal{P}(\beta) \}.$$

For  $0 \leq \beta < 1$ , we have that  $\mathcal{R}(\beta)$  is included in  $\mathcal{C}$ , but not in  $\mathcal{S}^*$ , and neither is the smaller class  $\mathcal{R}(\beta)$ . The question about inclusion of  $\mathcal{R}(\beta)$  (with  $\eta = 0$ ) in  $\mathcal{S}^*$  was raised by Zmorovič [52], and settled in the negative through an example by Krzyż [33].

The well-known univalence criteria for functions  $f \in \mathcal{A}$  can be divided roughly into two different types (for an extensive survey of univalence results, see [8–10]). The results of the first type give various sufficient conditions for  $f \in \mathcal{A}$  to be a member in one of the classes  $\mathcal{K}, \mathcal{S}^*, \mathcal{C}$  and such functions will be clearly univalent in view of (3.3). There are also studies pertaining to individual special functions and their membership in these classes.

The theorems of the second type give univalence criteria not in terms of the Maclaurin coefficients but in terms of conditions involving the Schwarzian derivative

$$S_f(z) = \frac{d}{dz} \left( \frac{f''(z)}{f'(z)} \right) - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2,$$

such as Nehari's condition

$$|S_f(z)(1 - |z|^2)| \leq 2, \quad |z| < 1,$$

or  $f''(z)/f'(z)$  such as Becker's condition (see [13,36])

$$(1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| < 1, \quad |z| < 1,$$

or some other quantities such as Epstein's [23] univalence criteria, a generalization of the last two criteria. There are many more important results of this type e.g. those due to Anderson and Hinkkanen [5], Chuaqui and Osgood [17] and see also [18]. It is clear that the above two univalence criteria impose constraints on the coefficients  $a_n$ . On the other hand, we do not know any conditions on the coefficients that would imply one of these conditions to become valid.

Our second problem deals with the stability of the subclasses of  $\mathcal{S}$  under a small perturbation. Observe first that condition (3.2) measures the nearness of  $f$  and  $g$ :  $f \in \mathcal{C}_g$  means that  $f$  is in some sense close to  $g$  and, in particular,  $f \in \mathcal{C}_f$ . Thus, a refined version of Problem 3.1 is the following one.

**Problem 3.2.** *Given a convex function  $g$  find sufficient conditions for  $f \in \mathcal{A}$  to be in  $\mathcal{C}_g$  in terms of the coefficients of the Maclaurin series of  $f$  and  $g$ .*

Many results on Problem 3.2 follow from the work of Ozaki [34]. More recently, Ponnusamy [38] extended the results of Ozaki [34] and proved the following interesting theorems which contain results concerning certain monotone conditions on the coefficients  $a_n$  when these coefficients are real and nonnegative. Ozaki's results are the case  $\beta = 0$  and  $\eta = 0$  in Lemma 3.3.

**Lemma 3.3.** *Let  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ . Then we have the following:*

- (i)  $\sum_{n \geq 1} |na_n - (n+1)a_{n+1}| \leq (1-\beta) \cos \eta$  implies that  $f \in \mathcal{C}(\beta; g)$  with  $g(z) = z/(1-z)$ .
- (ii)  $\sum_{n \geq 1} |(n-1)a_{n-1} - 2na_n + (n+1)a_{n+1}| \leq (1-\beta) \cos \eta$  implies that  $f \in \mathcal{C}(\beta; g)$  with  $g(z) = z/(1-z)^2$ .
- (iii)  $\sum_{n \geq 1} |(n-1)a_{n-1} - (n+1)a_{n+1}| \leq (1-\beta) \cos \eta$  implies that  $f \in \mathcal{C}(\beta; g)$  with  $g(z) = z/(1-z^2)$ .
- (iv)  $\sum_{n=1}^{\infty} |(n-1)a_{n-1} - na_n + (n+1)a_{n+1}| \leq (1-\beta) \cos \eta$  implies that  $f \in \mathcal{C}(\beta; g)$  with  $g(z) = z/(1-z+z^2)$ .

An important subclass of  $\mathcal{A}$  is described in the following classical result of Fejér [25] which deals with the case  $g(z) = z$ .

**Lemma 3.4** (Fejér [25]). *Let  $\{a_n\}$  be a sequence of nonnegative real numbers such that  $a_1 = 1$ , and that for  $n \geq 2$  the sequence  $\{a_n\}$  is a convex decreasing, i.e.  $0 \geq a_{n+2} - a_{n+1} \geq a_{n+1} - a_n$ , for all  $n \in \mathbb{N}$ . Then  $\operatorname{Re}\{\sum_{n=1}^{\infty} a_n z^{n-1}\} > 1/2$  for all  $z \in \Delta$ .*

We remark that the  $g$ 's in Lemmas 3.3 and 3.4 are associated with the functions from the set given by

$$\left\{ z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\}$$

and these are the only functions in  $\mathcal{S}$  having integer coefficients in the power series expansion of  $f \in \mathcal{S}$  (see [27]). We remark that each of these functions maps the disc  $\Delta$  onto a starlike domain. Further, Fejér [25] proved the following coefficient criteria for starlike functions.

**Lemma 3.5** (Fejér [25]). *Let  $\{a_n\}$  be a sequence of nonnegative real numbers such that  $a_1 = 1$ , and that, for  $n \geq 1$ , the quantities*

$$\underline{\Delta}a_n := na_n - (n+1)a_{n+1} \quad \text{and} \quad \underline{\Delta}a_n^2 := na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2}$$

*are nonnegative. Then  $f(z) = \sum_{n=1}^{\infty} a_n z^n$  is starlike in  $\Delta$ .*

Monotonicity of the sequence  $\{a_n\}$ ,  $a_n = (a, n)(b, n)/((c, n)n!)$ , was used in [2,45] to derive explicit bounds for the asymptotic behaviour of  $F(a, b; c; x)$  near  $x = 1$ . Several applications of the last three lemmas to special functions were obtained in a series of papers in [38–40,46,47,42,43] (see also [1]).

Recall that for two functions  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  and  $g(z) = \sum_{k=0}^{\infty} b_k z^k$  in  $\mathcal{H}$  we define the usual Hadamard product, or convolution, of  $f$  and  $g$  as  $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$ . Note that  $\mathcal{H} * \mathcal{H} \subset \mathcal{H}$  and  $\mathcal{A} * \mathcal{A} \subset \mathcal{A}$ . The Koebe function immediately shows that  $\mathcal{S} * \mathcal{S}$  is not included in  $\mathcal{S}$ .

Consider integral transforms of the type

$$V_\lambda(f) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad (3.4)$$

where  $\lambda(t)$  is a real valued nonnegative weight function normalized so that  $\int_0^1 \lambda(t) dt = 1$ . Properties of this integral transform have been studied under suitable restriction on the  $\lambda$ -function, see [26,40,41,11]. This interesting linear operator was also studied by Fournier and Ruscheweyh [26] to solve a special problem when  $\lambda(t) = (1 + \gamma)t^\gamma$ ,  $\gamma > -1$ , and the method used by them is the duality theory developed mainly by Ruscheweyh (see [49]). We want to look at a more general integral operator with this setting. For example, consider the convolution operator (see [39]) as taking the convolution between functions  $f(z) = \sum_{k=1}^{\infty} a_k z^k$  in  $\mathcal{A}$  and a normalized hypergeometric functions of the form  $zF(a, b; c; z)$ :

$$H_{a,b;c}(f(z)) := [H_{a,b;c}(f)](z) = zF(a, b; c; z) * f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} a_{n+1} z^{n+1}.$$

Note that  $H_{a,b;c}(z/(1-z)) = zF(a, b; c; z)$ . For  $a = 1$ , the Euler's representation immediately gives

$$H_{1,b;c}(f(z)) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \frac{f(tz)}{t} dt, \quad \operatorname{Re} c > \operatorname{Re} b > 0. \quad (3.5)$$

In [16], Carlson–Shaffer studied this operator under certain restrictions on the parameters  $b, c$ . On the other hand, the more general operator  $H_{a,b;c}(f)$  is well studied (see [39–41]). The simplest cases,  $H_{1,1;2}(f)$ ,  $H_{1,2;3}(f)$  and  $H_{1,b;b+1}(f)$ ,  $b > 0$ , are popularly known as Alexander, Libera and Bernardi operators, respectively. It is well known that

$$H_{1,1;2}(\mathcal{S}) \not\subset \mathcal{S}, \quad H_{1,2;3}(\mathcal{S}) \not\subset \mathcal{S}.$$

Here it is interesting to raise the following

**Problem 3.6.** Find the exact range of the triplets  $(a, b, c)$  so that  $H_{a,b;c}(\mathcal{S}) \subset \mathcal{S}$ .

Partial answer for this problem is found in [38,39,47]. Now, we recall the following well-known result from the work of Balasubramanian and Ponnusamy in [11].

**Theorem 3.7** (Balasubramanian and Ponnusamy [11, Theorem 1.4]). Let  $b, c > 0$  and  $\gamma \in [0, 1/2]$ . Suppose that  $b, c, \gamma$  are related by any one of the following conditions:

- (i)  $b \in (0, 3]$  and  $c \geq b + 1$  with  $\gamma = 0$ ,
- (ii)  $b \in (0, 2]$  and  $c \geq b + 1 + 2\gamma$  with  $\gamma \in (0, 1/2]$ .

If  $\beta_{b,c}$  is given by

$$\frac{\beta_{b,c}}{1 - \beta_{b,c}} = - \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \left[ \frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{(1-\gamma)} \frac{\log(1+t)}{t} \right] dt,$$

then  $H_{1,b;c}(\mathcal{R}(\beta)) \subset \mathcal{S}^*(\gamma)$ .

Now, our aim is to give a more general form of Theorem 3.7. For this, we define  $\phi$  by

$$\phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n,$$

where  $b_n \geq 0$  for each  $n \geq 1$ , and consider

$$P_{a,b;c}(f(z)) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt \quad \text{with } \lambda(t) = Ct^{b-1}(1-t)^{c-a-b}\phi(1-t),$$

where  $C$  is a constant satisfying the condition that  $\int_0^1 \lambda(t) dt = 1$ .

**Theorem 3.8.** Let  $a, b, c > 0$ ,  $\gamma \in [0, 1/2]$ ,  $\lambda(t)$  and  $\phi(1-t)$  be defined as above. Suppose that  $a, b, c, \gamma$  are related by any one of the following conditions:

- (i)  $b \in (0, 3]$  and  $c \geq a + b$  with  $\gamma = 0$ ,
- (ii)  $b \in (0, 2]$  and  $c \geq a + b + 2\gamma$  with  $\gamma \in (0, 1/2]$ .

Suppose that  $\beta = \beta_{a,b,c}$  is given by

$$\frac{\beta}{1 - \beta} = - \int_0^1 \lambda(t) \left[ \frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{(1-\gamma)} \frac{\log(1+t)}{t} \right] dt.$$

Then,  $P_{a,b;c}(\mathcal{R}(\beta)) \subset \mathcal{S}^*(\gamma)$ .

**Remark 3.9.** In the proof of Theorem 3.8, we observe that the condition of  $\phi(1-t)$  can be weakened. For instance, it suffices to assume that  $\phi(1-t) > 0$  and  $\phi'(1-t) \geq 0$  on  $(0, 1)$ .

The following Lemma is crucial in the proof of Theorem 3.8.

**Lemma 3.10** (Ponnusamy and Rønning [41]). Assume  $\Lambda$  is integrable on  $[0, 1]$  and positive on  $(0, 1)$ . Assume further that

$$\frac{\Lambda(t)}{(1+t)(1-t)^{1+2\gamma}},$$

is decreasing on  $(0, 1)$ . If  $\Lambda(t) = \int_t^1 \lambda(s) ds/s$ , and  $\beta = \beta(\lambda)$  is given by

$$\frac{\beta}{1 - \beta} = - \int_0^1 \lambda(t) \left[ \frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{1-\gamma} \frac{\log(1+t)}{t} \right] dt,$$

then we have  $V_\lambda(\mathcal{R}(\beta)) \subset \mathcal{S}^*(\gamma)$ ,  $0 \leq \gamma \leq 1/2$ .

The presence of the factor  $\phi(1-t)$  in Theorem 3.8 is important in the sense that special cases give some interesting applications as we see in the following two corollaries. For example, a special choice of  $\phi(1-t)$  yields the following result which generalizes Theorem 3.7.

**Corollary 3.11.** *Let  $a, b, c > 0$  and  $\gamma \in [0, 1/2]$ . Suppose that  $a, b, c, \gamma$  are related by any one of the following conditions:*

- (i)  $b \in (0, 3]$ ,  $a \in (0, 1]$  and  $c \geq a + b$  with  $\gamma = 0$ ,
- (ii)  $b \in (0, 2]$ ,  $a \in (0, 1]$  and  $c \geq a + b + 2\gamma$  with  $\gamma \in (0, 1/2]$ .

Suppose that  $\beta = \beta_{a,b,c}$  is given by

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \int_0^1 t^{b-1}(1-t)^{c-a-b} F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right) \\ \times \left[ \frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{(1-\gamma)} \frac{\log(1+t)}{t} \right] dt.$$

Then,  $H_{a,b;c}(\mathcal{R}(\beta)) \subset \mathcal{S}^*(\gamma)$ .

For the proof of the Corollary 3.11, we require the integral representation of  $H_{a,b;c}(f)$  in the form (3.4). To get this, we let  $f(z) = \sum_{k=1}^{\infty} a_k z^k$  and obtain

$$H_{a,b;c}(f(z)) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} a_{n+1} z^{n+1}.$$

Now,

$$\begin{aligned} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} &= F(c-a, 1-a; c-a+n+1; 1) \frac{\Gamma(b+n)}{\Gamma(c-a+n+1)} \\ &= \sum_{m=0}^{\infty} \frac{(c-a, m)(1-a, m)}{(c-a-b+1, m)(1, m)} \left( \frac{\Gamma(b+n)(c-a-b+1, m)}{(c-a+n+1, m)\Gamma(c-a+n+1)} \right) \\ &= \frac{1}{\Gamma(c-a-b+1)} \sum_{m=0}^{\infty} \frac{(c-a, m)(1-a, m)}{(c-a-b+1, m)(1, m)} \\ &\quad \times \left( \frac{\Gamma(b+n)\Gamma(c-a-b+m+1)}{\Gamma(c-a+n+m+1)} \right) \\ &= \frac{1}{\Gamma(c-a-b+1)} \sum_{m=0}^{\infty} \frac{(c-a, m)(1-a, m)}{(c-a-b+1, m)(1, m)} \\ &\quad \times \left( \int_0^1 t^{b+n-1}(1-t)^{c-a-b+m} dt \right) \end{aligned}$$

and therefore, we can write

$$\begin{aligned} H_{a,b;c}(f(z)) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \sum_{n=0}^{\infty} a_{n+1} \\ &\quad \times \int_0^1 \left( \sum_{m=0}^{\infty} \frac{(c-a,m)(1-a,m)}{(c-a-b+1,m)(1,m)} (1-t)^m t^{b-2} (1-t)^{c-a-b} (tz)^{n+1} dt \right) \\ &= \frac{\Gamma(c)(\Gamma(c-a-b+1))^{-1}}{\Gamma(a)\Gamma(b)} \\ &\quad \times \int_0^1 t^{b-2} (1-t)^{c-a-b} F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right) f(tz) dt, \end{aligned}$$

where we assume that  $\operatorname{Re} a > 0$ ,  $\operatorname{Re} b > 0$ , and  $\operatorname{Re}(c+1) > \operatorname{Re}(a+b)$ . Hence, we have the following integral representation of form (3.4)

$$H_{a,b;c}(f(z)) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt$$

where

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1} (1-t)^{c-a-b} F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right).$$

Now, the proof of Corollary 3.11 is a consequence of Theorem 3.8 if we choose

$$\phi(1-t) = F\left(\begin{matrix} c-a, 1-a \\ c-a-b+1 \end{matrix}; 1-t\right)$$

and the constant  $C$  as

$$C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}.$$

**Corollary 3.12.** *Let  $a > 0$ ,  $p \geq 1$ ,  $\gamma \in [0, 1/2]$  and  $a, p, \gamma$  be related by one of the following conditions:*

- (i)  $a \in (-1, 2]$  and  $p \geq 1$  with  $\gamma = 0$ ,
- (ii)  $a \in (-1, 1]$  and  $p \geq 1 + 2\gamma$  with  $\gamma \in (0, 1/2]$ .

Suppose that  $\beta$  is given by

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a (\log(1/t))^{p-1} \left[ \frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{(1-\gamma)} \frac{\log(1+t)}{t} \right] dt.$$

Then for  $f \in \mathcal{R}(\beta)$ , the function  $\Phi_p(a; z) * f(z)$  defined by

$$\Phi_p(a; z) * f(z) = \left( \sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n \right) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log 1/t)^{p-1} t^{a-1} f(tz) dt$$

belongs to  $\mathcal{S}^*(\gamma)$ .

Part (i) in Corollary 3.12 extends Theorem 2.3 in [32]. Indeed, the range of  $a$  obtained in [32] is  $(-1, 1]$  whereas in our result it is  $(-1, 2]$ . Further, Part (ii) of Corollary 3.12 is new and it improves Theorem 2.3 in [32]. Similarly, Corollary 3.11 extends and improves Theorem 2.4 in [32]. Moreover, our method of proof is slightly different from that of [32]. Several other properties of the integral transform defined by  $\Phi_p(a; z) * f(z)$  have been obtained in [42].

Now, we consider another integral transform studied by Ponnusamy in [40] and by Ponnusamy and Rønning in [41]. Define

$$\lambda(t) = \begin{cases} \frac{t^a(1-t^{b-a})}{b-a} & \text{for } b \neq a, a > -1, b > -1, \\ t^a \log(1/t) & \text{for all } b = a, a > -1. \end{cases} \quad (3.6)$$

Since  $G_f(a, b; z)$  defined by

$$G_f(a, b; z) = \left( \sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n \right) * f(z)$$

is symmetric in  $a$  and  $b$ , we can without loss of generality, assume  $b > a$  in the cases where  $b \neq a$ . That is we deal with the integral transform

$$V_\lambda(f) = G_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1-t^{b-a})f(tz) dt, & b > a > -1, \\ (a+1)^2 \int_0^1 t^{a-1} \log(1/t) f(tz) dt, & a > -1 \end{cases}$$

and in this case, we have

**Corollary 3.13.** *Let  $b > -1$ ,  $a > -1$ ,  $\gamma \in [0, 1/2]$  and  $a, b, \gamma$  are related by any one of the following conditions:*

- (i)  $a \in (-1, 2]$ ,  $b > a$  with  $\gamma = 0$ ,
- (ii)  $a \in (-1, 1]$ ,  $0 < b - a \leq 5/2$  with  $\gamma \in (0, 1/2]$ ,
- (iii)  $a \in (-1, 2]$ ,  $b = a$  with  $\gamma = 0$ ,
- (iv)  $a \in (-1, 1]$ ,  $b = a$  with  $\gamma \in (0, 1/2]$ .

Suppose that  $\lambda(t)$  is defined by (3.6), and  $\beta$  is given by

$$\frac{\beta}{1-\beta} = - \int_0^1 \lambda(t) \left[ \frac{1+\gamma-(1-\gamma)t}{(1-\gamma)(1+t)} - \frac{2\gamma}{(1-\gamma)} \frac{\log(1+t)}{t} \right] dt.$$

Then for  $f \in \mathcal{R}(\beta)$ , the function  $G_f(a, b; z) \in \mathcal{S}^*(\gamma)$ .

Corollary 3.13 gives the precise information about the order of starlikeness of  $G_f(a, b; z)$  which is discussed in Section 3 of [41] is only for certain special cases. The proof of Corollary 3.13 not only simplifies the proofs in [41] but also can be easily seen that it extends the range of these parameters for the case  $\gamma \in (0, 1/2]$ .



Next we recall

**Lemma 3.14.** *Let  $\beta_1 < 1$  and  $\beta_2 < 1$ . Then the following inclusions hold:*

- (i) *With  $1 - \delta = 2(1 - \beta_1)(1 - \beta_2)$ , we have  $\mathcal{R}(\beta_1) * \mathcal{R}_0(\beta_2) \subset \mathcal{R}(\delta)$ , or equivalently*
- (i)'  *$\mathcal{P}(\beta_1) * \mathcal{P}_0(\beta_2) \subset \mathcal{P}(\delta)$ ;*
- (ii)  *$\mathcal{R}(\delta) * \mathcal{R}_0(1/2) \subset \mathcal{R}(0) \cap \mathcal{S}^*$  for  $\delta = (1 - 2 \log 2)/(2 - 2 \log 2) \approx -0.629$ .*
- (iii)  *$\mathcal{R}(\beta_1) * \mathcal{R}(\beta_2) \subset \mathcal{S}^*$  provided  $(1 - \beta_1)(1 - \beta_2) < [4(1 - \log 2)]^{-1} \approx 0.8145$ .*

Part (i) of Lemma 3.14 is in [40], Part (ii) of Lemma 3.14 is a combination of Lemma C of [37] and Corollary 1 in [26] (see also [41]) whereas Part (iii) is in [44, Theorem 2, p. 141].

Now, we state our next results on geometric criteria for analytic functions.

**Theorem 3.15.** *Let  $a, b$  satisfy either  $a, b > 0$ , or  $a \in \mathbb{C} \setminus \{0\}$  with  $b = \bar{a}$ . Suppose that  $0 \neq c \geq \max\{0, a + b - 1, [(a + 1)(b + 1) - 2]/2\}$  and  $c$  satisfies the condition*

$$6c^2 + 6c(2 - ab - a - b) + 2(2 - a - b)(1 - a - b - 2ab) + ab(a - 1)(b - 1) \geq 0. \quad (3.7)$$

Let

$$\beta_0 = 1 - \frac{ab[4(c + 1) - (a + 1)(b + 1)]}{4c(c + 1)} < 1. \quad (3.8)$$

Define  $\phi(z)$  by the integral transform

$$\phi(z) = (1 - \lambda)z + \lambda \int_0^z F(a, b; c; t) dt, \quad \lambda = \frac{1}{1 - \beta}. \quad (3.9)$$

Then we have the following:

- (1) *For  $\beta \leq \beta_0$ , the function  $\phi(z)$  is starlike and close to convex w.r.t  $z$  and  $-\log(1 - z)$ .*
- (2) *For  $f \in \mathcal{R}(\beta_1)$ , the function  $\phi(z) * f(z)$  is starlike in  $\Delta$  and is in  $\mathcal{R}(2\beta_1 - 1)$  provided  $\beta_1 \geq (3 - 4 \log 2)/(4(1 - \log 2)) \approx 0.186$ .*

A counterpart of Theorem 3.15 for the confluent hypergeometric function is the following:

**Theorem 3.16.** *Let  $a, c > 0$  and  $\beta < 1$ . Suppose that*

$$\alpha_1 = \frac{2a - 19 + \sqrt{4a^2 + 84a + 181}}{10}, \quad \alpha_2 = \frac{3(a - 2) + \sqrt{3(a + 2)(a + 4)}}{6}. \quad (3.10)$$

Let

$$\beta_0 = 1 - \frac{a[4c + 3 - a]}{4c(c + 1)} \quad (3.11)$$

and  $c \geq \max\{\alpha_1, \alpha_2\}$ . Then, under the above hypotheses, the conclusions (1) and (2) in Theorem 3.15 hold for the integral transform

$$\phi(z) = (1 - \lambda)z + \lambda \int_0^z \Phi(a, b; c; t) dt, \quad \lambda = \frac{1}{1 - \beta},$$

$\beta_0$  and  $c$  as above.

#### 4. Proofs of geometric properties of hypergeometric function

**Proof of Theorem 3.15.** Note that the function  $\phi(z)$  defined by (3.9) can be rewritten as

$$\begin{aligned} \phi(z) &= (1 - \lambda)z + \lambda \frac{(c - 1)}{(a - 1)(b - 1)} [F(a - 1, b - 1; c - 1; z) - 1] \\ &= z + \sum_{n=2}^{\infty} a_n z^n, \end{aligned}$$

where  $a_1 = 1$  and

$$a_n = \frac{(a, n - 1)(b, n - 1)}{(1 - \beta)(c, n - 1)(1, n - 1)}, \quad \beta \leq \beta_0.$$

A simple computation gives

$$2a_1 - 4a_2 + 3a_3 = \frac{2}{1 - \beta} [\beta_0 - \beta],$$

which is positive. For  $n \geq 2$ , we have

$$\begin{aligned} \Delta a_n &= na_n - (n + 1)a_{n+1} \\ &= \frac{(a, n - 1)(b, n - 1)}{(1 - \beta)(c, n)(1, n)} [(n - 2)(c + 1 - a - b) + 2c + 2 - (a + 1)(b + 1)] \end{aligned}$$

and

$$\bar{\Delta} a_n = na_n - 2(n + 1)a_{n+1} + (n + 2)a_{n+2} = \frac{a_n}{(c + n)(c + n - 1)(n + 1)} M(n),$$

where

$$\begin{aligned} M(n) &= [(c + 2 - a - b)[n^2(c + 1 - a - b) + n(c - 1 + a + b - 2ab)] \\ &\quad - (a - 1)(b - 1)(2c - ab)]. \end{aligned}$$

From the proof of Lemma 3.3 in [39], it follows easily that the hypotheses of Lemmas 3.3 and 3.5 are satisfied. Thus, the function  $\phi$  is starlike and close to convex w.r.t  $z$  and  $-\log(1 - z)$ .

(2) Let  $f \in \mathcal{R}(\beta_1)$ . Define  $g(z) = ((1 - \lambda)z + \lambda zF(a, b; c; z)) * f(z)$ . Then, we have

$$g'(z) = ((1 - \lambda) + \lambda F(a, b; c; z)) * f'(z) = \phi'(z) * f'(z).$$

By (1), we find that  $\phi \in \mathcal{R}(0)$ . Therefore, by Lemma 3.14(iii) with  $\beta_2 = 0$ , it follows that the convolution  $\phi(z) * f(z)$  is starlike in  $\Delta$ . Using Lemma 3.14(i) with  $\beta_2 = 0$  and the fact that  $f \in \mathcal{R}(\beta_1)$ , we immediately get that the function  $g(z)$  belongs to  $\mathcal{R}(\delta)$ , with  $\delta = 1 - 2(1 - \beta_1)$ .  $\square$

**Proof of Theorem 3.16.** The proof of this theorem follows along the same lines as that of the proof of Theorem 3.15 and [39, Lemma 3.5]. So we omit the details.  $\square$

**Proof of Theorem 3.8.** Without loss of generality, we can omit the normalization constant  $C > 0$  and consider

$$A(t) = \int_t^1 \lambda(s) ds/s,$$

with

$$\lambda(t) = t^{b-1}(1-t)^{c-a-b}\phi(1-t), \quad (4.1)$$

where  $\phi(1-t)$  is described as in the hypothesis. In order to apply Lemma 3.10 with  $\gamma \in [0, 1/2]$ , it suffices to show that

$$g(t) = \frac{A(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on  $(0, 1)$ . Taking the logarithmic derivative of  $g(t)$  and using the fact that

$$A'(t) = -\frac{\lambda(t)}{t},$$

we have

$$\frac{g'(t)}{g(t)} = -\frac{\lambda(t)}{tA(t)} + \frac{2(\gamma + (1+\gamma)t)}{1-t^2}.$$

Thus, using the last expression for  $\lambda(t)$  we find that  $g'(t) \leq 0$  for  $t \in (0, 1)$  is equivalent to the inequality

$$\psi(t) = 2A(t) - \frac{(1-t^2)t^{-1}\lambda(t)}{\gamma + (1+\gamma)t} \leq 0 \quad \text{for } t \in (0, 1). \quad (4.2)$$

Clearly  $\psi(1) = 0$  and, we note that if  $\psi(t)$  is increasing on  $(0, 1)$  then  $g(t)$  is decreasing on  $(0, 1)$  and the proof will be complete. In view of this observation, it suffices to prove that  $\psi(t)$  is increasing on  $(0, 1)$ . To prove this, we compute  $\psi'(t)$  explicitly and after a simple calculation we find that

$$\psi'(t) = \frac{(1-t^2)t^{-1}}{\gamma + (1+\gamma)t} \left\{ \lambda(t) \left[ -\frac{2\gamma}{1-t} + \frac{1}{t} + \frac{1+\gamma}{\gamma + (1+\gamma)t} \right] - \lambda'(t) \right\}$$

and therefore,  $\psi'(t) \geq 0$  on  $(0, 1)$  is equivalent to the condition that

$$\frac{\lambda(t)}{t(1-t)} \left[ -2\gamma t + 1 - t + \frac{t(1-t)(1+\gamma)}{\gamma + (1+\gamma)t} \right] \geq \lambda'(t). \quad (4.3)$$

Using the definition of  $\lambda$  in (4.1), we find that

$$\lambda'(t) = t^{b-2}(1-t)^{c-a-b-1}[\phi(1-t)\{(b-1)(1-t) - (c-a-b)t\} - t(1-t)\phi'(1-t)]$$

and therefore, a simple calculation proves that (4.3) is equivalent to

$$(1-t)(2-b) + t(c-a-b-2\gamma) + \frac{t(1-t)(1+\gamma)}{\gamma + (1+\gamma)t} \geq -t(1-t)\frac{\phi'(1-t)}{\phi(1-t)}.$$

It remains to show that this inequality holds for all  $t \in (0, 1)$ . First we note that the right-hand side of the last inequality is nonpositive for all  $t \in (0, 1)$ . If we assume that  $\gamma = 0$ , then the left-hand side becomes  $(1-t)(3-b) + t(c-a-b)$  which is nonnegative for  $t \in (0, 1)$  whenever  $b \in (0, 3]$  and  $c \geq a+b$ . If  $\gamma \in (0, 1/2]$ , then, for the validity of the last inequality, the necessary conditions are  $0 < b \leq 2$  and  $c \geq a+b+2\gamma$ . Thus, we complete the proof.  $\square$

**Proof of Corollary 3.12.** Choose

$$\phi(1-t) = \left( \frac{\log(1/t)}{1-t} \right)^{p-1} = \left( \frac{-\log(1-(1-t))}{1-t} \right)^{p-1}$$

and

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (1-t)^{p-1} \phi(1-t).$$

The desired conclusion follows from Theorem 3.8.  $\square$

**Proof of Corollary 3.13.** Clearly, from the proof of Theorem 3.8, it suffices to verify inequality (4.3) for the  $\lambda(t)$  defined by (3.6).

Let  $\lambda(t)$  be defined by (3.6). If  $b > a > -1$ , then inequality (4.3) is equivalent to

$$(1-t^{b-a}) \left[ (1-t) - 2\gamma t + \frac{t(1-t)(1+\gamma)}{\gamma + (1+\gamma)t} \right] \geq (a - bt^{b-a})(1-t), \quad t \in (0, 1) \quad (4.4)$$

and if  $a = b$ , inequality (4.3) is equivalent to

$$\log(1/t) \left[ (1-t) - 2\gamma t + \frac{t(1-t)(1+\gamma)}{\gamma + (1+\gamma)t} \right] \geq (-1 + a \log(1/t))(1-t), \quad t \in (0, 1). \quad (4.5)$$

Thus, our aim is to verify these two inequalities for the four cases mentioned in Corollary 3.13. The cases where  $\gamma = 0$  are easy and the cases where  $\gamma \in (0, 1/2)$  need some work.

*Case (i):* Let  $a \in (-1, 2]$ ,  $b > a$  and  $\gamma = 0$ . For  $\gamma = 0$ , (4.3) is equivalent to

$$t^{b-a}(2-b) - (2-a) \leq 0$$

which is clearly true. Indeed, for  $b \geq 2$  and  $a \leq 2$ , the above inequality is trivial. If  $b < 2$ , we have

$$t^{b-a}(2-b) \leq (2-b) < 2-a$$

so the required inequality follows.

Case (ii): Assume  $a \in (-1, 1]$ ,  $b > a$  and  $\gamma \in (0, 1/2]$ . Then, inequality (4.3) is equivalent to

$$\frac{(b-a)t^{b-a}(1-t)}{1-t^{b-a}} \geq (1-t)(a-1) + 2\gamma t - \frac{t(1-t)(1+\gamma)}{\gamma + (1+\gamma)t}, \quad t \in (0, 1). \quad (4.6)$$

For  $\gamma \in (0, 1/2]$ , it is clear that

$$R(\gamma) := \frac{(1+\gamma)}{\gamma + (1+\gamma)t} \geq R(1/2) = \frac{3}{1+3t}$$

and therefore, (4.6) holds for each  $\gamma \in (0, 1/2]$  provided that it holds for  $\gamma = 1/2$ . Thus, we see that the inequality (4.6) for  $\gamma = 1/2$  is equivalent to

$$\frac{(b-a)(1-t)}{1-t^{b-a}} \geq (1-t)(b-1) + t - \frac{3t(1-t)}{1+3t} = (1-t)(b-a+a-1) + \frac{2(3t-1)}{3t+1} \quad (4.7)$$

and hence, it suffices to verify this inequality for all  $t \in (0, 1)$ . Allowing  $t \rightarrow 0$ , we observe that  $a \leq 1$  is a necessary condition for the truth of (4.6) for all  $t \in (0, 1)$ . Now, for the sake of simplicity, we make the substitution  $b-a=d$  and  $t=1-x$  so that (4.7) becomes

$$\frac{dx}{1-(1-x)^d} \geq (d+a-1)x + \frac{2(2-3x)}{4-3x} \quad (4.8)$$

and, since  $a \leq 1$ , it suffices to verify (4.7) for  $a=1$ , that is,

$$\frac{dx}{1-(1-x)^d} \geq dx + \frac{2(2-3x)}{4-3x} \quad \text{for } x \in (0, 1). \quad (4.9)$$

If the right-hand side of (4.9) is nonpositive, then it is clearly true. When the right-hand side of (4.9) is positive, we can rewrite this inequality as

$$\frac{dx}{1-(1-x)^d} \geq dx + \frac{2(2-3x)}{4-3x} \quad \text{for } x \in (0, 1).$$

After a simple computation, the last inequality is seen to be equivalent to

$$(1-x)^{d-1} - 1 + x(1-d) \leq \frac{3x^2d}{4(1-(3/2)x)}$$

or, by the power series representation,

$$\sum_{n=2}^{\infty} a_n x^n \geq 0 \quad (4.10)$$

with

$$a_n = \frac{3d}{4} \left(\frac{3}{2}\right)^{n-2} - \frac{(d-1)(d)(d+1) \cdots (d+n-2)}{n!}.$$

Note that

$$a_2 = \frac{d(5-2d)}{5} \geq 0 \quad \text{for } d \leq 5/2$$

and by induction (or directly) it can be easily seen that  $d \leq 5/2$  is a sufficient condition for  $a_n$  to be nonnegative for all  $n \geq 2$ . Thus, inequality (4.10) holds and we complete the proof of this case.

Case (iii): Let  $a \in (-1, 2]$ ,  $b = a$ , and  $\gamma = 0$ . Then, inequality (4.5) simplifies to an equivalent inequality  $(2 - a) \log(1/t) > -1$  which is trivially true for all  $t \in (0, 1)$  as  $a \in (-1, 2]$ .

Case (iv): Let  $a \in (-1, 1]$ ,  $a = b$  and  $\gamma \in (0, 1/2]$ . In this case, inequality (4.5) can be rewritten as

$$\frac{1-t}{\log(1/t)} \geq (a-1)(1-t) + 2\gamma t - \frac{t(1-t)(1+\gamma)}{\gamma + (1+\gamma)t}, \quad t \in (0, 1) \quad (4.11)$$

and, therefore, by the same reasoning as in Case (i), it suffices to verify (4.11) only for  $\gamma = 1/2$ . Putting  $\gamma = 1/2$  in (4.11), we obtain

$$\frac{1-t}{\log(1/t)} \geq (a-1)(1-t) - \frac{2t(1-3t)}{1+3t}, \quad t \in (0, 1). \quad (4.12)$$

Its behaviour near  $t=0$  shows that  $a \leq 1$  is necessary condition. Further, we note that (4.12) holds if this inequality is true for  $a=1$  since by assumption  $a \in (-1, 1]$ . Now, we need to prove (4.12) for  $a=1$ . However, for  $a=1$ , (4.12) is clear for  $t \in (0, 1/3]$  and therefore, it suffices to check the inequality

$$\frac{1-t}{\log(1/t)} \geq -\frac{2t(1-3t)}{1+3t} \quad \text{for } t \in (1/3, 1).$$

As in the proof of Case (i), it can be easily verified that this inequality holds for all  $t \in (0, 1)$ . The proof is complete.  $\square$

## 5. Conclusion

We conclude the paper with a remark and an open problem. In the recent past, several articles appeared proving that certain special functions (mainly related to gamma and polygamma functions) are not only positive, decreasing, and convex, but even completely monotonic. Hence, it is natural to raise the following

**Problem 5.1.** Does any of the investigated functions and also its various generalizations have also these properties?

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## References

- [1] A.P. Acharya, Univalence criteria for analytic functions and applications to hypergeometric functions, Ph.D. Thesis, Universität Würzburg, Germany, 1997.
- [2] G.D. Anderson, R.W. Barnard, K.C. Richards, M.K. Vamanamurthy, M. Vuorinen, Inequalities for zero-balanced hypergeometric functions, *Trans. Amer. Math. Soc.* 347 (1995) 1713–1723.
- [3] G.D. Anderson, S.-L. Qiu, M.K. Vamanamurthy, M. Vuorinen, Generalized elliptic integrals and modular equations, *Pacific J. Math.* 192 (2000) 1–37.
- [4] G.D. Anderson, M.K. Vamanamurthy, M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Wiley, New York, 1997.
- [5] J.M. Anderson, A. Hinkkanen, Univalence criteria and quasiconformal extensions, *Trans. Amer. Math. Soc.* 324 (1991) 823–842.
- [6] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [7] R. Askey, S. Ramanujan, and hypergeometric and basic hypergeometric series (Russian), Translated from English and with a remark by N.M. Atakishiev and S.K. Suslov, *Uspekhi Mat. Nauk* 45(1)(271) (1990) 33–76, 222; translation in *Russian Math. Surveys* 45(1) (1990) 37–86.
- [8] F.G. Avkhadiyev, L.A. Aksent'ev, The main results on sufficient conditions for an analytic function to be schlicht, *Russian Math. Surveys* 30 (4) (1975) 1–64.
- [9] F.G. Avkhadiyev, L.A. Aksent'ev, Progress and problems on sufficient conditions for finite valence of analytic functions, *Soviet Math. (Iz. VUZ)* 30 (1986) 1–20.
- [10] F.G. Avkhadiyev, L.A. Aksent'ev, A.M. Elizarov, Sufficient conditions for finite-valence of analytic functions, and their applications (Russian), in: P.V. Gamkrelidze (Ed.), *Mathematical Analysis*, Vol. 25, *Itogi Nauki i Tekhniki*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1987 pp. 3–121, 200.
- [11] R. Balasubramanian, S. Ponnusamy, Applications of duality principle to integral transforms of analytic functions, *Complex Variables: Theory Appl.* 38 (1998) 298–305.
- [12] H. Bateman, in: A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi (Eds.), *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.
- [13] J. Becker, Conformal mappings with quasiconformal extensions, in: D.A. Brannan, J.G. Clunie (Eds.), *Aspects of Contemporary Complex Analysis*, Academic Press, London, 1980, pp. 37–77.
- [14] S.D. Bernardi, *Bibliography of Schlicht Functions*, Mariner Publ. Co. Inc., Tampa, FL, 1983.
- [15] L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* 154 (1985) 137–152.
- [16] B.C. Carlson, D.B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.* 15 (1984) 737–745.
- [17] M. Chuaqui, B. Osgood, General univalence criteria in the disk: extensions and extremal functions, *Ann. Acad. Sci. Fenn. Math.* 23 (1) (1998) 101–132.
- [18] M. Chuaqui, B. Osgood, Ch. Pommerenke, John domains, quasidisks, and the Nehari class, *J. Reine Angew. Math.* 47 (1996) 77–114.
- [19] P. Deligne, G.D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, *IHES. Publ. Math.* 63 (1986) 5–106.
- [20] P. Deligne, G.D. Mostow, Commensurabilities Among Lattices in  $PU(1, n)$ , *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, 1993.
- [21] P.L. Duren, *Univalent Functions (Grundlehren der mathematischen Wissenschaften 259)*, Springer, Berlin, 1983.
- [22] E.B. Elliott, A formula including Legendre's  $EK' + KE' - KK' = \frac{1}{2}\pi$ , *Messenger of Math.* 33 (1904) 31–40.
- [23] C.L. Epstein, Univalence criteria and surfaces in hyperbolic space, *J. Reine Angew. Math.* 380 (1987) 196–214.
- [24] R.J. Evans, Ramanujan's second notebook: asymptotic expansions for hypergeometric series and related functions, in: G.E. Andrews, R.A. Askey, B.C. Berndt, R.G. Ramanathan, R.A. Rankin (Eds.), *Ramanujan Revisited: Proceedings of the Centenary Conference Univ. of Illinois at Urbana-Champaign*, Academic Press, Boston, 1988, pp. 537–560.
- [25] L. Fejér, Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge, *Acta Litterarum ac Scientiarum* 8 (1936) 89–115.
- [26] R. Fournier, St. Ruscheweyh, On two extremal problems related to univalent functions, *Rocky Mountain J. Math.* 24 (2) (1994) 529–538.
- [27] B. Frideman, Two theorems on schlicht functions, *Duke Math. J.* 13 (1946) 171–177.

- [28] A.W. Goodman, *Univalent Functions*, Vols. I & II, Mariner Publ. Co. Inc., Tampa, FL, 1983.
- [29] D.J. Hallenbeck, T.H. MacGregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Pitman Advanced Publishing Program, Boston, 1984.
- [30] E.A. Karatsuba, M. Vuorinen, On hypergeometric functions and generalizations of Legendre's relation, Reports of the Department of Mathematics of the University of Helsinki, Preprint, Vol. 216, 1999, 16pp., *J. Math. Anal. Appl.*, to appear.
- [31] M. Kashiwara, T. Miwa (Eds.), *Special Functions*, ICM90 Satellite Conferences, Proceedings of the Hayashibara Forum 1990 held in Fujisaki Institute, Okayama, Japan, August 16–20, 1990, Springer, Tokyo, 1991.
- [32] Y.C. Kim, F. Rønning, Integral transforms of certain subclasses of analytic functions, *J. Math. Anal. Appl.* 258 (2001) 466–489.
- [33] J. Krzyż, A counterexample concerning univalent functions, *Folia Societatis Scientiarum Lublinensis, Mat. Fiz. Chem.* 2 (1962) 57–58.
- [34] S. Ozaki, On the theory of multivalent functions, *Sci. Rep. Tokyo Bunrika Daigaku A* 2 (1935) 167–188.
- [35] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
- [36] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.
- [37] S. Ponnusamy, Differential subordination and starlike functions, *Complex Variables: Theory Appl.* 19 (1992) 185–194.
- [38] S. Ponnusamy, Close-to-convexity properties of Gaussian hypergeometric functions, *J. Comput. Appl. Maths.* 88 (1997) 327–337.
- [39] S. Ponnusamy, Hypergeometric transforms of functions with derivative in a half plane, *J. Comput. Appl. Maths.* 96 (1998) 35–49.
- [40] S. Ponnusamy, Inclusion theorems for convolution product of second order polylogarithms and functions with the derivative in a halfplane, *Rocky Mountain J. Math.* 28 (2) (1998) 695–733.
- [41] S. Ponnusamy, F. Rønning, Duality for Hadamard products applied to certain integral transforms, *Complex Variables: Theory Appl.* 32 (1997) 263–287.
- [42] S. Ponnusamy, S. Sabapathy, Polylogarithms in the theory of univalent functions, *Results in Mathematics* 30 (1996) 136–150.
- [43] S. Ponnusamy, S. Sabapathy, Geometric properties of generalized hypergeometric functions, *Ramanujan J.* 1 (2) (1997) 187–210.
- [44] S. Ponnusamy, V. Singh, Convolution properties of some classes analytic functions, *Zapiski Nauchnykh Seminarov POMI* 226 (1996) 138–154.
- [45] S. Ponnusamy, M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, *Mathematika* 44 (1997) 278–301.
- [46] S. Ponnusamy, M. Vuorinen, Univalence and convexity properties for Confluent hypergeometric functions, *Complex Variables: Theory Appl.* 36 (1998) 73–97.
- [47] S. Ponnusamy, M. Vuorinen, Univalence and convexity properties for Gaussian hypergeometric functions, *Rocky Mountain J. Math.* 31 (2001) 327–353.
- [48] E.D. Rainville, *Special functions*, Chelsea, New York, 1960.
- [49] St. Ruscheweyh, *Convolution in Geometric Function Theory*, Les Presses de l'Université de Montréal, Montréal, 1982.
- [50] A. Varchenko, Multidimensional hypergeometric functions and their appearance in conformal field theory, algebraic  $K$ -theory, algebraic geometry, etc., Proceedings of the International Congress on Kyoto, Japan, 1990, pp. 281–300.
- [51] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, 4th Edition, Cambridge University Press, Cambridge, 1958.
- [52] V.A. Zmorovič, On some problems in the theory of univalent functions (Russian), *Nauk. Zapiski, Kiev. Derjavnyi Univ.* (1952) 83–94.