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A meshless Galerkin method for Dirichlet problems using radial basis functions

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Abstract

In this paper, a numerical method is given for partial differential equations, which combines the use of Lagrange multipliers with radial basis functions. It is a new method to deal with difficulties that arise in the Galerkin radial basis function approximation applied to Dirichlet (also mixed) boundary value problems. Convergence analysis results are given. Several examples show the efficiency of the method using TPS or Sobolev splines.

MSC: 65N30; 65N12

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1. Introduction

In the last decade, meshless methods using radial basis functions have been extensively studied. Compared with classical numerical method such as finite element methods (FEM), meshless method will save computing time because generating the mesh for FEM is time consuming. This complication has been described recently by Griebel and Schweitzer [8] as "a very time-consuming portion of overall computation is the mesh generation from CAD input data. Typically, more than 70% of overall computation is spent by mesh generators". Numerical methods for partial differential equations using RBF are focused on for its 'meshless' property. Both collocation and the Galerkin method are discussed in a series of papers, for example [6,7,19,5]. The linear system in the unsymmetric collocation method may not be solvable. Examples are given in [9] where the coefficient matrix in the linear system is singular. In [19], the author gives an analysis for the Neumann problem using radial basis functions (RBF). FEM-like convergence estimates are given. However, the finite dimensional subspace spanned by RBF cannot coincide with the Dirichlet boundary condition, even by compactly supported RBF. Hence, there are lack of error analysis for both Dirichlet and the mixed problem.

The Lagrange multipliers method, which was first introduced by Ivo Babuška in [2], has been thoroughly studied by Pitkäranta [11–13], Barbosa [3], Brezzi [4], Stenberg [16]. This method is used to approximate the Dirichlet boundary

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conditions by using Lagrange multipliers. It was shown that the method had optimal convergence rates for the finite element spaces satisfying an inf–sup condition. In this paper, we combine the Lagrange multipliers method with RBF Galerkin approximation to obtain error estimates for Dirichlet problems. The traditional penalty method combined with RBFs is also introduced.

The rest of this paper is organized as follows. In Section 2, we introduce the Lagrange multiplier method. In Section 3, We consider the discrete form of the Lagrange multiplier method in the finite dimensional subspace spanned by translates of a radial basis function. Convergence rates for the approximate solutions are given. In Section 4, several numerical examples are given to show the efficiency of this method. The last section is devoted to a conclusion of this paper.

2. Lagrange multipliers method

Consider the model problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a d-dimensional domain, with a smooth boundary $\partial\Omega$,whose outer unit normal direction is denoted by $\mathbf{n}$, f is in $L^2(\Omega)$, The corresponding bilinear form is:

$$\mathscr{A}(u, v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,\mathrm{d}x \quad \forall u, v \in H^1_\Gamma(\Omega). \tag{2}$$

With $H^1_\Gamma(\Omega) = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega\}$. The Sobolev spaces $H^s(\Omega)$, $H^s(\partial\Omega)$ are defined in the standard way [1]. The norms are denoted $\|\cdot\|_{H^s(\Omega)}$, $\|\cdot\|_{H^s(\partial\Omega)}$, respectively. We recall that the well-known trace inequality and inverse trace inequality hold for suitable Sobolev spaces on Ω. We also use the space $H^{-1/2}(\partial\Omega)$, i.e., the dual space of $H^{1/2}(\partial\Omega)$, with the norm

$$\|\mu\|_{H^{-1/2}(\partial\Omega)} = \sup_{z \in H^{1/2}(\partial\Omega)} \frac{\langle \mu, z\rangle}{\|z\|_{H^{1/2}(\partial\Omega)}}, \tag{3}$$

where the duality pairing $\langle \mu, z\rangle = \int_{\partial\Omega} z\mu\,\mathrm{d}s$. The original problem is given in the following weak form: given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, find $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\partial\Omega)$ such that

$$\mathscr{B}(u, \lambda; v, \mu) = (f, v) + \langle g, \mu\rangle \quad \forall (v, \mu) \in H^1(\Omega) \times H^{-1/2}(\partial\Omega) \tag{4}$$

with the bilinear form defined by

$$\mathscr{B}(u, \lambda; v, \mu) = \mathscr{A}(u, v) - \langle \lambda, v\rangle - \langle \mu, u\rangle \tag{5}$$

$(f, v) = \int_\Omega fv\,\mathrm{d}x$ is the scalar product in $L^2(\Omega)$.

In the FEM case, the problem is transformed into: for finite element subspaces V_h, Λ_h with density h, find $(u_h, \lambda_h) \in V_h \times \Lambda_h$, such that

$$\mathscr{B}(u_h, \lambda_h; v, \mu) = (f, v) + \langle g, \mu\rangle. \tag{6}$$

The convergence rates of this method are given by the following theorem [2,4].

Theorem 2.1. *Suppose that the element of the finite element subspaces satisfy the conditions*

$$\sup_{v \in V_h \setminus \{0\}} \frac{\langle \mu, v\rangle}{\|v\|_{H^1(\Omega)}} \geqslant C\|\mu\|_{H^{-1/2}(\partial\Omega)}, \quad \forall \mu \in \Lambda_h$$

and

$$|v|^2_{H^1(\Omega)} \geqslant C\|v\|^2_{H^1(\Omega)}, \quad \forall v \in \{v \in V_h | \langle \mu, v\rangle = 0\ \forall \mu \in \Lambda_h\}.$$

For the solution (u_h, λ_h), *the following error estimate holds*

$$\|u - u_h\|_{H^1(\Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\partial\Omega)} \leqslant C(h^k \|u\|_{H^{k+1}(\Omega)} + h^{l+(3/2)} \|\lambda\|_{H^{l+1}(\partial\Omega)}) \tag{7}$$

when $u \in H^{k+1}(\Omega)$ *and* $\lambda \in H^{l+1}(\partial\Omega)$.

Remark 2.1. In general case, the densities of V_h, Λ_h are not the same. Without any diffusion, we do not distinguish them in notation here.

3. Approximated by RBF

Given a set of quasi-uniform [14,15] centers $X \subset \overline{\Omega}$ with density h_1, $X_1 \subset \partial\Omega$ with density h_2 and some RBFs $\phi(r)$, $\psi(r)$. We can define the finite dimensional subspace V_h of $H^1(\Omega)$ and Λ_h of $H^{-1/2}(\partial\Omega)$ by

$$V_h = \operatorname{span}\{\phi(\|x - x_j\|)\ x_j \in X\},$$

$$\Lambda_h = \operatorname{span}\{\psi(\|x - x_j\|)\ x_j \in X_1\},$$

where $\|\cdot\|$ is the Euclidean norm. For the sake of completeness, we list several radial basis function types:

Gaussian: $\exp(-c\|\cdot\|^2)$, $c > 0$,
Multiquadric: $(c^2 + \|\cdot\|^2)^\beta$, $\beta \in N$, $c \neq 0$,
Thin-plate spline: $\|\cdot\|^\beta \log \|\cdot\|$, $\beta \in 2N$.

The RBFs introduced above are global RBFs. One of the most popular classes of compactly supported RBFs (CS-RBFs) is the one introduced by Wendland [18]. Another is given by Wu [20]. These functions are strictly positive definite in R^d for all d less than or equal to some fixed value d_0, and can be constructed to have any desired amount of smoothness 2κ. For $l = [d/2] + \kappa + 1$ and $\kappa = 0, 1, 2, 3$, Wendland's function becomes

$$\phi_{l,0}(r) \doteq (1 - r)_+^l,$$

$$\phi_{l,1}(r) \doteq (1 - r)_+^{l+1}[(l + 1)r + 1],$$

$$\phi_{l,2}(r) \doteq (1 - r)_+^{l+2}[(l^2 + 4l + 3)r^2 + (3l + 6)r + 3],$$

$$\phi_{l,3}(r) \doteq (1 - r)_+^{l+3}[(l^3 + 9l^2 + 32l + 15)r^3 + (6l^2 + 36l + 45)r^2 + (15l + 45)r + 15].$$

Here and throughout, $\doteq$ denotes equality up to a constant factor. For example, the functions

$$\phi_{2,0} \doteq (1 - r)_+^2,$$

$$\phi_{3,1}(r) \doteq (1 - r)_+^4[4r + 1],$$

$$\phi_{4,2}(r) \doteq (1 - r)_+^6[35r^2 + 18r + 3],$$

$$\phi_{5,3}(r) \doteq (1 - r)_+^8[32r^3 + 25r^2 + 8r + 1],$$

which are in $\mathscr{C}^0, \mathscr{C}^2, \mathscr{C}^4, \mathscr{C}^6$, respectively, and strictly positive definite in R^3.

Wu's CS-RBF is:

$$\phi(r) = (1 - r)_+^6(5r^5 + 30r^4 + 72r^3 + 82r^2 + 36r + 6) \in \mathscr{C}^6.$$

Assuming that $\phi \in \mathscr{C}^{l_1}$, $\psi \in \mathscr{C}^l$ with Fourier transforms $\hat{\phi}$, $\hat{\psi}$ satisfying

$$\hat{\phi}(t) \sim (1 + \|t\|)^{-2l_1}, \quad \hat{\psi}(t) \sim (1 + \|t\|)^{-2l}$$

then the following theorem provides a bound on the interpolation error.

Theorem 3.1 (*RBF-interpolant error estimate theorem, Wendland [17,19]*). *Let $\Omega \subseteq R^d$ be an open and bounded domain, having Lipschitz boundary. Denote by S_u the interpolant on $X = \{x_1, x_2, \ldots, x_N\} \subseteq \Omega$ to $u \in H^k(\Omega)$ with $k > d/2$. Then there exists a constant $h_0 > 0$ such that for all X with $h < h_0$ where h is the density of X, the estimate*

$$\|u - s_u\|_{H^j(\Omega)} \leqslant Ch^{k-j}\|u\|_{H^k(\Omega)}$$

is valid for $0 \leqslant j \leqslant k$.

The inverse inequality for Λ_h is the key point in FEM version. In the RBF case, we have the following inverse inequality:

Theorem 3.2. *Assuming that ψ is positive definite and $\hat{\psi}(t) \sim (1 + \|t\|)^{-l-d}$, then for $\forall g \in \Lambda_h$, the following inequality holds*:

$$\|g\|_{H^{1/2}(\partial\Omega)} \leqslant Ch_2^{-2l-3d+3}\|g\|_{H^{-1/2}(\partial\Omega)}, \tag{8}$$

where C is a constant independent of the density h_2.

Proof. We will denote by C, a generic constant independent of h_2. According to the definition of $\|g\|_{H^{-1/2}(\partial\Omega)}$, we have

$$\|g\|_{H^{-1/2}(\partial\Omega)} = \sup_{\mu \in H^{1/2}(\partial\Omega)\setminus\{0\}} \frac{\langle g, \mu\rangle}{\|\mu\|_{H^{1/2}(\partial\Omega)}} \geqslant \frac{\langle g, g\rangle}{\|g\|_{H^{1/2}(\partial\Omega)}} = \frac{\|g\|^2_{H^0(\partial\Omega)}}{\|g\|_{H^{1/2}(\partial\Omega)}}.$$

Assuming that $g = \sum_i \psi(\|x - x_i\|)\lambda_i$, by using the technique in [10], we have

$$\|g\|^2_{H^0(\partial\Omega)} \geqslant Ch_2^{2l+2d-2}\|\vec{\lambda}\|^2$$

furthermore,

$$\|g\|^2_{H^1(\partial\Omega)} = \left\|\sum_i \psi(\|x - x_i\|)\lambda_i\right\|^2_{H^1(\partial\Omega)}$$

$$\leqslant \left(\sum_i \|\psi(\|x - x_i\|)\lambda_i\|_{H^1(\partial\Omega)}\right)^2 \leqslant \max_i \|\psi(\|x - x_i\|)\|^2_{H^1(\partial\Omega)}\left(\sum_i |\lambda_i|\right)^2$$

by using $(\sum_{i=1}^n |a_i|)^2 \leqslant n\sum_{i=1}^n |a_i|^2$, we have

$$\|g\|^2_{H^1(\partial\Omega)} \leqslant N \max_i \|\psi(\|x - x_i\|)\|^2_{H^1(\partial\Omega)}\|\vec{\lambda}\|^2 \leqslant Ch_2^{-d+1} \max_i \|\psi(\|x - x_i\|)\|^2_{H^1(\partial\Omega)}\|\vec{\lambda}\|^2.$$

This yields

$$\|g\|^2_{H^0(\partial\Omega)} \geqslant Ch_2^{2l+3d-3}\|g\|^2_{H^{1/2}(\partial\Omega)}.$$

Then

$$\|g\|_{H^{1/2}(\partial\Omega)} \leqslant Ch_2^{-2l-3d+3}\|g\|_{H^{-1/2}(\partial\Omega)}. \qquad \square$$

Remark 3.1. For others RBFs, when $\psi \in \mathscr{C}^\infty$, the inverse inequality will be quite different which depend on the lower bound of interpolation matrix. This situation can be dealt with in a same way.

Remark 3.2. The inverse inequality is much worse than which in FEM. The corresponding form in FEM is

$$\|g\|_{H^{1/2}(\partial\Omega)} \leqslant Ch_2^{-1}\|g\|_{H^{-1/2}(\partial\Omega)}.$$

Based on the two theorems above, we have the following error estimates:

Theorem 3.3. *Let $f \in H^k(\Omega)$, $k \geqslant 0$, $g \in H^s(\partial\Omega)$, $s \geqslant \frac{1}{2}$ and u the solution of the original problem. Further, let u_h, λ_h be the approximate solution of the RBF method with Lagrange multipliers. Assuming that the centers of both Ω and $\partial\Omega$ are quasi-uniform with densities h_1, h_2, respectively. $h_1 h_2^{-2l-3d+3}$ is sufficiently small. Then*

$$\|u - u_h\|_{H^1(\Omega)} + \left\|\lambda_h - \frac{\partial u}{\partial n}\right\|_{H^{-1/2}(\partial\Omega)} \leqslant C(h^{k+1}\|f\|_{H^k(\Omega)} + h^{s+(3/2)}\|g\|_{H^{s+1}(\partial\Omega)}), \tag{9}$$

where $h = \max\{h_1, h_2\}$, $C \sim \min\{h_1, h_2\}^{-2l-5d+6}$.

Proof. The proof is almost the same as in [2]. Let $(u, \lambda) \in V_h \times \Lambda_h$ be given. Denote by $w \in H^1(\Omega)$ the solution of the Neumann problem for the differential equation

$$\begin{cases} -\Delta w + w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = \lambda & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Using the well-known bound for the solution of the elliptic problem, we have

$$\|w\|_{H^2(\Omega)} \leqslant C\|\lambda\|_{H^{1/2}(\partial\Omega)}.$$

From Theorem 3.2, we have

$$\|w\|_{H^2(\Omega)} \leqslant C h_2^{-2l-3d+3}\|\lambda\|_{H^{-1/2}(\partial\Omega)}.$$

Let $z \in V_h$ satisfy

$$\|w - z\|_{H^1(\Omega)} \leqslant C h_1 \|w\|_{H^2(\Omega)} \leqslant C h_1 h_2^{-2l-3d+3}\|\lambda\|_{H^{-1/2}(\partial\Omega)}.$$

When $h_1 h_2^{-2l-3d+3}$ is sufficiently small, we have

$$\int_{\partial\Omega} z\lambda \,\mathrm{d}s \geqslant C\|\lambda\|^2_{H^{-1/2}(\partial\Omega)}.$$

The rest of the proof is just a simple repetition of the FEM-version. $\square$

Let

$$A = (\mathscr{A}(\phi(\|x - x_i\|), \phi(\|x - x_j\|)))_{N_1 \times N_1}, \quad x_i, x_j \in X$$

$$B = \left(\int_{\partial\Omega} \phi(\|x - x_i\|)\psi(\|x - x_j\|)\right)_{N_1 \times N_2}, \quad x_i \in X, \quad x_j \in X_1$$

$$\vec{f} = \left(\int_{\Omega} f\phi(\|x - x_i\|)\,\mathrm{d}x\right)_{N_1 \times 1}$$

$$\vec{g} = \left(\int_{\partial\Omega} g\psi(\|x - x_i\|)\,\mathrm{d}s\right)_{N_2 \times 1}$$

the linear system, induced by the bilinear form $\mathscr{B}$ and the finite dimensional space spanned by RBF, is

$$MC = \vec{F} \tag{11}$$

with

$$M = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & 0 \end{bmatrix}_{N \times N}, \quad \vec{F} = \begin{bmatrix} \vec{f} \\ \vec{g} \end{bmatrix}.$$

Remark 3.3. We can also consider the penalty method. For the general elliptic problem with homogeneous Dirichlet boundary condition, we have the approximation [5]

$$\begin{cases} \mathscr{L}u_\alpha = f & \text{in } \Omega, \\ \dfrac{\partial u_\alpha}{\partial n_L} + \alpha u_\alpha = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Then

$$\|u - u_{\alpha,h}\|_{H^1(\Omega)} \leqslant \frac{C}{\sqrt{\alpha}}\|f\|_{H^0(\Omega)} + C\alpha h^{k+1}\|f\|_{H^k(\Omega)}, \tag{13}$$

where $u_{\alpha,h}$ is the RBF-solution for Eq. (12).

Remark 3.4. We use Gaussian numerical integration. GMRES (a subroutine in MATLAB) is recommended for solving the linear system (11).

Remark 3.5. These method can also be used for mixed problems.

Remark 3.6. For global radial basis functions, the error induced by numerical integration just depends on the regularity of the basis function.

4. Numerical examples

Throughout this section, let

$$L^2_{\text{er}} = \frac{\|u_h - u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}}$$

denote the relative error (L^2-*norm* in Ω) of u_h, where u_h is the numerical solution using Lagrange multipliers and u is the exact solution. Let

$$L^\infty_{\text{er}} = \sup_{\Omega} \frac{|u_h - u|}{|u|}$$

be the maximum pointwise relative error.

Example 4.1. Consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

with $\Omega = [-1, 1; -1, 1]$, the right-hand side

$$f_1(x, y) = -e^x(y^2 - 1)(x^2 + 4x + 1) - 2e^x(x^2 - 1),$$

$$f_2(x, y) = 4(x^2 + y^2)\sin(x^2 - 1)\sin(y^2 - 1) - 2\sin(y^2 + x^2 - 2),$$

respectively, and corresponding exact solutions are

$$u_1(x, y) = (x^2 - 1)(y^2 - 1)e^x,$$

$$u_2(x, y) = \sin(x^2 - 1)\sin(y^2 - 1).$$

Take TPS $\phi(r) = r^6\log(r)$, $\phi(r) = r^5$ and Sobolev spline $\phi(r) = e^{-r}(3 + 3r + 3r^2)$ as the RBF. The centers are uniformly distributed in Ω with density $h = 0.2$.

$$V_h = \operatorname{span}\{\phi(\|x - x_i\|),\ x_i \in \bar{\Omega}\}, \quad \Lambda_h = \operatorname{span}\{\phi(\|x - x_i\|),\ x_i \in \partial\Omega\}.$$

Table 1
Numerical results for Poisson equation using $f_1(x, y)$

Radial basis function	L^2_{er}	L^∞_{er}
TPS	0.0048	0.0509
r^5	0.0040	0.0314
Sobolev spline	0.0187	0.0390

Table 2
Numerical results for Poisson equation using $f_2(x, y)$

Radial basis function	L^2_{er}	L^∞_{er}
TPS	0.0060	0.0286
Sobolev spline	0.0289	0.0822
r^5	0.0087	0.0183

Table 3
Numerical results for Example 4.2 using r^5

Density of the centers	L^2_{er}	L^∞_{er}
$n_1 = 5,\ n_2 = 5$	0.0044	0.0223
$n_1 = 5,\ n_2 = 8$	0.0039	0.0057
$n_1 = 5,\ n_2 = 10$	0.0039	0.0055

Values of the L^2-norm and *sup* norm relative errors for different choices of the RBF and different choice of the right hand side of Eq. (14). Tables 1–2.

Example 4.2. Consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

with $\Omega = \{(x, y) | x^2 + y^2 \leqslant 1\}$. Taking

$$f(x, y) = 4(x^2 + y^2)\sin(x^2 + y^2 - 1) - 4\cos(x^2 + y^2 - 1)$$

the exact solution of this problem is

$$u = \sin(x^2 + y^2 - 1).$$

The radial basis function is r^5. The centers are taken as

$$\left(\frac{i}{n_1}\cos\left(\frac{2\pi j}{n_2}\right), \frac{i}{n_1}\sin\left(\frac{2\pi j}{n_2}\right)\right), \quad i = 1, \ldots, n_1, \quad j = 1, \ldots, n_2.$$

See Table 3.

5. Conclusions

We have introduced and analyzed a meshless Galerkin method for Dirichlet problems. The inverse inequalities in the approximated subspace by RBFs play an important role in theoretical analysis. Our analysis is restricted to positive definite RBF satisfying Theorem 3.2. In [10], the authors had claimed that there are many kinds of RBF satisfying this condition, For example, Wendland's and Wu's functions. Those functions make the numerical integration be complicate, although the stiffness matrix will be sparse if the scaled CS-RBF is employed. In numerical examples, only Sobolev Spline is positive definite. TPS is conditionally positive definite. We use TPS to compute without additional polynomial, which ensure them to be the basis of suitable function space. It should be pointed out that higher accuracy is difficult to reach. This is due to bad condition number of linear system (11), which is induced by the *uncertainty* [14] property of RBF interpolation. Further work will focus on decreasing the condition number.

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