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Improvements of preconditioned AOR iterative method for L-matrices☆

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Abstract

In this paper, we improve the preconditioned AOR method of linear systems considered by Evans et al. [The AOR iterative method for new preconditioned linear systems, Comput. Appl. Math. 132 (2001) 461–466]. In Evans' paper, the coefficient matrix of linear system have to be an L-matrix with $a_{i,i+1}a_{i+1,i} > 0$, $i = 1, \ldots, n-1$ and $0 < a_{1n}a_{n1} < 1$. When $a_{i,i+1}a_{i+1,i} = 0$ for some $i \in N = \{1, \ldots, n-1\}$, the preconditioned method is invalid. In order to solve the above problem, a new preconditioner is presented. Meanwhile, some recent results are improved.

MSC: 65F10

Keywords: Preconditioner; Linear system; AOR iteration; L-matrix

1. Introduction

Consider the following linear system:

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ are given and $x \in R^n$ is unknown.

The preconditioned methods are often used to accelerate the convergence of iterative method solving the linear system (1). Recently, in Evans et al. [1] provided a preconditioner and improved the convergence rate of AOR iteration method for the original linear system. However, in [1], the coefficient matrix of linear system (1) A have to be an L-matrix with $a_{i,i+1}a_{i+1,i} > 0$, $i = 1, \ldots, n-1$ and $0 < a_{1n}a_{n1} < 1$, the assumptions are too strong in many cases. For example, when $a_{i,i+1}a_{i+1,i} = 0$ for some $i \in N = \{1, \ldots, n-1\}$, the preconditioned method of Evans et al. is invalid. In this paper, we present a new preconditioner, which overcomes the shortcomings of [1].

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For simplicity, without loss of generality, let

$$A = I - L - U,$$

where I is the identity matrix, L and U are strictly lower and upper triangular matrices obtained from A, respectively.

For any splitting, $A = M - N$ with $\det(M) \neq 0$, the basic iterative method for solving (1) is

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, \ldots,$$

and in [2], the AOR iterative method is defined

$$x^{(i+1)} = (I - rL)^{-1}[(1-w)I + (w-r)L + wU]x^{(i)} + (I - rL)^{-1}wb,$$

where $i = 0, 1, 2, \ldots$. Its iterative matrix is

$$L_{rw} = (I - rL)^{-1}[(1-w)I + (w-r)L + wU], \tag{2}$$

where w and r are real parameters with $w \neq 0$.

The spectral radius of the iterative matrix is decisive for the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. The effective method to decrease the spectral radius is to precondition the linear system (1), namely,

$$PAx = Pb,$$

where P is a non-singular matrix. The corresponding basic preconditioned iterative method is given in general by

$$x^{(i+1)} = M_p^{-1}N_px^{(i)} + M_p^{-1}Pb, \quad i = 0, 1, \ldots,$$

where $PA = M_p - N_p$. In particular, if we express PA as

$$PA = D^* - L^* - U^*,$$

then the preconditioned AOR iterative method is

$$x^{(i+1)} = (D^* - rL^*)^{-1}[(1-w)D^* + (w-r)L^* + wU^*]x^{(i)} + (D^* - rL^*)^{-1}wPb,$$

where $i = 0, 1, 2, \ldots$. Its iterative matrix is

$$L_{rw}^* = (D^* - rL^*)^{-1}[(1-w)D^* + (w-r)L^* + wU^*].$$

2. Preparatory knowledge

For convenience, we shall now briefly explain some of the terminologies used in the next sections. Let $C = (c_{ij}) \in R^{n\times n}$ be an $n \times n$ real matrix. By $\mathrm{diag}(C)$, we denote the $n \times n$ diagonal matrix coinciding in its diagonal with c_{ii}. For $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times n}$, we write $A \geqslant B$ if $a_{ij} \geqslant b_{ij}$ holds for all $i, j = 1, 2, \ldots, n$. Calling A nonnegative if $A \geqslant 0$ ($a_{ij} \geqslant 0$; $i, j = 1, \ldots, n$), we say that $A - B \geqslant 0$ if and only if $A \geqslant B$. These definitions carry immediately over to vectors by identifying them with $n \times 1$ matrices. $\rho(\cdot)$ denotes the spectral radius of a matrix.

Definition 1 (*Young [4]*). A matrix A is a L-matrix if $a_{ii} > 0$; $i = 1, \ldots, n$ and $a_{ij} < 0$, for all $i, j = 1, 2, \ldots, n$; $i \neq j$.

Definition 2 (*Varga [3]*). A matrix A is irreducible if the directed graph associated to A is strongly connected.

Now, we are going to cite several known results which are indispensable for our subsequent discussions.

Lemma 1 (Varga [3]). *Let $A \in R^{n\times n}$ be nonnegative and irreducible $n \times n$ matrix. Then*

(i) *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;*
(ii) *for $\rho(A)$, there corresponds an eigenvector $x > 0$;*

(iii) $\rho(A)$ *is a simple eigenvalue of* A;
(iv) $\rho(A)$ *increases when any entry of* A *increases.*

Lemma 2 (Varga [3]). *Let* $A = (a_{ij}) \geqslant 0$ *be an irreducible* $n \times n$ *matrix, and* P^* *be the hyperoctant of vectors* $x > 0$. *Then, for any* $x \in P^*$, *either*

$$\min_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right] < \rho(A) < \max_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right],$$

or

$$\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} = \rho(A) \quad \text{for all } 1 \leqslant i \leqslant n.$$

Moreover

$$\sup_{x \in P^*} \min_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right] < \rho(A) < \inf_{x \in P^*} \max_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right].$$

3. The [n, 1] (or [1, n]) preconditioned AOR iterative method

Now, we consider the preconditioned linear system,

$$\widetilde{A} x = \widetilde{b}, \tag{3}$$

where $\widetilde{A} = (I + \widetilde{S})A$ and $\widetilde{b} = (I + \widetilde{S})b$ with

$$\widetilde{S} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-a_{n1}}{\alpha} & 0 & \cdots & 0 \end{bmatrix}$$

and the preconditoined linear system

$$\widehat{A} x = \widehat{b}, \tag{4}$$

where $\widehat{A} = (I + \widehat{S})A$ and $\widehat{b} = (I + \widehat{S})b$ with

$$\widehat{S} = \begin{bmatrix} 0 & 0 & \cdots & \frac{-a_{1n}}{\alpha} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now, we express the coefficient matrix of (3) as

$$\widetilde{A} = \widetilde{D} - \widetilde{L} - \widetilde{U},$$

where $\widetilde{D} = \operatorname{diag}(\widetilde{A})$, $\widetilde{L}$ and $\widetilde{U}$ are strictly lower and upper triangular matrices obtained from $\widetilde{A}$, respectively. By calculation, we obtain that

$$\widetilde{D} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 - \frac{a_{1n} a_{n1}}{\alpha} \end{bmatrix}, \tag{5}$$

$$\widetilde{L} = \begin{bmatrix} 0 & & & & \\ -a_{21} & 0 & & & \\ -a_{31} & -a_{32} & 0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \left(\frac{1}{\alpha}-1\right)a_{n1} & \frac{a_{n1}a_{12}}{\alpha}-a_{n2} & \cdots & \frac{a_{n1}a_{1,n-1}}{\alpha}-a_{n,n-1} & 0 \end{bmatrix}, \tag{6}$$

$$\widetilde{U} = U = \begin{bmatrix} 0 & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ & 0 & -a_{23} & \cdots & -a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & -a_{n-1,n} \\ & & & & 0 \end{bmatrix}. \tag{7}$$

The coefficient matrix of (4) can be expressed as

$$\widehat{A} = \widehat{D} - \widehat{L} - \widehat{U}, \tag{8}$$

where $\widehat{D} = \mathrm{diag}(\widehat{A})$, $\widehat{L}$ and $\widehat{U}$ are strictly lower and upper triangular matrices obtained from $\widehat{A}$, respectively. By calculation, we also obtain that

$$\widehat{D} = \begin{bmatrix} 1-\frac{a_{1n}a_{n1}}{\alpha} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \tag{9}$$

$$\widehat{L} = L = \begin{bmatrix} 0 & & & \\ -a_{21} & 0 & & \\ \vdots & \vdots & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix}, \tag{10}$$

$$\widehat{U} = \begin{bmatrix} 0 & \frac{a_{1n}a_{n2}}{\alpha}-a_{12} & \frac{a_{1n}a_{n3}}{\alpha}-a_{13} & \cdots & \left(\frac{1}{\alpha}-1\right)a_{1n} \\ & 0 & -a_{23} & \cdots & -a_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & -a_{n-1,n} \\ & & & & 0 \end{bmatrix}. \tag{11}$$

Applying the AOR method to the preconditioned linear systems (3) and (4), respectively, we have the corresponding preconditioned AOR iterative methods whose iterative matrices are, respectively,

$$\widetilde{L}_{rw} = (\widetilde{D} - r\widetilde{L})^{-1}[(1-w)\widetilde{D} + (w-r)\widetilde{L} + w\widetilde{U}] \tag{12}$$

and

$$\widehat{L}_{rw} = (\widehat{D} - r\widehat{L})^{-1}[(1-w)\widehat{D} + (w-r)\widehat{L} + w\widehat{U}]. \tag{13}$$

4. Main results

Firstly, we present a lemma which is useful in the paper.

Lemma 3. *Let A, $\widetilde{A}$ and $\widehat{A}$ be the coefficient matrices of the linear systems* (1), (3) *and* (4), *respectively. If* $0 \leqslant r \leqslant w \leqslant 1$ $(w \neq 0)(r \neq 1)$ *and A is an irreducible L-matrix with* $0 < a_{1n}a_{n1} < \alpha$ $(\alpha \geqslant 1)$, *then the iterative matrices L_{rw}, $\widetilde{L}_{rw}$ and $\widehat{L}_{rw}$ associated to the AOR method applied to the linear systems* (1), (3) *and* (4), *respectively, are nonnegative and irreducible.*

Proof. From that A is an L-matrix, we have $L \geqslant 0$ is a strictly lower triangular matrix and $U \geqslant 0$ is a strictly upper triangular matrix. So $(I - rL)^{-1} = I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}$.

By (2), we have

$$\begin{aligned} L_{rw} &= (I - rL)^{-1}[(1-w)I + (w-r)L + wU] \\ &= [I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}][(1-w)I + (w-r)L + wU] \\ &= (1-w)I + (w-r)L + wU + rL(1-w)I + rL[(w-r)L + wU] \\ &\quad + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-w)I + (w-r)L + wU] \\ &= (1-w)I + w(1-r)L + wU + T, \end{aligned}$$

where

$$\begin{aligned} T &= rL[(w-r)L + wU] + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-w)I + (w-r)L + wU] \\ &\geqslant 0. \end{aligned}$$

So L_{rw} is nonnegative. We can also get that $(1-w)I + w(1-r)L + wU$ is irreducible for A is irreducible, hence L_{rw} is irreducible.

From (5)–(7), we have $\widetilde{D} > 0$ when $a_{1n}a_{n1} < \alpha$, $\widetilde{L} \geqslant 0$ when $\alpha \geqslant 1$, and $\widetilde{U} = U \geqslant 0$, respectively. So,

$$\begin{aligned} \widetilde{L}_{rw} &= (\widetilde{D} - r\widetilde{L})^{-1}[(1-w)\widetilde{D} + (w-r)\widetilde{L} + w\widetilde{U}] \\ &= (I - r\widetilde{D}^{-1}\widetilde{L})^{-1}[(1-w)I + (w-r)\widetilde{D}^{-1}\widetilde{L} + w\widetilde{D}^{-1}\widetilde{U}] \\ &= (1-w)I + w(1-r)\widetilde{D}^{-1}\widetilde{L} + w\widetilde{D}^{-1}\widetilde{U} + \widetilde{T} \geqslant 0, \end{aligned}$$

where

$$\begin{aligned} \widetilde{T} &= r\widetilde{D}^{-1}\widetilde{L}[(w-r)\widetilde{D}^{-1}\widetilde{L} + w\widetilde{D}^{-1}\widetilde{U}] + [r^2(\widetilde{D}^{-1}\widetilde{L})^2 + \cdots + r^{n-1}(\widetilde{D}^{-1}\widetilde{L})^{n-1}] \\ &\quad \times [(1-w)I + (w-r)\widetilde{D}^{-1}\widetilde{L} + w\widetilde{D}^{-1}\widetilde{U}] \\ &\geqslant 0. \end{aligned}$$

Furthermore $\widetilde{L}_{rw}$ is irreducible followed by the irreducibility of $(1-w)I + w(1-r)\widetilde{D}^{-1}\widetilde{L} + w\widetilde{D}^{-1}\widetilde{U}$ and $\widetilde{T} \geqslant 0$. Similarly, we can prove that $\widehat{L}_{rw}$ is nonnegative and irreducible too. □

Theorem 1. *Let L_{rw} and $\widetilde{L}_{rw}$ be defined by* (2) *and* (12), *respectively. If matrix A of* (1) *is an irreducible L-matrix with* $0 < a_{1n}a_{n1} < \alpha$ $(\alpha \geqslant 1)$, *and* $0 \leqslant r \leqslant w \leqslant 1$ $(w \neq 0)$ $(r \neq 1)$, *then*

(1) $\rho(\widetilde{L}_{rw}) < \rho(L_{rw})$, *if* $\rho(L_{rw}) < 1$;
(2) $\rho(\widetilde{L}_{rw}) = \rho(L_{rw})$, *if* $\rho(L_{rw}) = 1$;
(3) $\rho(\widetilde{L}_{rw}) > \rho(L_{rw})$, *if* $\rho(L_{rw}) > 1$.

Proof. From Lemma 3, L_{rw} and $\widetilde{L}_{rw}$ are nonnegative and irreducible matrices. Thus, from Lemma 1 there is a positive vector x, such that

$$L_{rw}x = \lambda x, \tag{14}$$

where $\rho(L_{rw}) = \lambda$ or, equivalently,

$$[(1-w)I + (w-r)L + wU]x = \lambda(I - rL)x \tag{15}$$

and

$$w\widetilde{U}x = wUx = (\lambda - 1 + w)x + (r - w - \lambda r)Lx.$$

From (12), for the positive vector x,

$$\widetilde{L}_{rw}x - \lambda x = (\widetilde{D} - r\widetilde{L})^{-1}[(1-w)\widetilde{D} + (w-r)\widetilde{L} + w\widetilde{U} - \lambda(\widetilde{D} - r\widetilde{L})]x. \tag{16}$$

Since

$$\widetilde{A} = (I + \widetilde{S})A = (I + \widetilde{S} - L - \widetilde{S}U) - U = \widetilde{D} - \widetilde{L} - \widetilde{U},$$

$$\widetilde{S}L = 0,$$

and

$$\begin{aligned}\lambda(\widetilde{D} - r\widetilde{L})x &= \lambda(1-r)\widetilde{D}x + \lambda r(\widetilde{D} - \widetilde{L})x \\ &= \lambda(1-r)\widetilde{D}x + \lambda r(I + \widetilde{S} - L - SU)x,\end{aligned}$$

then

$$\begin{aligned}\widetilde{L}_{rw}x - \lambda x &= (\widetilde{D} - r\widetilde{L})^{-1}[(1 - w - \lambda + \lambda r)\widetilde{D} + \lambda(I - rL) - (1-w)I \\ &\quad + (w-r)(\widetilde{L} - L) - \lambda r(I + \widetilde{S} - L - \widetilde{S}U)]x \\ &= (\widetilde{D} - r\widetilde{L})^{-1}[(1 - w - \lambda + \lambda r)\widetilde{D} + \lambda(I - rL) - (1-w)I \\ &\quad + (w-r)(\widetilde{D} - I - \widetilde{S} + \widetilde{S}U) - \lambda r(I + \widetilde{S} - L - \widetilde{S}U)]x \\ &= (\widetilde{D} - r\widetilde{L})^{-1}[(1-\lambda)(1-r)(\widetilde{D} - I) + (w - r + \lambda r)(\widetilde{S}U - \widetilde{S})]x \\ &= (\widetilde{D} - r\widetilde{L})^{-1}\{(1-\lambda)(1-r)(\widetilde{D} - I) - (w - r + \lambda r)\widetilde{S} \\ &\quad + r(\lambda - 1)\widetilde{S}U + \widetilde{S}[\lambda(I - rL) - (1-w)I - (w-r)L]\}x \\ &= (\widetilde{D} - r\widetilde{L})^{-1}[(1-\lambda)(1-r)(\widetilde{D} - I) - (1-r)(1-\lambda)\widetilde{S} \\ &\quad + r(\lambda - 1)\widetilde{S}U + (r - w - \lambda r)\widetilde{S}L]x \\ &= (\widetilde{D} - r\widetilde{L})^{-1}[(1-\lambda)(1-r)(\widetilde{D} - I - \widetilde{S}) + r(\lambda - 1)\widetilde{S}U]x.\end{aligned}$$

Let $B = (1-\lambda)(1-r)(\widetilde{D} - I) - (1-r)(1-\lambda)\widetilde{S} + r(\lambda - 1)\widetilde{S}U$. Obviously, $B \leqslant 0$ when $\lambda < 1$. By calculation, we obtain that $b_{n1} = (1-\lambda)(1-r)a_{n1}/\alpha$, $b_{nj} = -r(1-\lambda)a_{n1}a_{1j}/\alpha$ $(j = 2, \ldots, n-1)$, $b_{nn} = -(1-\lambda)a_{1n}a_{n1}/\alpha$, $b_{ij} = 0$ $(1 = 1, \ldots, n-1,\ j = 1, \ldots, n)$, respectively. Then we have

$$\widetilde{L}_{rw} - \lambda x = (\widetilde{D} - r\widetilde{L})^{-1}Bx. \tag{17}$$

Let

$$z = \widetilde{L}_{rw} - \lambda x = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \quad \text{and} \quad \widetilde{L}_{rw}x = (\widetilde{l}_{ij})_{n\times n}.$$

When $\lambda < 1$, from (17), we have

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \leqslant 0$$

and

$$z_n \leqslant \frac{(\lambda-1)a_{1n}a_{n1}x_n}{\alpha - a_{1n}a_{n1}} < 0.$$

Therefore,

$$\widetilde{L}_{rw}x = \lambda x + z = \lambda x + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \leqslant \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_{n-1} \\ \lambda x_n + z_n \end{bmatrix}.$$

Let $z_n/x_n = -h \ (h > 0)$. Then

$$\widetilde{L}_{rw}x = \lambda x + z = \lambda x + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \leqslant \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_{n-1} \\ (\lambda - h)x_n \end{bmatrix}.$$

Further, we get

$$\frac{\sum_{j=1}^{n} \widetilde{l}_{ij}x_j}{x_i} \leqslant \lambda \ (i = 1, \ldots, n-1) \quad \text{and} \quad \max_{1 \leqslant i \leqslant n-1} \left[\frac{\sum_{j=1}^{n} \widetilde{l}_{ij}x_j}{x_i} \right] \leqslant \lambda,$$

$$\frac{\sum_{j=1}^{n} \widetilde{l}_{nj}x_j}{x_n} \leqslant \lambda - h.$$

Using Lemma 2, we have that

$$\min_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} \widetilde{l}_{ij}x_j}{x_i} \right] < \rho(\widetilde{L}_{rw}) < \max_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} \widetilde{l}_{ij}x_j}{x_i} \right],$$

or,

$$\frac{\sum_{j=1}^{n} \widetilde{l}_{ij}x_j}{x_i} = \rho(\widetilde{L}_{rw}), \quad 1 \leqslant i \leqslant n.$$

By the above discussion, we get

$$\rho(\widetilde{L}_{rw}) < \lambda.$$

When $\lambda = 1$, from (17), we have

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = 0.$$

So

$$\widetilde{L}_{rw}x = \lambda x.$$

From Lemma 2, it is obvious that

$$\rho(\widetilde{L}_{rw}) = \frac{\sum_{j=1}^{n} \widetilde{l}_{ij} x_j}{x_i} = \lambda.$$

When $\lambda > 1$, from (17), we have

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \geqslant 0$$

and

$$z_n \geqslant \frac{(\lambda - 1) a_{1n} a_{n1} x_n}{\alpha - a_{1n} a_{n1}} > 0.$$

So

$$\widetilde{L}_{rw} x = \lambda x + z = \lambda x + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \geqslant \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_{n-1} \\ \lambda x_n + z_n \end{bmatrix}.$$

Let $z_n / x_n = h \ (h > 0)$. Then

$$\widetilde{L}_{rw} x = \lambda x + z = \lambda x + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \geqslant \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_{n-1} \\ (\lambda + h) x_n \end{bmatrix}.$$

So we get

$$\frac{\sum_{j=1}^{n} \widetilde{l}_{ij} x_j}{x_i} \geqslant \lambda \ (i = 1, \ldots, n-1) \quad \text{and} \quad \min_{1 \leqslant i \leqslant n-1} \left[\frac{\sum_{j=1}^{n} \widetilde{l}_{ij} x_j}{x_i} \right] \geqslant \lambda,$$

$$\frac{\sum_{j=1}^{n} \widetilde{l}_{nj} x_j}{x_n} \geqslant \lambda + h.$$

Using Lemma 2, we have

$$\min_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} \widetilde{l}_{ij} x_j}{x_i} \right] < \rho(\widetilde{L}_{rw}) < \max_{1 \leqslant i \leqslant n} \left[\frac{\sum_{j=1}^{n} \widetilde{l}_{ij} x_j}{x_i} \right],$$

or $\sum_{j=1}^{n} \widetilde{l}_{ij} x_j / x_i = \rho(\widetilde{L}_{rw})$, for all $1 \leqslant i \leqslant n$.

$$\rho(\widetilde{L}_{rw}) = \frac{\sum_{j=1}^{n} \widetilde{l}_{nj} x_j}{x_n} \geqslant \lambda + h.$$

So

$$\rho(\widetilde{L}_{rw}) > \lambda.$$

Thus, we get the required results. □

Similarly, we can obtain the following theorem.

Theorem 2. *Let L_{rw} and $\widehat{L}_{rw}$ be the iterative matrices of the AOR method given* (2) *and* (13), *respectively. If the matrix A of* (1) *is an irreducible L-matrix with* $0<a_{1n}a_{n1}<\alpha$ $(\alpha\geqslant 1)$ *and* $0\leqslant r\leqslant w\leqslant 1$ $(w\neq 0)$ $(r\neq 1)$, *then*

(1) $\rho(\widehat{L}_{rw})<\rho(L_{rw})$, *if* $\rho(L_{rw})<1$;
(2) $\rho(\widehat{L}_{rw})=\rho(L_{rw})$, *if* $\rho(L_{rw})=1$;
(3) $\rho(\widehat{L}_{rw})>\rho(L_{rw})$, *if* $\rho(L_{rw})>1$.

In (2), (12) and (13), take $w=r$, we obtain the iterative matrix of the successive overrelaxation (SOR) method associated to (2), (12) and (13). Therefore, we have the following corollary from Theorems 1 and 2.

Corollary 1. *Let L_w, $\widetilde{L}_w$ and $\widehat{L}_w$ be the iterative matrices of the SOR iterative method associated to* (1), (3) *and* (4), *respectively. If the matrix A of* (1) *is an irreducible L-matrix with* $0<a_{1n}a_{n1}<\alpha$ $(\alpha\geqslant 1)$ *and* $0<w<1$, *then*

(1) $\rho(\widetilde{L}_w)<\rho(L_w)$ *and* $\widehat{L}_w<L_w$ *if* $\rho(L_w)<1$;
(2) $\rho(\widetilde{L}_w)=\widehat{L}_w=\rho(L_w)$, *if* $\rho(L_w)=1$;
(3) $\rho(\widetilde{L}_w)>\rho(L_w)$ *and* $\rho(\widehat{L}_w)>\rho(L_w)$ *if* $\rho(L_w)>1$.

Similarly, let $w=1$ and $r=0$ in (2), (12) and (13), we can obtain the iteration matrices of Jacobi method associated to (1), (3) and (4). Therefore, we also have the following corollary.

Corollary 2. *Let B, $\widetilde{B}$ and $\widehat{B}$, be the iterative matrices of the Jacobi iterative method associated to* (1), (3) *and* (4), *respectively. If the matrix A of* (1) *is an irreducible L-matrix with* $0<a_{1n}a_{n1}<\alpha$ $(\alpha\geqslant 1)$, *then*

(1) $\rho(\widetilde{B})<\rho(B)$ *and* $\rho(\widehat{B})<\rho(B)$ *if* $\rho(B)<1$;
(2) $\rho(\widetilde{B})=\rho(\widehat{B})=\rho(B)$, *if* $\rho(B)=1$;
(3) $\rho(\widetilde{B})>\rho(B)$ *and* $\rho(\widehat{B})>\rho(B)$, *if* $\rho(B)>1$.

Remark 1. From our proof,we have weakened $a_{i,i+1}a_{i+1,i}>0$ of [1]. That is, our results are more general than those of [1].

Remark 2. From above results, we know that the convergence rate of the AOR (SOR, Jacobi, respectively) method can be accelerated when $\rho(L_{rw})$ ($\rho(L_w)$, $\rho(B)$, respectively) <1 if we apply the preconditioned methods to the preconditioned linear systems (3) and (4). We also know that the preconditioned methods are invalid when $\rho(L_{rw})$ ($\rho(L_w)$, $\rho(B)$, respectively) $\geqslant 1$.

5. Numerical example

Now let us consider the following example to illustrate the results obtained from Theorems 1 and 2.

Example. The coefficient matrix A of (1) is given by

$$A=\begin{bmatrix} 1 & \frac{1}{20}-\frac{1}{20} & \frac{1}{30}-\frac{1}{20} & \frac{1}{40}-\frac{1}{20} & \cdots & \frac{1}{10n}-\frac{1}{20}\\ \frac{1}{20}-\frac{1}{20} & 1 & \frac{1}{30}-\frac{1}{20} & \frac{1}{40}-\frac{1}{20} & \cdots & \frac{1}{10n}-\frac{1}{20}\\ \frac{1}{30}-\frac{1}{20} & \frac{1}{20}-\frac{1}{20} & 1 & \frac{1}{40}-\frac{1}{20} & \cdots & \frac{1}{10n}-\frac{1}{20}\\ \frac{1}{40}-\frac{1}{20} & \frac{1}{30}-\frac{1}{20} & \frac{1}{20}-\frac{1}{20} & 1 & \cdots & \frac{1}{10n}-\frac{1}{20}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{10n}-\frac{1}{20} & \frac{1}{10n}-\frac{1}{20} & \frac{1}{10(n-1)}-\frac{1}{20} & \frac{1}{10(n-2)}-\frac{1}{20} & \cdots & 1 \end{bmatrix}.$$

It is easy to know that $a_{i+1,i}=0$ from the above matrix A, then $a_{i,i+1}a_{i+1,i}=0$. The preconditioned method in [1] is invalid. However, we get the following table by using our methods. The digital of following table is formed by Matlab 6.51 program.

n	w	r	α	$\rho(L_{wr})$	$\rho(\widetilde{L}_{wr})$	$\rho(\widehat{L}_{wr})$
10	0.9	0.8	1	0.2465	0.2428	0.2385
15	0.95	0.75	2	0.3847	0.3835	0.3844
20	0.8	0.65	3	0.6843	0.6839	0.6842
25	0.7	0.6	4	0.9234	0.9233	0.9233
30	0.6	0.5	2	1.1219	1.1221	1.1220

For the SOR method, we have the following results.

n	w	r	α	$\rho(L_w)$	$\rho(\widetilde{L}_w)$	$\rho(\widehat{L}_w)$
10	0.9	0.9	1	0.2271	0.2235	0.2265
15	0.8	0.8	2	0.4723	0.4712	0.4720
20	0.7	0.7	3	0.7172	0.7169	0.7171
25	0.6	0.6	1	0.9434	0.9341	0.9342
30	0.5	0.5	2	1.1016	1.1017	1.1017

Remark 3. From the above table, it is easy to know that $\rho(\widetilde{L}_{wr}) < \rho(L_{wr})$ and $\rho(\widehat{L}_{wr}) < \rho(L_{wr})$ when $\rho(L_{wr}) < 1$. $\rho(\widetilde{L}_{wr}) > \rho(L_{wr})$ and $\rho(\widehat{L}_{wr}) > \rho(L_{wr})$ when $\rho(L_{wr}) > 1$. From the above numerical experiments, we get that the results are in concord with Theorems 1 and 2.

References

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