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Quicksort algorithm: Application of a fixed point theorem in intuitionistic fuzzy quasi-metric spaces at a domain of words

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ABSTRACT

In this paper, we apply an intuitionistic fuzzy quasi-metric version of a fixed point theorem, to obtain the existence of solution for a recurrence equation associated with the analysis of Quicksort algorithms.

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1. Introduction and preliminaries

In this section, using the idea of intuitionistic fuzzy metric spaces introduced in [1–3] we define the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t-representable (see [4]).

Lemma 1.1 ([5]). *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ and $x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 1.2 ([6]). An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u))|u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

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We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Classically, a triangular norm $* = T$ on $[0, 1]$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \longrightarrow [0, 1]$ satisfying $T(1, x) = 1 * x = x$, for all $x \in [0, 1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \longrightarrow [0, 1]$ satisfying $S(0, x) = 0 \diamond x = x$, for all $x \in [0, 1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 1.3 ([5]). A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$, satisfying the following conditions:
$(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)
$(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$, (commutativity)
$(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$, (associativity)
$(\forall (x, x', y, y') \in (L^*)^4)\ (x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$. (monotonicity)

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*}, then $\mathcal{T}$ is said to be a *continuous* t-norm.

Definition 1.4 ([5]). A continuous t-norm $\mathcal{T}$ on L^* is called continuous *t-representable* if and only if, there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now, define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \ldots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \ldots, x^{(n)}), x^{(n+1)})$$

for $n \geq 2$ and $x^{(i)} \in L^*$.

For example, $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* is a continuous t-representable.

Definition 1.5. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \longrightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined as, for all $x \in [0, 1]$, $N_s(x) = 1 - x$. We show $(N_s(\lambda), \lambda) = \mathcal{N}_s(\lambda)$.

Definition 1.6. The t-norm $\mathcal{T}$ is *Hadžić type* if for given $\varepsilon \in (0, 1)$ there is $\delta \in (0, 1)$, such that

$$\mathcal{T}^m(\mathcal{N}_s(\delta), \ldots, \mathcal{N}_s(\delta)) >_{L^*} \mathcal{N}_s(\varepsilon), \quad m \in \mathbf{N}.$$

A typical example of such t-norms is

$$\wedge(a, b) = (\min(a_1, b_1), \max(a_2, b_2)),$$

in which $a = (a_1, a_2)$ and $b = (b_1, b_2)$ are two elements of L^*. The pair (X, d) is said to be a *quasi metric space*, if X is an arbitrary (non-empty) set, and d is a mapping from X^2 into $[0, +\infty)$, satisfying the following conditions for every $x, y, z \in X$:

(i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 1.7. Let M, N are fuzzy sets from $X^2 \times [0, +\infty)$ to $[0, 1]$ such that $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy quasi-metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-representable and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times [0, +\infty) \to L^*$ (an intuitionistic fuzzy set, see Definition 1.2) satisfying the following conditions for every $x, y, z \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x, y, 0) = 0_{L^*}$;
(b) $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t) = 1_{L^*}$ if and only if $x = y$;
(c) $\mathcal{M}_{M,N}(x, y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x, z, t), \mathcal{M}_{M,N}(z, y, s))$;
(d) $\mathcal{M}_{M,N}(x, y, \cdot) : [0, \infty) \longrightarrow L^*$ is left continuous.

In this case, $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy quasi-metric*. Here,

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)).$$

Note that an intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfying for all $x, y \in X$ and $t > 0$ the symmetry axiom $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$ is an intuitionistic fuzzy metric space [4].

If the intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ satisfies the condition:

$$\lim_{t \to \infty} \mathcal{M}_{M,N}(x, y, t) = \lim_{t \to \infty} \mathcal{M}_{M,N}(y, x, t) = 1_{L^*},$$

then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called *Menger* intuitionistic fuzzy quasi-metric space. If $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy quasi-metric space then $(X, \mathcal{M}^{-1}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy quasi-metric space, where $\mathcal{M}^{-1}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$. Moreover, if we denote by $\mathcal{M}^i_{M,N}$ the fuzzy set in $X^2 \times [0, +\infty)$ given by

$$\mathcal{M}^i_{M,N}(x, y, t) = \mathcal{T}(\mathcal{M}_{M,N}(x, y, t), \mathcal{M}^{-1}_{M,N}(x, y, t)),$$

then $(X, \mathcal{M}^i_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy quasi-metric space. For $t > 0$, define the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}_{M,N}(x, y, t) >_{L^*} (N_s(r), r) = \mathcal{N}_s(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$ for example $\emptyset$ and X are two open sets.

Proposition 1.8. *Let $U_1, \ldots, U_n$ be n open sets in an intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Then, $\bigcap_{i=1}^n U_i$ is open in $(X, \mathcal{M}_{M,N}, \mathcal{T})$.*

Proof. Let $x \in \bigcap_{i=1}^n U_i$, then $x \in U_i$ for every $i = 1, 2, \ldots, n$. Since U_i's $(i = 1, 2, \ldots, n)$ are open then there are $r_i \in (0, 1)$ and $t_i > 0$, such that $B(x, r_i, t_i) \subseteq U_i$. Put $r = \min\{r_i\}$ and $t = \min\{t_i\}$ in which $i = 1, 2, \ldots, n$. Then, we have $B(x, r, t) \subseteq B(x, r_i, t_i) \subseteq U_i$ for every $i = 1, 2, \ldots, n$. Hence $B(x, r, t) \subseteq \bigcap_{i=1}^n U_i$. □

Proposition 1.9. *Let $\{U_n\}_{n\in\mathbf{N}}$ be a sequence of open sets in an intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Then $\bigcup_{n\in\mathbf{N}} U_n$ is open in $(X, \mathcal{M}_{M,N}, \mathcal{T})$.*

Proof. The proof is the same as classical case see [7]. □

Let $\tau_{\mathcal{M}_{M,N}}$ denote the family of all open subsets of X. Propositions 1.8 and 1.9 imply that $\tau_{\mathcal{M}_{M,N}}$ is a topology. $\tau_{\mathcal{M}_{M,N}}$ is called the *topology induced by intuitionistic fuzzy quasi-metric*.

A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a *Cauchy sequence* if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbf{N}$, such that

$$\mathcal{M}_{M,N}(x_n, x_m, t) >_{L^*} \mathcal{N}_s(\varepsilon),$$

and for each $n, m \geq n_0$. The sequence $\{x_n\}$ is said to be *convergent* to $x \in X$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and denoted by $x_n \xrightarrow{\mathcal{M}_{M,N}} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \longrightarrow 1_{L^*}$ whenever $n \longrightarrow \infty$ for every $t > 0$. An intuitionistic fuzzy metric space is said to be *complete* if and only if, every Cauchy sequence is convergent (see [4,1]).

A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a *G-Cauchy sequence* if $\lim_{n\to\infty} \mathcal{M}_{M,N}(x_n, x_{n+p}, t) = 1_{L^*}$ for each $t > 0$ and $p \in \mathbf{N}$. An intuitionistic fuzzy metric space is said to be *G-complete* if and only if, every *G*-Cauchy sequence is convergent.

Definition 1.10. A sequence $\{x_n\}$ in an intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is *G*-Cauchy if it is *G*-Cauchy in the intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,N}, \mathcal{T})$.

Definition 1.11. An intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called *G*-bicomplete if the intuitionistic fuzzy metric space $(X, \mathcal{M}^i_{M,N}, \mathcal{T})$ is *G*-complete. In this case, we say that $\mathcal{M}_{M,N}$ is a *G*-bicomplete intuitionistic fuzzy quasi-metric on X.

Lemma 1.12. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy quasi-metric space. Then, $\mathcal{M}_{M,N}(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Proof. The proof is the same as intuitionistic fuzzy metric space (see [4]). □

Example 1.13. Let (X, d) be a quasi-metric space. Let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}^d_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right).$$

Then, $(X, \mathcal{M}^d_{M,N}, \wedge)$ is an intuitionistic fuzzy quasi-metric space.

2. The Banach fixed point theorem in intuitionistic fuzzy quasi-metric space

A *IB*-contraction on an intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is a self mapping f on X, such that there is a constant $k \in (0, 1)$ satisfying

$$\mathcal{M}^i_{M,N}(f(x), f(y), kt) \geq_{L^*} \mathcal{M}^i_{M,N}(x, y, t), \tag{2.1}$$

for all $x, y \in X$ and $t > 0$.

The following theorem is an extension of Alaca et al.'s theorem [8].

Theorem 2.1. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a G-bicomplete Menger intuitionistic fuzzy quasi-metric space. Then every IB-contraction on X has a unique fixed point.*

Proof. Let $x \in X$ and $x_n = T^n x$ $(n \in \mathbb{N})$. By (2.1) and induction we get

$$\mathcal{M}^i_{M,N}(x_n, x_{n+1}, t) \geq_{L^*} \mathcal{M}^i_{M,N}\left(x, x_1, \frac{t}{k^n}\right)$$

for all n and $t > 0$. Thus for any positive integer p we have

$$\begin{aligned}
\mathcal{M}^i_{M,N}(x_n, x_{n+p}, t) &= \mathcal{T}(\mathcal{M}_{M,N}(x_n, x_{n+p}, t), \mathcal{M}^{-1}_{M,N}(x_n, x_{n+p}, t)) \\
&\geq_{L^*} \mathcal{T}\left[\mathcal{T}^{p-1}\left(\mathcal{M}_{M,N}\left(x_n, x_{n+1}, \frac{t}{p}\right)\right), \ldots, \mathcal{M}_{M,N}\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right),\right. \\
&\quad \left.\mathcal{T}^{p-1}\left(\mathcal{M}^{-1}_{M,N}\left(x_n, x_{n+1}, \frac{t}{p}\right)\right), \ldots, \mathcal{M}^{-1}_{M,N}\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right)\right] \\
&\geq_{L^*} \mathcal{T}\left[\mathcal{T}^{p-1}\left(\mathcal{M}_{M,N}\left(x, x_1, \frac{t}{pk^n}\right)\right), \ldots, \mathcal{M}_{M,N}\left(x, x_1, \frac{t}{pk^n}\right),\right. \\
&\quad \left.\mathcal{T}^{p-1}\left(\mathcal{M}^{-1}_{M,N}\left(x, x_1, \frac{t}{pk^n}\right)\right), \ldots, \mathcal{M}^{-1}_{M,N}\left(x, x_1, \frac{t}{pk^n}\right)\right]
\end{aligned}$$

taking $n \to \infty$ in the above inequality, then we have

$$\lim_{n\to\infty} \mathcal{M}^i_{M,N}(x_n, x_{n+p}, t) = 1_{L^*},$$

i.e., $\{x_n\}$ is a G-Cauchy sequence, hence convergent. Call the limit y. Then we have

$$\begin{aligned}
\mathcal{M}^i_{M,N}(Ty, y, t) &= \mathcal{T}(\mathcal{M}_{M,N}(Ty, y, t), \mathcal{M}^{-1}_{M,N}(Ty, y, t)) \\
&\geq_{L^*} \mathcal{T}\left[\mathcal{T}\left(\mathcal{M}_{M,N}\left(Ty, Tx_n, \frac{t}{2}\right)\right), \mathcal{T}\left(\mathcal{M}_{M,N}\left(x_{n+1}, y, \frac{t}{2}\right)\right),\right. \\
&\quad \left.\mathcal{T}\left(\mathcal{M}^{-1}_{M,N}\left(Ty, Tx_n, \frac{t}{2}\right)\right), \mathcal{T}\left(\mathcal{M}^{-1}_{M,N}\left(x_{n+1}, y, \frac{t}{2}\right)\right)\right] \\
&\geq_{L^*} \mathcal{T}\left[\mathcal{T}\left(\mathcal{M}_{M,N}\left(y, x_n, \frac{t}{2k}\right)\right), \mathcal{T}\left(\mathcal{M}_{M,N}\left(x_{n+1}, y, \frac{t}{2}\right)\right),\right. \\
&\quad \left.\mathcal{T}\left(\mathcal{M}^{-1}_{M,N}\left(y, x_n, \frac{t}{2k}\right)\right), \mathcal{T}\left(\mathcal{M}^{-1}_{M,N}\left(x_{n+1}, y, \frac{t}{2}\right)\right)\right],
\end{aligned}$$

taking $n \to \infty$ in the above inequality, then we get $Ty = y$, a fixed point. To show uniqueness, assume $Tz = z$ for some $z \in X$. Then

$$1_{L^*} \geq_{L^*} \mathcal{M}^i_{M,N}(y, z, t) = \mathcal{M}^i_{M,N}(Ty, Tz, t) \geq_{L^*} \mathcal{M}^i_{M,N}\left(y, z, \frac{t}{k}\right) \geq_{L^*} \cdots \geq_{L^*} \mathcal{M}^i_{M,N}\left(y, z, \frac{t}{k^n}\right) \longrightarrow 1_{L^*}$$

as $n \longrightarrow \infty$. Since

$$\mathcal{M}^i_{M,N}(y, z, t) = \mathcal{T}(\mathcal{M}_{M,N}(y, z, t), \mathcal{M}^{-1}_{M,N}(y, z, t)) = 1_{L^*},$$

we have $\mathcal{M}_{M,N}(y, z, t) = \mathcal{M}^{-1}_{M,N}(y, z, t) = 1_{L^*}$. Hence $z = y$. □

3. *G*-bicompleteness in non-Archimedian intuitionistic fuzzy quasi metric space

Definition 3.1. If in the definition of intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ the triangle inequality, (c) of Definition 1.7, is replaced by:

$$\mathcal{M}_{M,N}(x, y, t) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x, z, t), \mathcal{M}_{M,N}(z, y, t)),$$

for all $x, y, z \in X$ and $t > 0$, then it is called a *non-Archimedian* intuitionistic fuzzy quasi-metric space.

Example 3.2. Let (X, d) be a quasi-metric space. It is immediate to show that (X, d) is a non-Archimedian quasi-metric space if and only if, $(X, \mathcal{M}^d_{M,N}, \wedge)$ is a non-Archimedian intuitionistic fuzzy quasi-metric space.

Theorem 3.3. *Each G-Cauchy sequence in a non-Archimedian intuitionistic fuzzy quasi-metric space* $(X, \mathcal{M}_{M,N}, \mathcal{T})$*, where* $\mathcal{T}$ *is Hadžić type, is a Cauchy sequence.*

Proof. Since $\mathcal{T}$ is Hadžić type, for given $\varepsilon \in (0, 1)$ there is $\delta \in (0, 1)$, such that

$$\mathcal{T}^m(\mathcal{N}_s(\delta), \ldots, \mathcal{N}_s(\delta)) >_{L^*} \mathcal{N}_s(\varepsilon), \quad m \in \mathbf{N}.$$

Let $\{x_n\}$ be a G-Cauchy sequence in the non-Archimedian intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Fix $\varepsilon \in (0, 1)$ and $t > 0$ and consider n_0, such that $\mathcal{M}^i_{M,N}(x_n, x_{n+1}, t) >_{L^*} \mathcal{N}_s(\varepsilon)$ for all $n \geq n_0$.

Then, for all $n \geq n_0$ and $j > 0$ we have

$$\begin{aligned}\mathcal{M}^i_{M,N}(x_n, x_{n+j}, t) &\geq_{L^*} \mathcal{T}^{j-1}(\mathcal{M}^i_{M,N}(x_n, x_{n+1}, t), \ldots, \mathcal{M}^i_{M,N}(x_{n+j-1}, x_{n+j}, t))\\ &\geq_{L^*} \mathcal{T}^{j-1}(\mathcal{N}_s(\delta), \ldots, \mathcal{N}_s(\delta))\\ &>_{L^*} \mathcal{N}_s(\varepsilon).\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in $(X, \mathcal{M}_{M,N}, \mathcal{T})$. □

Theorem 3.4. *Each bicomplete non-Archimedian intuitionistic fuzzy quasi-metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$, where $\mathcal{T}$ is Hadžić type, is G-complete.*

Proof. Let $\{x_n\}$ be a G-Cauchy sequence in the non-Archimedian intuitionistic fuzzy quasi-metric space. By Theorem 3.3, $\{x_n\}$ is a Cauchy sequence in $(X, \mathcal{M}_{M,N}, \mathcal{T})$. Then, there exists $x \in X$ such that $\lim_{n\to\infty} \mathcal{M}^i_{M,N}(x, x_n, t) = 1_{L^*}$ for all $t > 0$. Hence, $(X, \mathcal{M}^i_{M,N}, \mathcal{T})$ is G-complete, i.e., $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is G-bicomplete. □

4. Quicksort algorithm

Let n be the size of input. For a given algorithm, $A(n)$ is defined as the average (expected value) of number of times the algorithm does the basic operation for an input size of n.

Now, we look at a sorting algorithm, called *Quicksort*, that was developed by Hoare (1962) (for more details see [9,10]). In Quicksort, the sort is accomplished by dividing the array into partitions, and then sorting each partition recursively.

Average-Case Time Complexity

Bascic operation: the comparison of $S[i]$ with pivotitem in partition.

Input size: n, the number of items in the array S.

We will assume that we have no reason to believe that the numbers in the array are in any particular order, and therefore that the value of pivotpoint returned by partition is equally likely to be any of the numbers from 1 through n. If there was reason to believe a different distribution, this analysis would not be applicable. The average obtained is, therefore, the average sorting time when every possible ordering is sorted the same number of times. In this case, the average-case time complexity is given by the following recurrence:

$$A(n) = \sum_{p=1}^{n} \overbrace{\frac{1}{n}}^{\text{Probability pivotpoint is } p} \underbrace{[A(p-1) + A(n-p)]}_{\substack{\text{Average time to}\\ \text{sort subarrays when}\\ \text{pivotponit is } p}} + \underbrace{n-1}_{\substack{\text{Times to}\\ \text{partition}}}. \tag{4.1}$$

Therefore

$$\sum_{p=1}^{n}[A(p-1) + A(n-p)] = 2\sum_{p=1}^{n} A(p-1). \tag{4.2}$$

Plugging Equality (4.2) into Equality (4.1) yields

$$A(n) = \frac{2}{n}\sum_{p=1}^{n} A(p-1) + n - 1.$$

Multiplying by n we have

$$nA(n) = 2\sum_{p=1}^{n} A(p-1) + n(n-1). \tag{4.3}$$

Applying Equality (4.3) to $n-1$ gives

$$(n-1)A(n-1) = 2\sum_{p=1}^{n-1} A(p-1) + (n-1)(n-2). \tag{4.4}$$

Subtracting Equality (4.4) from Equality (4.3) yields

$$nA(n) - (n-1)A(n-1) = 2A(n-1) + 2(n-1),$$

which simplifies to

$$A(1) = 0, \qquad A(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}A(n-1), \qquad n \geq 2.$$

5. Applications to the domain of words

Actually, we will establish our results in the more general framework of intuitionistic fuzzy quasi-metric spaces because, in this context, the measurement of the distance from a word x to other word y automatically, indicates if x is a prefix of y or not.

Let Σ be a nonempty alphabet. Let Σ^∞ be the set of all finite and infinite sequence ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞.

Denote by $\sqsubseteq$, the prefix order on Σ^∞, i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y. For each $x \in \Sigma^\infty$ denote by $l(x)$ the length of x. Then, $l(x) \in [1, \infty]$ whenever $x \neq \phi$ and $l(\phi) = 0$. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of x and y.

Thus, the function $d_\sqsubseteq$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$\begin{cases} d_\sqsubseteq(x,y) = 0, & \text{if } x \sqsubseteq y, \\ d_\sqsubseteq(x,y) = 2^{-l(x\sqcap y)}, & \text{otherwise,} \end{cases}$$

is a quasi-metric on Σ^∞. We adopt the convention that $2^{-\infty} = 0$ (for more details see [11]).

Let $\mathcal{M}_{M,N}^{d_\sqsubseteq 1}(x, y, t)$ be defined as:

$\mathcal{M}_{M,N}^{d_\sqsubseteq 1}(x, y, 0) = 0_{L^*}$, for all $x, y \in \Sigma^\infty$;

$\mathcal{M}_{M,N}^{d_\sqsubseteq 1}(x, y, t) = 1_{L^*}$ if x is a prefix of y, and $t > 0$;

$\mathcal{M}_{M,N}^{d_\sqsubseteq 1}(x, y, t) = \mathcal{N}_s(2^{-l(x\sqcap y)})$ if x is not a prefix of y, and $t \in (0, 1]$;

$\mathcal{M}_{M,N}^{d_\sqsubseteq 1}(x, y, t) = 1_{L^*}$ if x is not a prefix of y, and $t > 1$.

Theorem 5.1. *$(\Sigma^\infty, \mathcal{M}_{M,N}^{d_\sqsubseteq 1}, \wedge)$ is a bicomplete non-Archimedian intuitionistic fuzzy quasi-metric space.*

Proof. See [11]. □

Let $\mathcal{M}_{M,N}^{d_\sqsubseteq}(x, y, t)$ be defined as:

$$\mathcal{M}_{M,N}^{d_\sqsubseteq}(x,y,t) = \begin{cases} 0_{L^*}, & \text{if } t = 0, \text{ for all } x, y \in \Sigma^\infty, \\ \left(\dfrac{t}{t + 2^{-l(x\sqcap y)}}, \dfrac{2^{-l(x\sqcap y)}}{t + 2^{-l(x\sqcap y)}} \right) & \text{if } t > 0, \text{ if } x \text{ is not a prefix of } y, \\ 1_{L^*} & \text{if } t > 0, \text{ if } x \text{ is a prefix of } y. \end{cases}$$

Theorem 5.2. *$(\Sigma^\infty, \mathcal{M}_{M,N}^{d_\sqsubseteq}, \wedge)$ is a bicomplete non-Archimedian intuitionistic fuzzy quasi-metric space.*

Next, we apply Theorem 2.1 to the complexity analysis of Quicksort algorithms. The following recurrence equation (see [9–12])

$$A(1) = 0, \qquad A(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}A(n-1), \quad n \geq 2,$$

is obtained in the average case analysis of Quicksort algorithms.

Consider as an alphabet Σ the set of nonnegative real numbers, i.e., $\Sigma = [0, \infty)$. We associate with A the functional $\Phi : \Sigma^\infty \longrightarrow \Sigma^\infty$ given by $(\Phi(x))_1 = A(1)$ and

$$(\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}x_{n-1}$$

for all $n \geq 2$ (if $x \in \Sigma^\infty$ has length $n < \infty$, we write $x := x_1x_2\ldots x_n$, and if x is an infinite word, we write $x := x_1x_2\ldots$).

Now, we show that Φ is an IB-contractive mapping on the bicomplete non-Archimedean intuitionistic fuzzy quasi-metric space $(\Sigma^\infty, \mathcal{M}_{M,N}^{d_\sqsubseteq}, \wedge)$ with contraction constant $1/2$.

By construction, we have $l(\Phi(x)) = l(x) + 1$ for all $x, y \in \Sigma^\infty$ (in particular, $l(\Phi(x)) = \infty$ whenever $l(x) = \infty$). Furthermore, it is clear that $x \sqsubseteq y$ if and only if, $\Phi(x) \sqsubseteq \Phi(y)$ and consequently, $\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y)$ for all $x, y \in \Sigma^\infty$. Hence $l(\Phi(x \sqcap y)) \leq l(\Phi(x) \sqcap \Phi(y))$ for all $x, y \in \Sigma^\infty$.

From the preceding observations, we deduce that if x is a prefix of y, then

$$\mathcal{M}^{d_{\sqsubseteq}}_{M,N}(\Phi(x), \Phi(y), t/2) = \mathcal{M}^{d_{\sqsubseteq}}_{M,N}(x, y, t) = 1_{L^*},$$

and if x is not a prefix of y, then

$$\begin{aligned}
\mathcal{M}^{d_{\sqsubseteq}}_{M,N}(\Phi(x), \Phi(y), t/2) &= \left(\frac{t/2}{(t/2)+2^{-l(\Phi(x)\sqcap\Phi(y))}}, \frac{2^{-l(\Phi(x)\sqcap\Phi(y))}}{(t/2)+2^{-l(\Phi(x)\sqcap\Phi(y))}}\right) \\
&\geq_{L^*} \left(\frac{t/2}{(t/2)+2^{-l(\Phi(x\sqcap y))}}, \frac{2^{-l(\Phi(x\sqcap y))}}{(t/2)+2^{-l(\Phi(x\sqcap y))}}\right) \\
&= \left(\frac{t/2}{(t/2)+2^{-(l(x\sqcap y)+1)}}, \frac{2^{-(l(x\sqcap y)+1)}}{(t/2)+2^{-(l(x\sqcap y)+1)}}\right) \\
&= \left(\frac{t}{t+2^{-(l(x\sqcap y))}}, \frac{2^{-(l(x\sqcap y))}}{t+2^{-(l(x\sqcap y))}}\right) \\
&= \mathcal{M}^{d_{\sqsubseteq}}_{M,N}(x, y, t),
\end{aligned}$$

for all $t > 0$. Similarly, we have

$$\mathcal{M}^{d_{\sqsubseteq}}_{M,N}(\Phi(y), \Phi(x), t/2) \geq_{L^*} \mathcal{M}^{d_{\sqsubseteq}}_{M,N}(y, x, t).$$

Therefore Φ is a IB-contraction on $(\Sigma^\infty, \mathcal{M}^{d_{\sqsubseteq}}_{M,N}, \wedge)$ with contraction constant $1/2$. By Theorem 2.1, Φ has a unique fixed point $z = z_1 z_2 \ldots$, which is the unique solution to the recurrence equation A, i.e., $z_1 = 0$ and

$$z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1},$$

for all $n \geq 2$.

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