Journal of Computational and Applied Mathematics 229 (2009) 7–15

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Moment information for probability distributions, without solving the moment problem, II: Main-mass, tails and shape approximation

P.N. Gavriliadis *, G.A. Athanassoulis [1]

*National Technical University of Athens, Department of Naval Architecture and Marine Engineering, Heroon Polytechniou 9, 157 73 Zografos, Athens, Greece*

ARTICLE INFO

*Article history:*
Received 6 September 2007
Received in revised form 4 April 2008

*Keywords:*
Chebyshev–Stieltjes–Markov inequality
Moments
Distribution functions
Tail
Christoffel function
Approximation

ABSTRACT

How much information does a small number of moments carry about the unknown distribution function? Is it possible to explicitly obtain from these moments some useful information, e.g., about the support, the modality, the general shape, or the tails of a distribution, without going into a detailed numerical solution of the moment problem? In this, previous and subsequent papers, clear and easy to implement answers will be given to some questions of this type. First, the question of how to distinguish between the main-mass interval and the tail regions, in the case we know only a number of moments of the target distribution function, will be addressed. The answer to this question is based on a version of the Chebyshev–Stieltjes–Markov inequality, which provides us with upper and lower, moment-based, bounds for the target distribution. Then, exploiting existing asymptotic results in the main-mass region, an explicit, moment-based approximation of the target probability density function is provided. Although the latter cannot be considered, in general, as a satisfactory solution, it can always serve as an initial approximation in any iterative scheme for the numerical solution of the moment problem. Numerical results illustrating all the theoretical statements are also presented.



## 1. Introduction

An important question arising in many problems in Physics, Technology and Finance, is how the information contained in the sequence of moments (of an unknown probability distribution) can be used to determine the corresponding distribution function or important features of it. This is, in essence, the classical moment problem of mathematical analysis, initiated by Thomas J. Stieltjes in 1894–95.

From the theoretical point of view, the univariate moment problem has been extensively studied and solved many years ago. Various famous mathematicians have contributed to the analysis and solution of this problem. Among them we refer P. Chebyshev, A. Markov, J. Shohat, M. Frechet, H. Hamburger, M. Riesz, F. Hausdorff, M. Krein. Necessary and sufficient conditions under which a probability density function (pdf) $f(x)$ can be recovered from its moments, either uniquely or not, can be found in the mathematical literature; see, e.g., various special monographs, such as [1,7,15,23] and the more recent book [24] containing excellent surveys on the moment problems. See also the recent works [4,13,18,19,22,25].

From the numerical point of view, the moment problem is universally recognized as a difficult *inverse problem* which leads to the solution of highly ill-posed systems of equations (see, e.g., [5,6,8,30,31] for a general overview of ill-posed problems). Thus, a fundamental question is how to reformulate the numerical moment problems in a way permitting the reliable and efficient determination (approximation) of the underlying pdfs. In accordance with the general principles concerning the regularization of ill-posed problems (see, e.g., [5,6,8,29]), the *a priori* characterization of the *data space* (i.e. the moment

* Corresponding author. Tel.: +30 210 7721139.
*E-mail addresses:* pmath@central.ntua.gr, takis.pmath@gmail.com (P.N. Gavriliadis), mathan@central.ntua.gr (G.A. Athanassoulis).
[1] Tel.: +30 210 7721136, +30 210 7721137; fax: +30 210 7721397.

0377-0427/$ – see front matter 
doi:10.1016/j.cam.2008.10.011

space) and the *solution space* (i.e. the admissible pdf class) have been found to be of fundamental importance in implementing a well-posed numerical solution scheme to the moment problem.

The *a priori* characterization of the (*moment*) *data space* can be achieved by using appropriate restrictions on the moment data, ensuring that the given moment sequence, does belong to the appropriate moment space [9]. The characterization of the *solution space* is mainly controlled by the choice of an appropriate representation for the function to be recovered [5,29, 2,11]. In this way, it is possible to exploit much (if not all) of existing a priori knowledge about the unknown pdf, resulting in regularized numerical solution schemes. For instance, the positivity of (any) probability distribution function is an obvious restriction, the implementation of which greatly enhances and accelerates the solution scheme [2,11]. In some cases the boundary conditions (end-point values, tail behavior) [3,11], or modality information (unimodality/multimodality, mode localization) [10] might be known and contribute to the efficiency of the numerical solution scheme.

In this work, we exploit two more properties that can be used as *a priori* information in order to improve the efficiency of any numerical algorithm for the solution of the moment problem: the "localization" of the target pdf (i.e. the identification of the main-mass interval and of the tail intervals) and an explicitly given approximant pdf, that can be used as an initial approximate solution for any numerical solution scheme.

Thus, the main questions examined in this work are the following: Given the $2k$ moments $\mu_0 = 1, \mu_1 = \int x\mathrm{d}F(x), \ldots, \mu_{2k} = \int x^{2k}\mathrm{d}F(x)$ of an unknown cumulative distribution function (cdf) $F(x)$, with unknown support, how can we identify the right-tail and left-tail delimiters, defined as, e.g.,

$$x_* : F(x_*) < \varepsilon_*,$$
$$x^* : F\left(x^*\right) > 1 - \varepsilon^*,$$

where $\varepsilon_*$ and $\varepsilon^*$ are small positive quantities? And further, having defined the tail delimiters $x_*, x^*$ (or equivalently, having defined the main-mass interval $[x_*, x^*]$), is it possible to construct (explicit) moment-based approximants for the target pdf in $(x_*, x^*)$?

Before dealing with the above questions, we briefly summarize, in Section 2, some theoretical background we need for our purposes. Section 3 provides moment-based bounds for the probability distribution which are the main tools for the separation of the main-mass interval and the tail interval of it. Furthermore, in Section 4, a known theoretical asymptotic result is reviewed and numerically exploited to construct explicit pdf approximations, demonstrating that a few moments are able to provide us with valuable shape information for the main-mass pattern. The usefulness of all theoretical results discussed herewith is illustrated in Section 5, by means of various numerical examples. Finally, a general discussion and some fundamental conclusions are presented in Section 6.

## 2. Preliminaries

To formulate bounds for the cdf use will be made of the sequence of orthonormal (with respect to $\mathrm{d}F(x)$) polynomials $P_0(x), P_1(x), \ldots, P_k(x), \ldots,$ which can be also expressed in terms of moments of the cdf $F(x)$ as follows (see, e.g., [23,26]):

$$P_k(x) = \frac{1}{\sqrt{H_{2k}H_{2k-2}}} D_k(x), \quad k = 0, 1, \ldots, \tag{1}$$

where

$$D_k(x) = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \vdots & \vdots & & \vdots \\ \mu_{k-1} & \mu_k & \cdots & \mu_{2k-1} \\ 1 & x & \cdots & x^k \end{bmatrix}, \quad \text{and} \quad H_{2k} = \begin{vmatrix} \mu_0 & \cdots & \mu_k \\ \vdots & & \vdots \\ \mu_k & \cdots & \mu_{2k} \end{vmatrix},$$

with $H_{-2} = H_0 = 1$.

The following properties of $P_k(x)$ (see, e.g., [21,26]) provides us with useful information concerning the roots of the polynomials $P_k(x)$.

- *Properties of the polynomials* $P_k$

Let $x_{k,1}, x_{k,2}, \ldots, x_{k,k}$ be the roots of the polynomial $P_k(x)$, and $\tilde{x}_{k+1,1}, \tilde{x}_{k+1,2}, \ldots, \tilde{x}_{k+1,k}, \tilde{x}_{k+1,k+1}$ be the roots of the polynomial $P_{k+1}(x)$, both taken in ascending order. Then

(i) The zeros of $P_k(x)$ are all *real and simple*. If the support of $F(x)$ is an interval $[a, b]$, then all the zeros of $P_k(x)$ are in $[a, b]$ and $\lim_{k\to\infty} x_{k,1} = a$ and $\lim_{k\to\infty} x_{k,k} = b$.
(ii) Two consecutive polynomials $P_k(x)$ and $P_{k+1}(x)$ have *no common zeros*.
(iii) The roots $x_{k,i}$, $i = 1, 2, \ldots, k$, of the polynomial $P_k(x)$ separate the roots $\tilde{x}_{k+1,i}$, $i = 1, 2, \ldots, k+1$ of the polynomial $P_{k+1}(x)$; that is, $\tilde{x}_{k+1,i} < x_{k,i} < \tilde{x}_{k+1,i+1}$. The last property is usually called the *interlacing property*.

In order to formulate the bounds of a cdf, it is useful to consider the *Christoffel function* $\lambda_k(x)$ associated with $\mathrm{d}F$, which is defined by

$$\lambda_k(x) = \left[\sum_{n=0}^{k} |P_n(x)|^2\right]^{-1}, \quad k = 1, 2, \ldots. \tag{2}$$

The values of the Christoffel function at two roots $x_{k,i}$, $x_{k,j}$, $j > i$, of polynomials $P_k(x)$ can provide us with useful information for the "estimation" of probability mass lying between them. This is made precise in the following.

**Theorem 2.1.** *Let* $\mu_0 = 1, \mu_1 = \int x \, \mathrm{d}F(x), \dots, \mu_{2k} = \int x^{2k} \mathrm{d}F(x)$ *be the moments of a cdf* $F(x)$, *and* $x_{k,1}, x_{k,2}, \dots, x_{k,k}$, *be the roots of polynomial* $P_k(x)$, *taken* in ascending order. *Then, for two any distinct roots* $x_{k,l} < x_{k,m}$, $1 \le l < m \le k$, *we have*

$$\sum_{i=l+1}^{m-1} \lambda_k\left(x_{k,i}\right) \le \int_{x_{k,l}}^{x_{k,m}} \mathrm{d}F(x) \le \sum_{i=l}^{m} \lambda_k\left(x_{k,i}\right). \quad \blacksquare \tag{3}$$

The proof can be found in various classical treatises as, e.g., [1,16,17,23].

## 3. Lower and upper bounds for the cdf

Exploiting Eq. (3) it is not difficult to obtain strict upper and lower bounds for the values of $F\left(x_{k,i}\right)$.

**Theorem 3.1.** *With the same notation as in Theorem 2.1, for any set of roots* $\left\{x_{k,i}, i = 1, 2, \dots, k\right\}$, *we have*

$$L_{k,i}(F) \le F\left(x_{k,i}\right) \le U_{k,i}(F), \tag{4a}$$

*where*

$$L_{k,i}(F) = L_{k,i}\left(\{\mu_n\}_{n=1}^{2k}\right) = 1 - \sum_{j=i}^{k} \lambda_k\left(x_{k,j}\right) \tag{4b}$$

$$\textit{and} \quad U_{k,i}(F) = U_{k,i}\left(\{\mu_n\}_{n=1}^{2k}\right) = \sum_{j=1}^{i} \lambda_k\left(x_{k,j}\right). \tag{4c}$$

**Proof.** By taking the right hand of inequality (3) for the first root $x_{k,l} = x_{k,1}$ of the polynomials $P_k(x)$ and assuming $\lim_{k\to\infty} x_{k,1} = a$, with $F(a) = 0$ (see the first property of the polynomials $P_k(x)$), we get $F\left(x_{k,i}\right) \le \sum_{j=1}^{i} \lambda_k\left(x_{k,j}\right)$. Applying the same procedure for the last root $x_{k,m} = x_{k,k}$ and assuming that $\lim_{k\to\infty} x_{k,k} = b$, with $F(b) = 1$, we have $1 - \sum_{j=i}^{k} \lambda_k\left(x_{k,j}\right) \le F\left(x_{k,i}\right)$. $\blacksquare$

Theorem 3.1 permits us to easily construct lower and upper bounds ($L_{k,i}(F)$ and $U_{k,i}(F)$, respectively) for the unknown cdf, based exclusively on the knowledge of moments $\mu_1, \dots, \mu_{2k}$. Furthermore, the values of $L_{k,i}$ and $U_{k,i}$ calculated at the first and last roots, $x_{k,1}$, $x_{k,k}$, respectively, provide us with information about the total probability mass $PR_{mass}$ included in the main-mass interval and the total probability mass $PR_{tail}$ for the tails intervals. More precisely, we have the following estimates for the probability of the main-mass $PR_{mass}^{est}$ and tails $PR_{tail}^{est}$:

$$L_{k,k} - U_{k,1} \le PR_{mass}^{est} \le U_{k,k} - L_{k,1}, \tag{5a}$$

$$1 - U_{k,k} \le PR_{right\ tail}^{est} \le 1 - L_{k,k}, \tag{5b}$$

$$L_{k,1} \le PR_{left\ tail}^{est} \le U_{k,1}. \tag{5c}$$

In other words, the total probability mass lying on the $\left[x_{k,1}, x_{k,k}\right]$ interval, estimated only by means of the $2k$ given moments, will be

$$PR_{mass}^{est} = PR_{mass}^{(k)} \pm \mathrm{ER}^{(k)}, \tag{6a}$$

with central value

$$PR_{mass}^{(k)} = \frac{U_{k,k} - U_{k,1} + L_{k,k} - L_{k,1}}{2} \tag{6b}$$

and range defined by means of

$$\mathrm{ER}^{(k)} = \frac{U_{k,k} + U_{k,1} - L_{k,k} - L_{k,1}}{2}. \tag{6c}$$

The estimation of the tail probability(ies) is most valuable for cdfs with unbounded support.

The above results permit us to formulate a clear and easy to implement algorithm for the definition of tail delimiters:

*- Algorithm for the definition of tail delimiters*

Assume that the $2k$ moments $\mu_1, \dots, \mu_{2k}$ of a target pdf are known. Then,

(i) Define the polynomials $P_0(x), P_1(x), \dots, P_k(x)$, by means of Eq. (1).
(ii) Find the $k$ roots $x_{k,1}, x_{k,2}, \dots, x_{k,k}$ of the last polynomial $P_k(x)$.
(iii) Define the Christoffel function $\lambda_k(x)$ by means of Eq. (2).

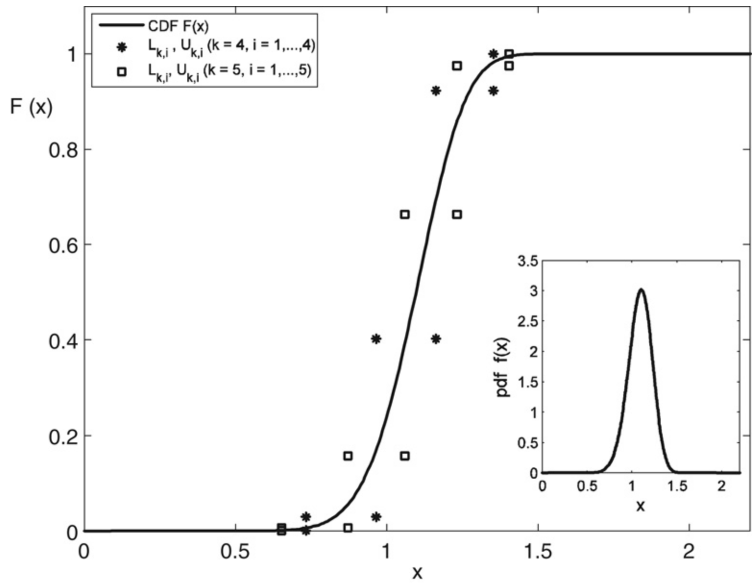


**Fig. 1.** Localization of the main-mass and tail intervals of a finitely-supported, bimodal cdf (a generalized Gamma distribution supported on $[0, +\infty)$), by means of the cdf bounds (Eq. (4)), for $2k = 8$, 10 moments.

(iv) Calculate the Christoffel function $\lambda_k(x)$ at $x = x_{k,1}, x_{k,2}, \ldots, x_{k,k}$.
(v) Calculate the lower and upper bounds $L_{k,i}$ and $U_{k,i}$ of the cdf $F(x)$ at each $x_{k,1}, x_{k,2}, \ldots, x_{k,k}$, by means of Eq. (4).
(vi) Consider $x_{k,1}$ and $x_{k,k}$ as left- and right-tail delimiters, respectively.

In Section 5, the above algorithm is applied to various pdfs under the assumption that a number of exact moments are known.

## 4. An explicit approximation of the target pdf

An explicitly given, moment-based, continuous approximant of the target pdf, valid in the main-mass region, and carrying over its main shape characteristics, will be described in this section. This approximant is based on the asymptotic behaviour of the Christoffel functions (cf. [14], referring to a similar approach). In this connection, we recall an asymptotic result formulated and proved by Totik [32].

**Theorem 4.1.** *Let $f(x)$ be a positive function on the real line. If $I = (a, b)$ is an interval, in which the function $f(x)$ is bounded and $f(x) > 0$, then, for almost all $x \in I$,*

$$\lim_{k\to\infty} k\lambda_k(x) = \pi\sqrt{(x-a)(b-x)}\, f(x). \quad \blacksquare \tag{7}$$

Thus, if $k$ is sufficiently large, the target pdf can be approximated as follows:

$$f(x) \approx f_{AP,k}(x) = \frac{k}{c_0\,\pi\sqrt{(x-a)(b-x)}}\lambda_k(x), \quad x \in (a, b), \tag{8a}$$

where $c_0$ is a normalized factor. Besides, by integrating (8a), we obtain an approximant for the target cdf:

$$F(x) \approx F_{AP,k}(x) = \frac{k}{\pi c_0}\int_a^x \frac{1}{\sqrt{(u-a)(b-u)}}\lambda_k(u)\,\mathrm{d}u, \quad x \in (a, b). \tag{8b}$$

The interval $I = (a, b)$ should be contained in the support of the main-mass of the target pdf. Thus, if $F(x)$ is strictly monotone, the end points $a$ and $b$ can be taken as $a = x_{k,1}$ and $b = x_{k,k}$, provided that $L_{k,1}(F)$ is sufficiently near to zero (e.g., $L_{k,k}(F) \approx 0.01$), while $U_{k,k}(F)$ is sufficiently near to unity (e.g., $U_{k,k}(F) \approx 0.99$), defined by means of the upper and lower bounds.

For simple, unimodal pdfs, approximation (8a) and (8b) may be very satisfactory for $k = 5$–7, while for more difficult, bimodal densities typical values of $k$ providing a good first shape approximation are $k = 7$–9. Numerical illustrations are given in the next section.

## 5. Application and numerical examples

Some illustrative examples of applications of the theoretical results presented above are given in Figs. 1–4. The tail delimiters' algorithm, presented in Section 3, was applied to a unimodal and to three bimodal probability distributions,

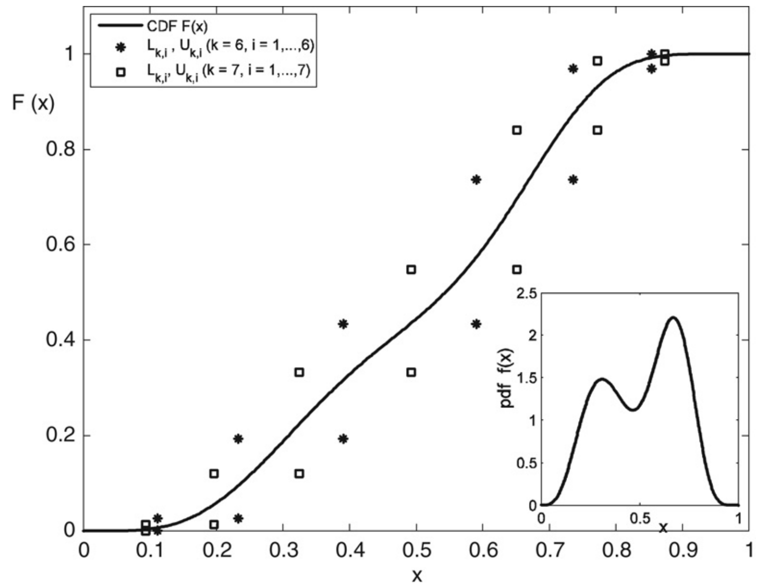


**Fig. 2.** Localization of the main-mass and tail intervals of a finitely-supported, bimodal cdf (a mixture of Beta distributions supported on [0, 1]), by means of the cdf bounds (Eq. (4)), for $2k = 12$, 14 moments.

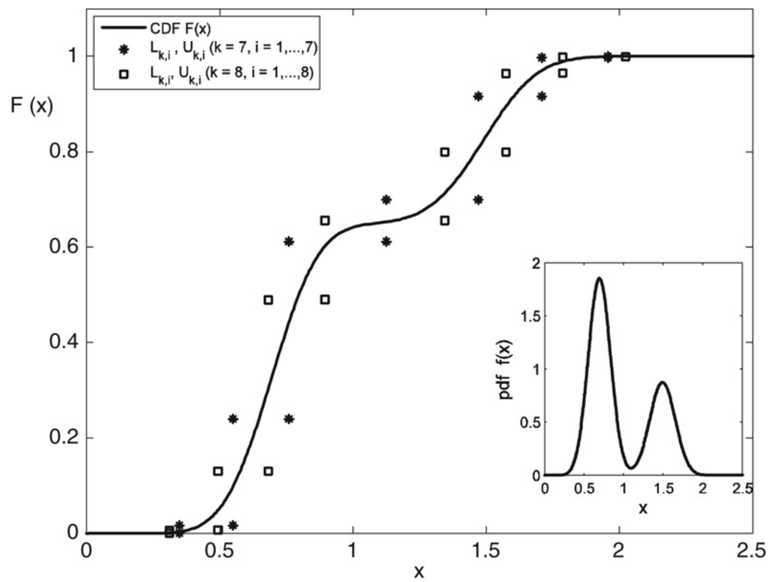


**Fig. 3.** Localization of the main-mass and tail intervals of a semi-infinite supported, bimodal cdf (a mixture of generalized Gamma distributions supported on $[0, +\infty)$), by means of the cdf bounds (Eq. (4)), for $2k = 14$, 16 moments.

supported, the first and the third on a semi-infinite interval, the second on a finite interval, and the fourth on the full line. In all figures the corresponding lower and upper bounds, $L_{k,i}(F)$ and $U_{k,i}(F)$, have been plotted, along with the target cdfs. To each figure (Figs. 1–4) a table is associated, containing the first and last roots of the polynomials $P_k(x)$, the values of the lower and upper bounds, and the values of the cdf at the aforementioned roots.

In the first case (Fig. 1; semi-infinite supported cdf; a generalized gamma distribution), the interval $\left[x_{k,1}, x_{k,k}\right]$, for $k = 4$, 5, (total number of moments $2k = 8$, 10, respectively), gives a very good estimate of the location of the main-mass region. For example, the exact probability mass on $\left[x_{5,1} = 0.651824343, x_{5,5} = 1.403742415\right]$ is $PR\left[x_{5,1}, x_{5,7}\right] = F\left(x_{5,5}\right) - F\left(x_{5,1}\right) = 0.993684031$, while the estimate for the probability on this interval is $PR^{est}_{mass} = 0.984527843 \pm 0.015467541$ (see Table 1).

In the second case (Fig. 2; finitely-supported cdf on [0, 1]; a mixture of two Beta distributions), the interval $\left[x_{k,1}, x_{k,k}\right]$, for $k = 6$, 7, (total number of moments $2k = 12$, 14, respectively), also gives a very good estimate of the location of the

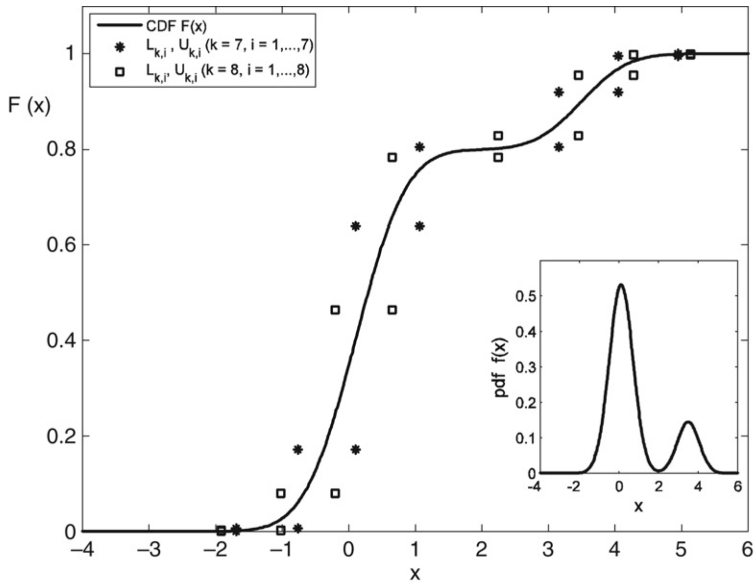


**Fig. 4.** Localization of the main-mass and tail intervals of a full-line supported, bimodal cdf (a mixture of Gaussian distributions supported on $\mathbb{R}$), by means of the cdf bounds (Eq. (4)), for $2k = 14, 16$ moments.

**Table 1**
The values of the lower and upper bounds ($L_{k,i}$ and $U_{k,i}$) and of the cdf $F(x)$, calculated at the first and last roots of the $k$th-order polynomial $P_k(x)$, for the case shown in Fig. 1.

| Degree of $P_k(x)$ | Roots of $P_k(x)$ | Lower bound $L_{k,i}$ | CDF $F(x)$ | Upper bound $U_{k,i}$ |
|---|---|---|---|---|
| $k = 4$ | $x_{k,1} = 0.733272893$ | 0.000000006 | 0.005906508 | 0.028684244 |
| | $x_{k,4} = 1.352372074$ | 0.922883717 | 0.982101219 | 0.999999993 |
| $k = 5$ | $x_{k,1} = 0.651824343$ | 0.000002307 | 0.001137028 | 0.006073012 |
| | $x_{k,5} = 1.403742415$ | 0.975133314 | 0.994821059 | 0.999997692 |

**Table 2**
The values of the lower and upper bounds ($L_{k,i}$ and $U_{k,i}$) and of the cdf $F(x)$, calculated at the first and last roots of the $k$th-order polynomial $P_k(x)$, for the case shown in Fig. 2.

| Degree of $P_k(x)$ | Roots of $P_k(x)$ | Lower bound $L_{k,i}$ | CDF $F(x)$ | Upper bound $U_{k,i}$ |
|---|---|---|---|---|
| $k = 6$ | $x_{k,1} = 0.111817212$ | 0.000000052 | 0.006604550 | 0.025604921 |
| | $x_{k,6} = 0.854013216$ | 0.969160474 | 0.992512687 | 0.999999947 |
| $k = 7$ | $x_{k,1} = 0.093952631$ | 0.000002500 | 0.003190823 | 0.012845605 |
| | $x_{k,7} = 0.873687877$ | 0.985847005 | 0.996727699 | 0.999997499 |

**Table 3**
The values of the lower and upper bounds ($L_{k,i}$ and $U_{k,i}$) and of the cdf $F(x)$, calculated at the first and last roots of the $k$th-order polynomial $P_k(x)$, for the case shown in Fig. 3.

| Degree of $P_k(x)$ | Roots of $P_k(x)$ | Lower bound $L_{k,i}$ | CDF $F(x)$ | Upper bound $U_{k,i}$ |
|---|---|---|---|---|
| $k = 7$ | $x_{k,1} = 0.349307776$ | 0.000011506 | 0.003332787 | 0.015955041 |
| | $x_{k,7} = 1.956211834$ | 0.996756156 | 0.999413348 | 0.999988493 |
| $k = 8$ | $x_{k,1} = 0.310819746$ | 0.000053632 | 0.001255635 | 0.006290070 |
| | $x_{k,8} = 2.023690195$ | 0.999200098 | 0.999865473 | 0.999946367 |

main-mass region. For example, the exact probability mass on $\left[x_{7,1}=0.093952631\,, x_{7,7}=0.873687877\right]$ is $PR\left[x_{7,1}, x_{7,7}\right] = F\left(x_{7,7}\right) - F\left(x_{7,1}\right) = 0.993536875$, while the estimate for the probability on this interval is $PR^{est}_{mass} = 0.986498199 \pm 0.013496800$ (see Table 2).

Figs. 3 and 4 show how the method works for semi-infinite supported and full-line supported cdfs. In Fig. 3 (semi-infinite supported cdf; a mixture of two generalized gamma distributions), the interval $[x_{8,1} = 0.310819746, x_{8,8} = 2.023690195]$, containing 99.6% of the total probability mass ($PR^{est}_{mass} = 0.995918684 \pm 0.003008657$; see Table 3), gives a good estimate for the location of the main-mass region, leaving the semi-infinite interval $[x_{8,8}, +\infty)$ as the right-tail region. In Fig. 4

**Table 4**
The values of the lower and upper bounds ($L_{k,i}$ and $U_{k,i}$) and of the cdf $F(x)$, calculated at the first and last roots of the $k$th-order polynomial $P_k(x)$, for the case shown in Fig. 4.

| Degree of $P_k(x)$ | Roots of $P_k(x)$ | Lower bound $L_{k,i}$ | CDF $F(x)$ | Upper bound $U_{k,i}$ |
|---|---|---|---|---|
| $k = 7$ | $x_{k,1} = -1.691774734$ | 0.000000548 | 0.001129534 | 0.006305578 |
| | $x_{k,7} = 4.949479184$ | 0.995616602 | 0.999159659 | 0.999999451 |
| $k = 8$ | $x_{k,7} = -1.914953174$ | 0.000000549 | 0.000313744 | 0.001880435 |
| | $x_{k,8} = 5.135913296$ | 0.998381421 | 0.999706426 | 0.999999452 |

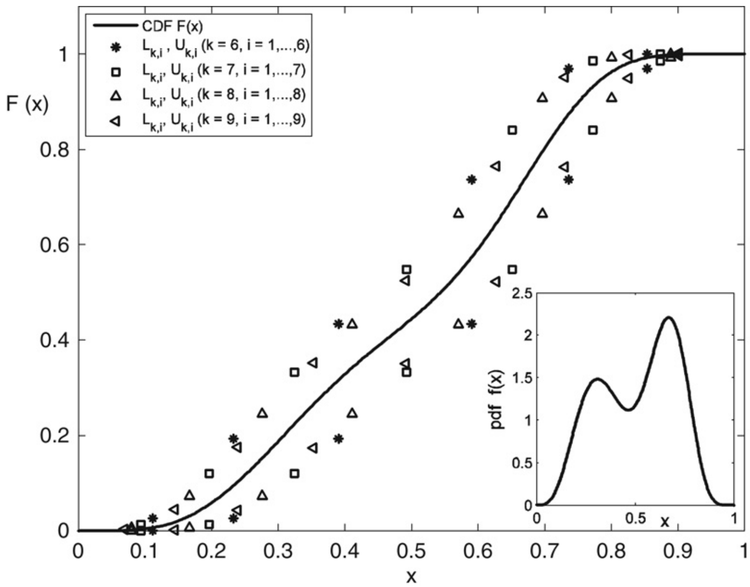


**Fig. 5.** Localization of the main-mass and tail intervals of a finitely-supported, bimodal cdf (the same with Fig. 2), by means of the cdf bounds (Eq. (4)), for $2k = 12$, 14, 16 and 18 moments.

(full-line supported cdf; a mixture of two Gaussian distributions), the interval $[x_{8,1} = -1.914953174, x_{8,8} = 5.135913296]$, containing 99.8% of the total probability mass ($PR^{est}_{mass} = 0.999824994 \pm 0.001748958$; see Table 4), is able to play the role of the main-mass interval.

In Fig. 5, we have applied the bounding procedure for the same case as in Fig. 2, using $2k = 12$, 14 (as in Fig. 2), and also $2k = 16$ and 18 moments. From this figure, we can conclude that, the points $(x_{k,i}, L_{k,i})$ and $(x_{k,i}, U_{k,i})$, as the number of roots $x_{k,i}$ increases, form an envelope, within which the target cdf lies. The shape of this envelope reveals in a clear way the basic shape features of the target cdf (e.g., the bimodality).

The explicit, moment-based approximations of the target pdf in the main-mass region, which is provided by the exploitation of the asymptotic results (8a) and (8b), are illustrated in Figs. 6–8. In Fig. 6, the target unimodal pdf (the same pdf as in Fig. 1) is reconstructed by using $2k = 8$, 10 and 12 moments. The general conclusion, valid for unimodal pdfs is that the use of 10–14 moments in the reconstruction procedure gives satisfactory results.

In Figs. 7 and 8, the target pdfs (the same as in Figs. 3 and 4, respectively) are plotted against the approximant pdfs, which are obtained from Eq. (8a), by using $2k = 12$, 14 and 16 moments. Although, the obtained approximants cannot be considered as satisfactory final solutions of the moment problem, they do provide us with valuable initial guess of the target pdf (obtained explicitly), which is of great importance for any numerical solution of the reconstruction problem.

## 6. Discussion and conclusions

In this paper we have focused on the investigation how the partial information contained in a few moments can be used to "dig out" various distributional characteristics and properties without going into the business of solving numerically the (inverse) moment problem (see, e.g., [2,11,20,27,28]). We have demonstrated that the tail delimiters $x_*, x^*$, can be explicitly (and easily) calculated in terms of a small number of moments, providing us with the main-mass interval $[x_*, x^*]$. Within this interval an explicitly given, moment-based, continuous approximant of the target pdf, valid in the main-mass region $[x_*, x^*]$, and carrying over its main shape characteristics, is presented. Although the latter cannot be considered, in general, as a satisfactory solution, it can always serve as an initial approximation in any iterative scheme for the numerical solution

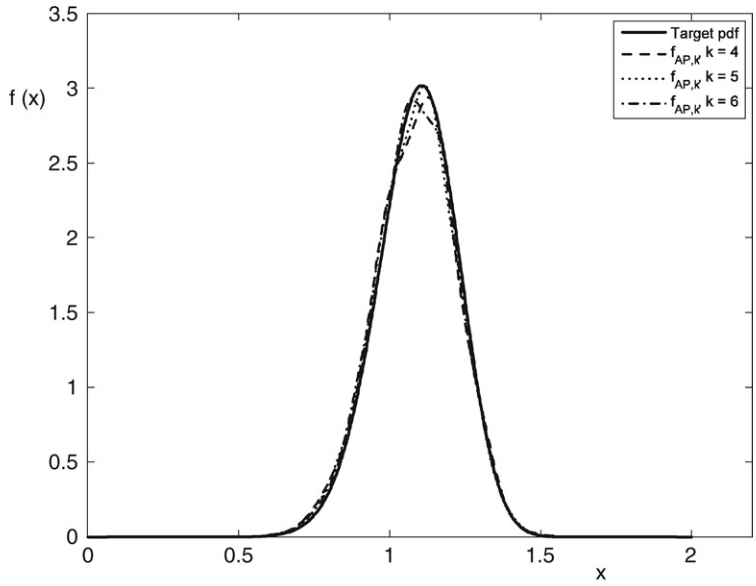


**Fig. 6.** The pdf approximants are plotted, by means of Eq. (8a), for $2k = 8$, 10, and 12 moments, respectively, along with the target pdf $f(x)$ (the same with Fig. 1).

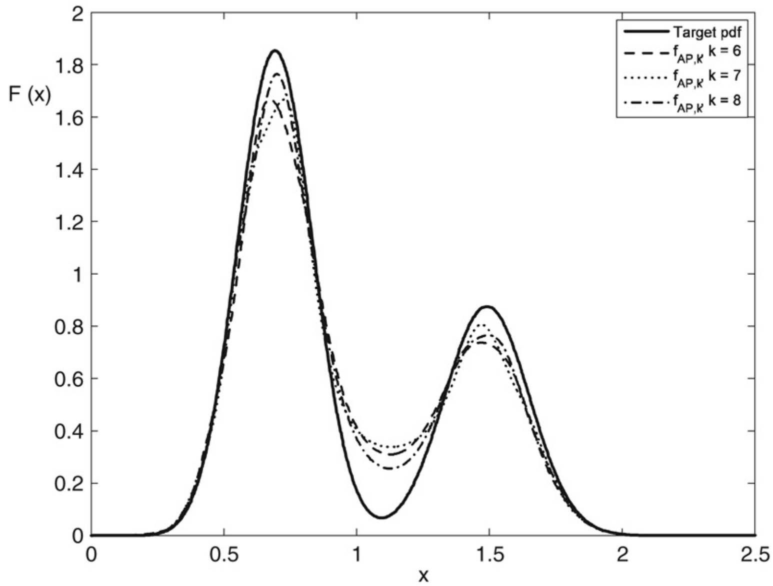


**Fig. 7.** The pdf approximants are plotted, by means of Eq. (8a), for $2k = 12$, 14, and 16 moments, respectively, along with the target pdf $f(x)$ (the same with Fig. 3).

of the moment problem. In this sense, a robust characterization of the solution space of the moment problem becomes possible, enabling a very efficient regularization of the moment data inversion.

In a forthcoming work [12], explicit moment-based approximants for the tails of the target pdf in $(-\infty, x_*)$and $(x^*, +\infty)$ will be presented. These results might be combined with the results presented herewith in order to provide an explicit initial approximation of the target pdf, which will be "uniformly valid" not only in $[a, b]$, but also in $(0, +\infty)$ and over the whole real axis $(-\infty, +\infty)$.

## Acknowledgements

We would like to thank Professors Jordan Stoyanov, Aldo Tagliani and Vilmos Totik for helpful discussions and comments. We further thank the anonymous reviewers for constructive comments.

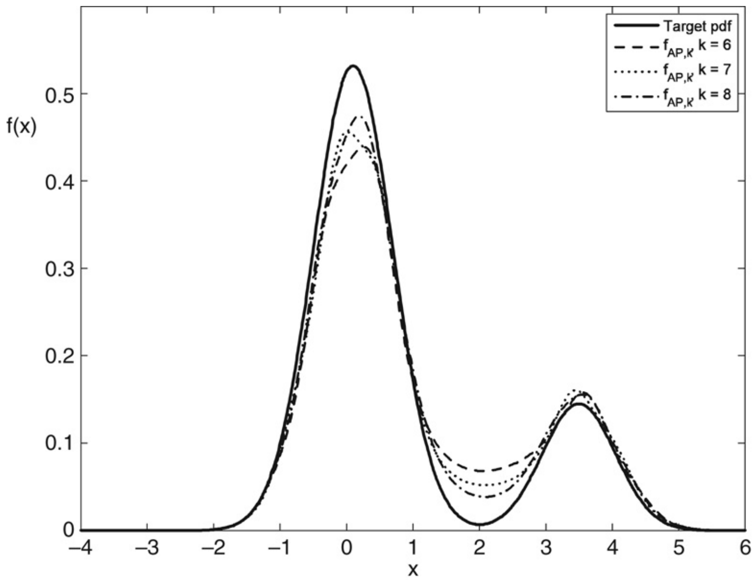


**Fig. 8.** The pdf approximants are plotted, by means of Eq. (8a), for $2k = 12$, 14, and 16 moments, respectively, along with the target pdf $f(x)$ (the same with Fig. 4).

## References


[1] N.I. Akhiezer, The Classical Moment Problem, Oliver and Boyd, London, 1965.
[2] G.A. Athanassoulis, P.N. Gavriliadis, The Hausdorff moment problem solved by using kernel density functions, Probab. Eng. Mech. 17 (2002) 273–291.
[3] K. Bandyopadhyay, K. Bhattacharyya, A.K. Bhattacharyya, On some problems of the maximum entropy ansatz, Pranama J. Phys. 54 (2000) 365–375.
[4] C. Berg, Moment problems and polynomial approximation, Ann. Fac. Sci. Toulouse Math. 6 (1996) 9–32 (special issue).
[5] M. Bertero, C. De Mol, E.R. Pike, Linear inverse problems with discrete data, I: General formulation and singular system analysis, Inverse Problems 1 (1985) 301–330.
[6] M. Bertero, C. De Mol, E.R. Pike, Linear inverse problems with discrete data, II: Stability and regularization, Inverse Problems 4 (1988) 573–594.
[7] H. Dette, W.J. Studden, The Theory of Canonical Moments with Applications in Statistics, Probability and Analysis, Jon Wiley and Sons, New York, 1997.
[8] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer, Dordrecht, 1996.
[9] P.N. Gavriliadis, G.A. Athanassoulis, Moment data can be analytically completed, Probab. Eng. Mech. 8 (2003) 329–338.
[10] P.N. Gavriliadis, Moment information for probability distributions, without solving the moment problem, I: Where is the mode? Comm. Statist. Theory Methods 37 (2008) 671–681.
[11] P.N. Gavriliadis, G.A. Athanassoulis, The Stieltjes moment problem solved by using kernel density functions (submitted for publication).
[12] P.N. Gavriliadis, G.A. Athanassoulis, Asymptotic analysis of the moment sequence (in preparation).
[13] A. Gut, On the moment problem, Bernoulli 8 (2002) 407–421.
[14] G. Inglese, Christoffel functions and finite moment problems, Inverse Problems 11 (1995) 949–960.
[15] S. Karlin, L.S. Shapley, Geometry of Moment Spaces, in: AMS Memoirs, vol. 12, Providence RI, 1953.
[16] M.G. Krein, The ideas of P. Chebysev and A. Markov in the Theory of Limiting Values of Integrals and their further development, AMS, Providence RI, 1959.
[17] M.G. Krein, A.A. Nudelman, The Markov Moment Problem and Extremal Problems, AMS, Providence RI, 1977.
[18] G.D. Lin, On the moment problems, Statist. Probab. Lett. 35 (1997) 85–90.
[19] G.D. Lin, On the moment problems, Statist. Probab. Lett. 50 (2000) 205.
[20] L. Mead, N. Papanicolaou, Maximum entropy in the problem of moments, J. Math. Phys. 25 (1984) 2404–2417.
[21] P. Nevai, Geza Freud, Orthogonal polynomials and Christoffel functions. A case study, J. Approx. Theory 48 (1986) 3–167.
[22] A.G. Pakes, Structure of Stieltjes classes of moment-equivalent probability laws, J. Math. Anal. Appl 326 (2007) 1268–1290.
[23] J.A. Shohat, J.D. Tamarkin, The Problem of Moments, American Mathematical Survey, 1943.
[24] J.M. Stoyanov, Counterexamples in Probability, Wiley, Chichester, 2000.
[25] J.M. Stoyanov, Krein condition in probabilistic moment problems, Bernoulli 6 (2000) 939–949.
[26] G. Szego, Orthogonal Polynomials, vol. 23, American Mathematical Society, 1959.
[27] A. Tagliani, On the application of maximum entropy to the moments problem, J. Math. Phys 34 (1993) 326–337.
[28] A. Tagliani, Inverse two-sided Laplace transform for probability density functions, J. Comput. Appl. Math 90 (1998) 157–170.
[29] G. Talenti, Recovering a function from a finite number of moments, Inverse Problems 3 (1987) 501–517.
[30] A.N. Tikhonov, V.Y. Arsenin, Solutions of Ill-Posed Problems, Wiley, New York, 1977.
[31] A.N. Tikhonov, A.V. Goncharsky, Ill-posed Problems in the Natural Sciences, Moscow, Mir Pub, 1987 (English translation).
[32] V. Totik, Asymptotics for Christoffel functions for general measures on the real line, J. Anal. Math. 81 (2000) 283–303.