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Exponential decay of errors of a fundamental solution method applied to a reduced wave problem in the exterior region of a disc[☆]

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ABSTRACT

This paper concerns a fundamental solution method (FSM, in abbreviation) applied to a reduced wave problem in the exterior region of a disc. The convergent rate of approximate solutions to the exact one is proven to be asymptotically exponentially decreasing with respect to the number N of collocation points employed in an approximate problem. Using obtained FSM solutions we add two numerical tests: numerical estimate of errors including cases of high wave numbers; and visualization of total waves appeared in the scattering phenomena around a circular obstacle in the cases of $\kappa = 50$ and $\kappa = 100$, where κ is a normalized wave number, defined through κ = length of wave number vector × radius of the disc. In the second test, the total waves almost vanish behind the disc, seemingly corresponding to the phenomenon of shadowing in the classical literature of physics.

1. Introduction

This paper concerns an error estimate for an approximation method for solving the Dirichlet boundary value problem $(\mathrm{E_f})$ of the 2-dimensional reduced wave equation in the exterior region of a disc. We employ a fundamental solution method (FSM, in abbreviation) to derive the approximate problem $(\mathrm{E_f^{(N)}})$.

Under a fairly general condition of the unique solvability for the problem $(\mathrm{E_f})$, it is shown that the rate of convergence of approximate solutions $u^{(N)}$ of approximate problems $(\mathrm{E_f^{(N)}})$ to the solution u of the original problem $(\mathrm{E_f})$ is asymptotically exponentially decaying with respect to N on the whole exterior region of the disc considered, where N is the number of collocation points on the boundary Γ_a of the disc, if the boundary data f belongs to a fairly general class which includes the case of the boundary data being a plane wave.

Let a be the radius of Γ_a and let ρ be the radius of a circle which is concentric and interior to the circle Γ_a, containing all the source points employed in $(\mathrm{E_f^{(N)}})$. We adopt a way of arranging the collocation points and source points, called the equi-distant equally phased arrangement of source points and collocation points in this paper. Our main theorem shows that the convergent rate of $u^{(N)}$ to u has the form of $O(\gamma^{N/2}N^{-1})$ with $\gamma = \frac{\rho}{a}$.

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Fundamental solution methods have long been recognized as a useful computational technique for solving reduced wave problems in unbounded domains. Among various computational results, the works of Sánchez-Sesma and Rosenblueth [1] and Sánchez-Sesma [2] were the earliest ones so far as the authors know.

Rigorous mathematical studies on FSM applied to 2-dimensional exterior reduced wave problems, however, had been few when the second author of the present paper formulated an FEM–FSM combined method for 2-dimensional reduced wave problem in the exterior region of a general scattering body in Ushijima [3] as a natural extension of the FEM–CSM combined method for the planar exterior Laplace problems discussed in Ushijima [4]. Here CSM is the abbreviation of charge simulation method. Then the authors started jointly a series of works to investigate mathematically and numerically the properties of FSM applied to reduced wave problems outside a disc. Some of preliminary results were reported in Ushijima [5]. In Ushijima and Chiba [6] the unique solvability of the problems ($\mathrm{E}_\mathrm{f}^{(\mathrm{N})}$) was investigated. Our paper [7] gave discussions of error estimates from theoretical and practical points of view which include the cases of high frequency problems. The previous paper [8] in Japanese treated the case of Neumann boundary value problem with the emphasis on the effectiveness of multiple-precision arithmetic with arbitrary many digits.

The paper of Katsurada and Okamoto [9] is a pioneering work for mathematical analysis of CSM applied to the Laplace equations in a disc. The CSM for the Laplace equations is one of typical examples of FSM. Our theoretical result in this work may be considered as a natural extension of that in [9] to the Helmholtz equation in a domain exterior to a disc.

In Ushijima and Chiba [7], we announced two theorems concerning error estimate, one of which treated the problem with boundary data of finitely many Fourier modes, and the other one treated the problem with boundary data of plane waves, separately. Since Theorem 4 of the present paper, a main theorem of this work, gives a sharper estimate which covers the above two cases, the authors have decided to withhold the publication of the proof of these theorems announced in [7].

The organization of the rest of the paper is as follows. In Section 2, the setting of the continuous problem (E_f) and approximate problems ($\mathrm{E}_\mathrm{f}^{(\mathrm{N})}$) are described. In Section 3, our previous results on the solvability of ($\mathrm{E}_\mathrm{f}^{(\mathrm{N})}$) are summarized. In Section 4, the main theorem of this work is stated under two assumptions, one of which concerns the solvability of the continuous problem (E_f), and the other one of which gives a description of the class of Dirichlet data treated in the main theorem. In Section 5, a Fourier series expansion of the approximation error $u - u^{(N)}$ is given. Various estimations of quantities related to the Fourier series expansion are shown in Section 6. A proof of the main theorem is completed in Section 7. In Section 8, we add two numerical tests: numerical estimate of errors including cases of high wave numbers; and two examples of total waves in circular obstacle scattering problems with high frequency incident plane waves. In the final part acknowledgements are stated.

2. A reduced wave problem and its FSM approximation

2.1. A reduced wave problem with Dirichlet boundary condition in the exterior region of a disc

Let Γ_a be the circle in the plane $\mathbb{R}^2$ with radius a having the origin of the plane as its center. Let k be the length of the wave number vector considered. Let Ω_e be the exterior domain of the circle Γ_a. We use the notation $\boldsymbol{r} = \boldsymbol{r}(\theta)$ for the point in the plane corresponding to the complex number $r\mathrm{e}^{\mathrm{i}\theta}$ with $r = |\boldsymbol{r}|$, where $|\boldsymbol{r}|$ is the Euclidean norm of $\boldsymbol{r} \in \mathbb{R}^2$. Similarly we use $\boldsymbol{a} = \boldsymbol{a}(\theta)$, and $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta)$, corresponding to $a\mathrm{e}^{\mathrm{i}\theta}$ with $a = |\boldsymbol{a}|$, and to $\rho\mathrm{e}^{\mathrm{i}\theta}$ with $\rho = |\boldsymbol{\rho}|$, respectively.

We consider the following inhomogeneous Dirichlet boundary value problem of the reduced wave equation in the region Ω_e as our continuous problem (E_f):

$$(\mathrm{E}_\mathrm{f}) \begin{cases} -\Delta u - k^2 u = 0 \quad \text{in } \Omega_e, \\ u = f \quad \text{on } \Gamma_a, \\ \lim\limits_{r\to\infty} \sqrt{r}\left\{\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right\} = 0. \end{cases}$$

The solution $u = u(\boldsymbol{r})$ is assumed to satisfy the Sommerfeld outgoing radiation condition at infinity. The boundary data $f = f(\boldsymbol{a}(\theta))$ is a complex valued continuous function on Γ_a.

Let f_n be the Fourier coefficient defined through

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{a}(\theta))\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta \quad \text{for } n \in \mathbb{Z}.$$

Then the solution $u(\boldsymbol{r})$ is formally represented through the following formula (1):

$$u(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} f_n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)}\mathrm{e}^{\mathrm{i}n\theta} \quad \text{for } r \geq a. \tag{1}$$

In the above formula (1), $H_n^{(1)}(x)$ is the nth order Hankel function of the first kind.

Note 1. It should be noted that, under certain decaying conditions on the Fourier coefficient f_n as $|n|$ tends to infinity, the infinite differentiability of u defined by (1) with respect to r, $r > a$, and $\theta \in \mathbb{R}$, and the continuity of all derivatives of u up to $r \geq a$ may be established, which in turn implies that the function u defined by (1) is the unique classical solution of the problem $(\mathrm{E_f})$. One of such decaying conditions is the following exponentially decaying condition $(\boldsymbol{\beta})$.

$$(\boldsymbol{\beta}) \begin{cases} \text{There are positive constants } F \text{ and } \beta \in (0, 1) \text{ such that} \\ |f_n| \leq F\beta^{|n|} \quad \text{for } n \in \mathbb{Z}. \end{cases}$$

2.2. Approximate problems to the reduced wave problem through a fundamental solution method

Let N be an arbitrary fixed positive integer. Then we use the notation θ_j for $j \in \mathbb{Z}$ through

$$\theta_1 = \frac{2\pi}{N}, \qquad \theta_j = j\theta_1 \quad \text{for } j \in \mathbb{Z}.$$

For fixed positive numbers ρ and a such that $0 < \rho < a$, $\boldsymbol{\rho}_j$ and $\boldsymbol{a}_j$ are defined as follows.

$$\boldsymbol{\rho}_j = \boldsymbol{\rho}(\theta_j), \qquad \boldsymbol{a}_j = \boldsymbol{a}(\theta_j), \quad 0 \leq j \leq N - 1.$$

The points $\boldsymbol{\rho}_j$ and $\boldsymbol{a}_j$ are said to be the source and the collocation points, respectively. The arrangement of the set of source points and collocation points introduced above is called the equi-distant equally phased arrangement of source points and collocation points in this paper.

Now we introduce an approximate problem $(\mathrm{E_f^{(N)}})$ to $(\mathrm{E_f})$ through a fundamental solution method, FSM, in the setting of the equi-distant equally phased arrangement of source points and collocation points. We consider the following problem:

$$(\mathrm{E_f^{(N)}}) \begin{cases} u^{(N)}(\boldsymbol{r}) = \displaystyle\sum_{j=0}^{N-1} Q_j G_j(\boldsymbol{r}), \\ u^{(N)}(\boldsymbol{a}_j) = f(\boldsymbol{a}_j), \quad 0 \leq j \leq N - 1. \end{cases}$$

In the problem $(\mathrm{E_f^{(N)}})$, we use basis functions $G_j(\boldsymbol{r})$ as follows,

$$G_j(\boldsymbol{r}) = H_0^{(1)}(k|r\mathrm{e}^{\mathrm{i}\theta} - \rho\mathrm{e}^{\mathrm{i}\theta_j}|), \quad 0 \leq j \leq N - 1.$$

It is noted that $G_j(\boldsymbol{r})$ is a constant multiple of the fundamental solution of Helmholtz equation with the singularity at $\boldsymbol{r} = \boldsymbol{\rho}_j$ satisfying the Sommerfeld outgoing radiation condition at infinity. The problem $(\mathrm{E_f^{(N)}})$ is understood in that the unknown N quantities Q_j, $0 \leq j \leq N - 1$, should be determined by the collocation condition described as the second equation of $(\mathrm{E_f^{(N)}})$.

Hereafter the following notation is employed:

$$\gamma = \frac{\rho}{a}, \qquad \delta = \frac{r}{a}, \qquad \kappa = ka.$$

These numbers are characteristic numbers of the relevant problem, normalized by the radius a. Using this notation we can rewrite the basis function $G_j(\boldsymbol{r})$ as follows.

$$G_j(\boldsymbol{r}) = H_0^{(1)}(\kappa|\delta - \gamma\mathrm{e}^{-\mathrm{i}(\theta - \theta_j)}|), \quad 0 \leq j \leq N - 1.$$

3. Solvability of the FSM approximate problems

For a fixed real number $\kappa > 0$ and a fixed real number $\gamma \in (0, 1)$, let us define the kernel function $g(\theta)$ through

$$g(\theta) = H_0^{(1)}(\kappa|1 - \gamma\mathrm{e}^{-\mathrm{i}\theta}|).$$

We understand that the problem $(\mathrm{E_f})$ is to find a density function $q(\theta)$ satisfying the following equality $(\mathbf{E_f})$:

$$(\mathbf{E_f}) f(\boldsymbol{a}(\theta)) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta - \varphi)q(\varphi)\mathrm{d}\varphi.$$

The function $f(\boldsymbol{a}(\theta))$ is represented with the kernel function $g(\theta)$ through the formula of convolution $(\mathbf{E_f})$. It is to be noted that we have

$$G_j(\boldsymbol{a}(\theta)) = g(\theta - \theta_j), \quad 0 \leq j \leq N - 1.$$

Hence we may consider the problem $(\mathrm{E_f^{(N)}})$ is an approximate problem of the integral equation $(\mathbf{E_f})$. Unknown quantities Q_j, $0 \leq j \leq N - 1$ in $(\mathrm{E_f^{(N)}})$ should be considered as approximate values of $\frac{1}{N}q(\theta_j)$, $0 \leq j \leq N - 1$.

Define a circulant matrix G and an inhomogeneous vector $\boldsymbol{f}$, and an unknown vector $\boldsymbol{q}$ as follows.

$$G_{ij} = \left(g(\theta_{i-j})\right)_{0\le i,j\le N-1}, \qquad \boldsymbol{f} = (f(\boldsymbol{a}_i))_{0\le i\le N-1}, \qquad \boldsymbol{q} = (Q_i)_{0\le i\le N-1}.$$

Then the problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ is equivalent to solving the linear equation:

$$G\boldsymbol{q} = \boldsymbol{f}. \tag{2}$$

Let $\omega = \mathrm{e}^{\mathrm{i}\theta_1}$, and introduce vectors $\boldsymbol{\omega}_n$ for $n \in \mathbb{Z}$ with $\boldsymbol{\omega}_n = (\omega^{jn})_{0\le j\le N-1}$. Since G is circulant, vectors $\boldsymbol{\omega}_n$ are eigenvectors of G. Denote the eigenvalue of G corresponding to $\boldsymbol{\omega}_n$ by λ_n for $n \in \mathbb{Z}$. Let $F_n^{(N)}$, and $G_n^{(N)}$, be the discrete Fourier coefficients of $f(\boldsymbol{a}(\theta))$, and $g(\theta)$, for $n \in \mathbb{Z}$, defined through

$$F_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} f(\boldsymbol{a}_j)\mathrm{e}^{-\mathrm{i}n\theta_j} \quad \text{and} \quad G_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} g(\theta_j)\mathrm{e}^{-\mathrm{i}n\theta_j}, \tag{3}$$

respectively. Although $F_n^{(N)}$ and $G_n^{(N)}$ should be called discrete Fourier coefficients with size N, we drop the phrase of "with size N" here and hereafter. By definition it follows that discrete Fourier coefficients $F_n^{(N)}$ and $G_n^{(N)}$ have the period N with respect to the suffix n. And an elemental calculation yields the relation:

$$\lambda_n = NG_n^{(N)} \quad \text{for } n \in \mathbb{Z}.$$

Hence a unique solvability condition of $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ is represented as follows.

$$(\mathbf{G}^{(\mathbf{N})}) \quad G_n^{(N)} \neq 0 \quad \text{for } n \in \mathbb{Z}.$$

Under the condition $(\mathbf{G}^{(\mathbf{N})})$ we can express the solution $\boldsymbol{q}$ of (2) as follows.

$$Q_n = \frac{1}{N}\sum_{j=0}^{N-1} \frac{F_j^{(N)}}{G_j^{(N)}}\mathrm{e}^{\mathrm{i}n\theta_j}, \quad 0 \le n \le N-1. \tag{4}$$

Due to Graf's addition formula in p. 361 of Watson [10], we have the Fourier series expansion of $g(\theta)$ as follows.

$$g(\theta) = \sum_{n=-\infty}^{\infty} g_n \mathrm{e}^{\mathrm{i}n\theta},$$

with

$$g_n = H_n^{(1)}(\kappa)J_n(\gamma\kappa) \quad \text{for } n \in \mathbb{Z}. \tag{5}$$

Now we introduce the condition (**g**) on the kernel function $g(\theta)$ through

$$(\mathbf{g})\ g_n \neq 0 \quad \text{for } n \in \mathbb{Z}.$$

This condition (**g**) has been denoted by (**G1**) in our previous paper [6], in which we have shown the following theorem as Theorem 3.

Theorem 1. *Let κ be an arbitrary positive number, and let $\gamma \in (0, 1)$ be fixed. Suppose that the kernel function $g(\theta)$ with the parameters κ and γ satisfies the condition* (**g**). *Then there is a positive integer N_1 depending on κ and γ such that the condition $(\mathbf{G}^{(\mathbf{N})})$ holds for any $N \ge N_1$.*

Since $H_n^{(1)}(\kappa)$ never vanishes for any $\kappa > 0$ as will be shown in the proof of Proposition 6 in Section 6, the condition (**g**) is equivalent to the condition that the nth order Bessel function $J_n(x)$ never vanishes at $x = \gamma\kappa$ for any positive integer n. From the properties of zeros of Bessel functions, we can conclude the following:

For fixed κ, except for the finite number of values of $\gamma \in (0, 1)$ depending on κ, the condition (**g**) *holds for any remaining $\gamma \in (0, 1)$. Especially if κ is less than or equal to the smallest positive zero of $J_0(x)$, the condition* (**g**) *holds for any $\gamma \in (0, 1)$.*

4. Main theorem

We assume the following Assumptions 2 and 3 throughout this section and the consecutive sections.

Assumption 2. Let κ be fixed as an arbitrary positive number. Choose $\gamma \in (0, 1)$ appropriately so that the kernel function $g(\theta)$ with parameters κ and γ may satisfy the condition (**g**).

Assumption 3. Let f_n and g_n be Fourier coefficients of $f(\boldsymbol{a}(\theta))$ and $g(\theta)$ for $n \in \mathbb{Z}$ defined through

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{a}(\theta))\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta \quad \text{and} \quad g_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta,$$

respectively. Under Assumption 2, define quantities q_n for $n \in \mathbb{Z}$ through

$$q_n = \frac{f_n}{g_n}.$$

Suppose that the following quantity $|||q|||$ is finite for the Dirichlet data f of the Problem $(\mathrm{E_f})$.

$$|||q||| = \sup_{n\in\mathbb{Z}} |q_n|.$$

Theorem 4. *Under Assumptions 2 and 3, there is a positive integer N_2 such that the following estimate is valid:*

$$\sup_{|\boldsymbol{r}|\geq a} \left|u(\boldsymbol{r}) - u^{(N)}(\boldsymbol{r})\right| < \frac{900|||q|||}{\pi(1-\gamma)}\frac{\gamma^{N/2}}{N} \quad \text{for } N \geq N_2.$$

The positive integer N_2 depends on κ and γ, but does not depend on the Dirichlet data f.

5. A Fourier series expansion of the approximation error

Throughout Sections from 5 to 7, the symbol N means a generic positive integer satisfying

$$N \geq \max(N_1, 2),$$

where N_1 is a positive integer determined in Theorem 1.

Theorem 5. *The solution $u^{(N)}(\boldsymbol{r})$ of $(\mathrm{E_f^{(N)}})$ is represented as follows.*

$$u^{(N)}(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \frac{F_n^{(N)}}{G_n^{(N)}} g_n \mathcal{H}_n(\delta)\mathrm{e}^{\mathrm{i}n\theta}, \tag{6}$$

where $\mathcal{H}_n(\delta)$ is defined through

$$\mathcal{H}_n(\delta) = \frac{H_n^{(1)}(\delta\kappa)}{H_n^{(1)}(\kappa)} \quad \text{for } n \in \mathbb{Z}. \tag{7}$$

Proof. For an arbitrarily fixed $r \geq a$, the basis function $G_j(\boldsymbol{r})$, $0 \leq j \leq N-1$, is expanded to the following Fourier series with respect to θ due to Graf's addition formula (See Watson [10] p. 361),

$$G_j(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(\kappa\delta)J_n(\gamma\kappa)\mathrm{e}^{\mathrm{i}n(\theta-\theta_j)} \quad \text{for } \boldsymbol{r} = \boldsymbol{r}(\theta) \text{ with } |\boldsymbol{r}| = r. \tag{8}$$

Inserting the series above into the formula of $u^{(N)}(\boldsymbol{r})$ in $(\mathrm{E_f^{(N)}})$, we obtain

$$u^{(N)}(\boldsymbol{r}) = \sum_{j=0}^{N-1} Q_j \left\{ \sum_{n=-\infty}^{\infty} H_n^{(1)}(\kappa\delta)J_n(\gamma\kappa)\mathrm{e}^{\mathrm{i}n(\theta-\theta_j)} \right\}. \tag{9}$$

Since the series (8) is absolutely and uniformly convergent with respect to θ, we have

$$u^{(N)}(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \left\{ \sum_{j=0}^{N-1} Q_j \mathrm{e}^{-\mathrm{i}n\theta_j} \right\} H_n^{(1)}(\kappa)J_n(\gamma\kappa)\frac{H_n^{(1)}(\kappa\delta)}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}.$$

(The absolute and uniform convergency of (8) will be admitted after one sees Proposition 6 and the proof of Proposition 7 in Section 6.) The representation formula (4) of Q_n yields

$$\sum_{j=0}^{N-1} Q_j \mathrm{e}^{-\mathrm{i}n\theta_j} = \frac{F_n^{(N)}}{G_n^{(N)}} \quad \text{for } n \in \mathbb{Z}. \tag{10}$$

From (10), (5) and (7) and the definition of $\mathcal{H}_n(\delta)$, we have the representation in the statement of Theorem 5. □

The approximation error, namely the difference between the exact solution (1) and the FSM approximate one (6) is given as follows.

$$u(\boldsymbol{r}) - u^{(N)}(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} \left(\frac{f_n}{g_n} - \frac{F_n^{(N)}}{G_n^{(N)}} \right) g_n \, \mathcal{H}_n(\delta) \, \mathrm{e}^{\mathrm{i}n\theta}. \tag{11}$$

6. Estimates of quantities appearing in the Fourier series expansion of the approximation error

Proposition 6. *We have*

$$0 < |\mathcal{H}_n(\delta)| \le 1 \quad \textit{for } \kappa > 0,\ \delta \ge 1 \textit{ and } n \in \mathbb{Z}.$$

Proof. Let x be a positive real variable, and let $Y_n(x)$ be the nth order Neumann function. The nth order Hankel function of the first kind $H_n^{(1)}(x)$ is represented as follows.

$$H_n^{(1)}(x) = J_n(x) + \mathrm{i}Y_n(x) \quad \text{for } x > 0 \text{ and } n \in \mathbb{Z}.$$

Using Nicholson's integral in p. 444 of Watson [10], we introduce the function $P_n(x)$ as follows.

$$P_n(x) = |H_n^{(1)}(x)|^2 = J_n^2(x) + Y_n^2(x) = \frac{8}{\pi^2} \int_0^\infty K_0(2x \sinh t) \cosh 2nt \, \mathrm{d}t \quad \text{for } x > 0 \text{ and } n \in \mathbb{Z},$$

with

$$K_0(x) = \int_0^\infty \mathrm{e}^{-x\cosh t} \, \mathrm{d}t \quad \text{for } x > 0,$$

where $K_0(x)$ is the zeroth order modified Bessel function of the second kind (see p. 446 of Watson [10]). The formula above indicates that $P_n(x)$ is a positive decreasing function of $x > 0$. Namely we have

$$0 < P_n(\delta\kappa) \le P_n(\kappa) \quad \text{for } \delta \ge 1,\ \kappa > 0 \text{ and } n \in \mathbb{Z}.$$

Definition (7) yields the following estimate.

$$0 < \frac{P_n(\delta\kappa)}{P_n(\kappa)} = \frac{|H_n^{(1)}(\delta\kappa)|^2}{|H_n^{(1)}(\kappa)|^2} = |\mathcal{H}_n(\delta)|^2 \le 1 \quad \text{for } \delta \ge 1,\ \kappa > 0 \text{ and } n \in \mathbb{Z}.$$

Therefore, the statement of the proposition is obtained. □

Proposition 7. *There exists a positive integer L, depending on κ and γ, such that*

$$|g_n| \le \frac{3}{2|n|\pi} \gamma^{|n|} \quad \textit{provided that } |n| \ge L.$$

Proof. This statement comes from Lemma 1 of Ushijima and Chiba [6]. Key steps of the proof are rewritten here for the sake of convenience. The following asymptotic estimates are written on p. 365 of Abramowitz–Stegun [11], which are valid for a fixed positive x as $n \to \infty$.

$$J_n(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{\mathrm{e}x}{2n} \right)^n, \qquad H_n^{(1)}(x) \sim -\mathrm{i}\sqrt{\frac{2}{\pi n}} \left(\frac{\mathrm{e}x}{2n} \right)^{-n}. \tag{12}$$

Then the asymptotic estimate below holds for fixed positive γ and κ as $n \to \infty$.

$$g_n = H_n^{(1)}(\kappa) J_n(\gamma\kappa) \sim -\frac{\mathrm{i}\gamma^n}{\pi n}.$$

As in Lemma 1 of [6], we understand that the above asymptotic behavior is equivalent to the following statement:

For any positive ϵ, there exists a positive integer $L(\epsilon)$ such that

$$\left| \frac{g_n}{-\frac{\mathrm{i}\gamma^n}{\pi n}} - 1 \right| \le \epsilon \quad \text{for } n \ge L(\epsilon).$$

Let $L = L(1/2)$. Then we have

$$\left| \frac{g_n}{-\frac{\mathrm{i}\gamma^n}{\pi n}} - 1 \right| \le \frac{1}{2} \quad \text{for } n \ge L.$$

Hence the following inequality holds.

$$|g_n| \le \frac{3}{2n\pi}\gamma^n \quad \text{for } n \ge L.$$

On the other hand the following formulas hold.

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{and} \quad H_{-n}^{(1)}(x) = (-1)^n H_n^{(1)}(x) \quad \text{for } n \in \mathbb{Z}. \tag{13}$$

Hence $g_{-n} = g_n$ for $n \in \mathbb{Z}$. Thus the result of Proposition 7 is obtained. □

Lemma 8. *Let $\psi(\theta)$ be a 2π periodic continuous function. Suppose that the derivative $\psi'(\theta)$ exists almost everywhere, and that it belongs to $L^2(0, 2\pi)$. Let ψ_n and $\Psi_n^{(N)}$ be the nth Fourier coefficient of ψ, and the nth discrete Fourier coefficients of ψ, respectively. Then the following equality holds.*

$$\Psi_n^{(N)} - \psi_n = \sum_{p \in \mathbb{Z}-\{0\}} \psi_{n+Np} \quad \textit{for } n \in \mathbb{Z}. \tag{14}$$

Proof. The function $\psi(\theta)$ is expanded in the following uniformly absolutely convergent Fourier series:

$$\psi(\theta) = \sum_{n=-\infty}^{\infty} \psi_n \mathrm{e}^{\mathrm{i}n\theta}. \tag{15}$$

The nth discrete Fourier coefficient of ψ is given as follows.

$$\Psi_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} \psi(\theta_j)\mathrm{e}^{-\mathrm{i}n\theta_j}, \quad \theta_j = \frac{2\pi j}{N},\ n \in \mathbb{Z} \tag{16}$$

Inserting (15) into the right-hand of (16), we obtain (14). □

Proposition 9. *There exists a positive integer L, depending on κ and γ, with the following property: If $N \ge L$, then*

$$\left|G_n^{(N)} - g_n\right| \le \frac{6}{N\pi}\left(\gamma^{N+|n|} + \gamma^{N-|n|}\right) \quad \textit{for n with } |n| \le N/2.$$

Proof. Fix a positive integer L_1 arbitrarily. Suppose that integers N, n and p satisfy

$$N \ge L_1, \qquad |n| \le N/2, \quad p \ne 0.$$

Then the following inequality holds.

$$|n + Np| \ge L_1/2. \tag{17}$$

In fact we have

$$|n + Np| \ge |Np| - |n| \ge N - |n| \ge N - N/2 = N/2 \ge L_1/2.$$

Let $L_1/2$ equal to the integer L determined in Proposition 7. Then we have

$$|g_n| \le \frac{3\gamma^{|n|}}{2|n|\pi} \quad \text{for any } n \text{ with } |n| \ge L_1/2. \tag{18}$$

Lemma 8 yields the following inequality.

$$\left|G_n^{(N)} - g_n\right| \le \sum_{p \in \mathbb{Z}-\{0\}} |g_{n+Np}|.$$

If $N \ge L_1$ and $|n| \le N/2$, we can insert the estimate (18) into the right-hand side of the above equality due to (17). Hence we obtain

$$\left|G_n^{(N)} - g_n\right| \le \frac{3}{2\pi}\sum_{p \in \mathbb{Z}-\{0\}} \frac{\gamma^{|n+Np|}}{|n+Np|} \quad \text{for } n \text{ with } |n| \le N/2 \tag{19}$$

provided that $N \ge L_1$. We note the following equality:

$$\frac{3}{2\pi}\sum_{p \in \mathbb{Z}-\{0\}} \frac{\gamma^{|n+Np|}}{|n+Np|} = \frac{3}{2\pi}\sum_{p=1}^{\infty}\left(\frac{\gamma^{Np+|n|}}{Np+|n|} + \frac{\gamma^{Np-|n|}}{Np-|n|}\right) \quad \text{for } n \text{ with } |n| \le N/2.$$

Further we have

$$Np + |n| \geq Np - |n| \geq N - |n| \geq N - N/2 = N/2$$

in the the right hand side of the equality above. Therefore we can calculate for n with $|n| \leq N/2$ in the following way.

$$\sum_{p=1}^{\infty}\left(\frac{\gamma^{Np+|n|}}{Np+|n|} + \frac{\gamma^{Np-|n|}}{Np-|n|}\right) \leq \frac{2}{N}\sum_{p=1}^{\infty}\left(\gamma^{Np+|n|} + \gamma^{Np-|n|}\right) = \frac{2(\gamma^{N+|n|} + \gamma^{N-|n|})}{N(1-\gamma^N)}.$$

We take a positive integer L_2 so as to satisfy

$$\frac{1}{1-\gamma^N} \leq 2 \quad \text{for } N \geq L_2.$$

Let $L = \max(L_1, L_2)$. Then we have

$$\frac{2(\gamma^{N+|n|} + \gamma^{N-|n|})}{N(1-\gamma^N)} \leq \frac{4(\gamma^{N+|n|} + \gamma^{N-|n|})}{N} \quad \text{for } n \text{ with } |n| \leq N/2$$

provided that $N \geq L$. Summing up the above estimations starting from (19), we have the conclusion of Proposition 9. It is to be noted that L_1 depends on κ and γ, and that L_2 depends on γ. □

Corollary 10. *Let L be the positive integer determined in Proposition 9. Then we have*

$$\left|G_n^{(N)} - g_n\right| \leq \frac{12}{N\pi}\gamma^{N/2} \quad \textit{for } n \textit{ with } |n| \leq N/2$$

provided that $N \geq L$.

Proof. Since we have

$$N + |n| \geq N - |n| \geq N - N/2 = N/2 \quad \text{for } n \text{ with } |n| \leq N/2,$$

Proposition 9 implies Corollary 10. □

Proposition 11. *Let L be the positive integer determined in Proposition 9. Then we have*

$$\left|F_n^{(N)} - f_n\right| \leq \frac{6}{N\pi}|||q|||\left(\gamma^{N+|n|} + \gamma^{N-|n|}\right) \quad \textit{for } n \textit{ with } |n| \leq N/2$$

provided that $N \geq L$.

Corollary 12. *Let L be the positive integer determined in Proposition 9. Then we have*

$$\left|F_n^{(N)} - f_n\right| \leq \frac{12}{N\pi}|||q|||\gamma^{N/2} \quad \textit{for } n \textit{ with } |n| \leq N/2$$

provided that $N \geq L$.

Proof of Proposition 11 and Corollary 12. Due to Assumptions 2 and 3, we have for any $n \in \mathbb{Z}$

$$|f_n| \leq |||q|||\,|g_n|.$$

Proposition 7 yields

$$|f_n| \leq \frac{3|||q|||}{2\pi|n|}\gamma^{|n|} \quad \text{for } n \text{ with } |n| \geq L,$$

where L is determined in Proposition 9. The estimate above assures that the function $u(\boldsymbol{r})$ represented in the form (1) is the unique classical solution of the problem ($\mathrm{E_f}$), and especially that $f(\theta) = u(\boldsymbol{a}(\theta))$ is a 2π-periodic continuous function having the derivative $f'(\theta) \in L^2(0, 2\pi)$. Hence Lemma 8 yields

$$F_n^{(N)} - f_n = \sum_{p\in\mathbb{Z}-\{0\}} f_{n+Np} \quad \text{for } n \in \mathbb{Z}.$$

Therefore we have

$$|F_n^{(N)} - f_n| \leq \sum_{p\in\mathbb{Z}-\{0\}} |f_{n+Np}| \leq |||q||| \sum_{p\in\mathbb{Z}-\{0\}} |g_{n+Np}|.$$

Hence Proposition 11, and Corollary 12, are established through the same arguments as are employed in the proof of Proposition 9, and that of Corollary 10, respectively. □

Proposition 13. *There exists a positive integer L depending on κ and γ such that*

$$\left|G_n^{(N)}\right| \geq \frac{\gamma^{N/2}}{2N\pi} \quad \text{for } n \in \mathbb{Z} \text{ provided that } N \geq L.$$

Proof. Reduction of the proof of Proposition 13 to the proof of Theorem 3 in Ushijima and Chiba [6] is as follows. Temporarily the integer L determined in Proposition 7 is denoted by L_7, and integers N_i, $0 \leq i \leq 4$, are employed in accordance with those in the proof of Theorem 3 in [6]. Let $N_1 = L_7$ and let

$$N_2 = -\frac{\log 2}{\log \gamma}.$$

Define

$$G_3 = \min_{0 \leq n \leq N_1} |g_n|.$$

Due to Assumption 2, G_3 is positive. If

$$G_3 \leq \frac{24}{\pi} \gamma^{1/2},$$

then let

$$N_3 = \frac{2}{\log \gamma} \times \log \frac{\pi G_3}{24},$$

otherwise let

$$N_3 = 1.$$

Let N_4 be the largest zero of the following equation for the real variable x:

$$6x\gamma^{x/2} = \frac{1}{2}.$$

Define

$$N_0 = \max_{1 \leq i \leq 4} N_i.$$

In Step 3 of the proof of Theorem 3 in [6], we have shown that if $N \geq N_i$ for $1 \leq i \leq 3$, then

$$|G_n^{(N)}| \geq \frac{12}{\pi} \gamma^{N/2} \quad \text{for } n \in \left[0, \frac{N_1}{2}\right].$$

In Step 4 of the proof of Theorem 3 in [6], we have shown that if $N \geq N_i$ for $1 \leq i \leq 4$, then

$$|G_n^{(N)}| \geq \frac{\gamma^{N/2}}{2\pi N} \quad \text{for } n \in \left[\frac{N_1}{2}, \frac{N}{2}\right].$$

Combining the above 2 estimates, and noticing the equality: $G_{-n}^{(N)} = G_n^{(N)}$ for any $n \in \mathbb{Z}$, we have

$$|G_n^{(N)}| \geq \frac{\gamma^{N/2}}{2\pi N} \quad \text{for } n \in \left[-\frac{N}{2}, \frac{N}{2}\right]$$

provided that $N \geq N_0$. Since the discrete Fourier coefficient $G_n^{(N)}$ has a period N with respect to the suffix n, we have

$$|G_n^{(N)}| \geq \frac{\gamma^{N/2}}{2\pi N} \quad \text{for } n \in \mathbb{Z} \text{ provided that } N \geq N_0.$$

For the positive integer L nearest to N_0 from above, the statement of Proposition 13 is valid. □

Proposition 14. *Let L_9, and L_{13} be positive integers determined in Proposition 9, and in Proposition 13, respectively. Let $L = \max(L_9, L_{13})$. If $N \geq L$, then*

$$\left|\frac{g_n}{G_n^{(N)}}\right| \leq 25 \quad \text{for any } n \text{ with } |n| \leq N/2.$$

Proof. Since

$$\left|\frac{g_n}{G_n^{(N)}}-1\right|=\left|\frac{g_n}{G_n^{(N)}}-\frac{G_n^{(N)}}{G_n^{(N)}}\right|=\left|\frac{1}{G_n^{(N)}}\right|\left|g_n-G_n^{(N)}\right|,$$

Propositions 9 and 13 yield

$$\left|\frac{g_n}{G_n^{(N)}}-1\right|\leq\frac{2N\pi}{\gamma^{N/2}}\times\frac{12}{N\pi}\gamma^{N/2}=24\quad\text{for } n \text{ with } |n|\leq N/2$$

if $N\geq L$. Hence we have

$$\left|\frac{g_n}{G_n^{(N)}}\right|\leq\left|\frac{1}{G_n^{(N)}}\right|\left|g_n-G_n^{(N)}\right|+1\leq 24+1=25\quad\text{for } n \text{ with } |n|\leq N/2$$

if $N\geq L$. □

Proposition 15. *Let L be a positive integer determined in Proposition 14. If $N\geq L$, then*

$$\left|\frac{F_n^{(N)}}{G_n^{(N)}}\right|\leq 49|||q|||\quad\text{for } n\in\mathbb{Z}.$$

Proof. Discrete Fourier coefficients $F_n^{(N)}$ and $G_n^{(N)}$ have the period N with respect to the suffix $n\in\mathbb{Z}$. Then there exists an integer m such that

$$|m|\leq N/2\quad\text{and}\quad\frac{F_m^{(N)}}{G_m^{(N)}}=\frac{F_n^{(N)}}{G_n^{(N)}}.$$

Further we have

$$\begin{aligned}\left|\frac{F_m^{(N)}}{G_m^{(N)}}-\frac{f_m}{g_m}\right|&=\left|\frac{F_m^{(N)}}{G_m^{(N)}}-\frac{f_m}{G_m^{(N)}}+\frac{f_m}{G_m^{(N)}}-\frac{f_m}{g_m}\right|\\&=\left|\frac{1}{G_m^{(N)}}\right|\left|\left(F_m^{(N)}-f_m\right)+\frac{f_m}{g_m}\left(g_m-G_m^{(N)}\right)\right|\\&\leq\left|\frac{1}{G_m^{(N)}}\right|\left\{\left|F_m^{(N)}-f_m\right|+\left|\frac{f_m}{g_m}\right|\left|g_m-G_m^{(N)}\right|\right\}.\end{aligned}$$

Due to Corollary 12, Assumption 3 and Corollary 10, we have

$$\begin{aligned}\left|\frac{F_n^{(N)}}{G_n^{(N)}}\right|&=\left|\frac{F_m^{(N)}}{G_m^{(N)}}\right|\leq\left|\frac{1}{G_m^{(N)}}\right|\left\{\left|F_m^{(N)}-f_m\right|+\left|\frac{f_m}{g_m}\right|\left|g_m-G_m^{(N)}\right|\right\}+\left|\frac{f_m}{g_m}\right|\\&\leq 2N\pi\gamma^{-N/2}\times\left(\frac{12|||q|||}{N\pi}\gamma^{N/2}+|||q|||\times\frac{12}{N\pi}\gamma^{N/2}\right)+|||q|||\\&=49|||q|||\end{aligned}$$

if $N\geq L$. □

7. Proof of the main theorem

The difference (11) is divided into terms I, II and III in the following fashion:

$$\begin{aligned}u(\mathbf{r})-u^{(N)}(\mathbf{r})&=\sum_{-N/2\leq n\leq N/2}I_n\mathcal{H}_n(\delta)\mathrm{e}^{\mathrm{i}n\theta}+\sum_{n>N/2}II_n\mathcal{H}_n(\delta)\mathrm{e}^{\mathrm{i}n\theta}+\sum_{n<-N/2}III_n\mathcal{H}_n(\delta)\mathrm{e}^{\mathrm{i}n\theta}\\&=I+II+III,\end{aligned}$$

where I_n, II_n and III_n represent the quantities

$$\left(\frac{f_n}{g_n}-\frac{F_n^{(N)}}{G_n^{(N)}}\right)g_n \tag{20}$$

with integer n running in the corresponding ranges specified to the terms I, II and III, respectively.

7.1. Estimation of the term I

The term I_n defined in (20) in Section 6 is represented as follows.

$$\begin{aligned} I_n &= \left(\frac{f_n}{g_n} - \frac{f_n}{G_n^{(N)}} + \frac{f_n}{G_n^{(N)}} - \frac{F_n^{(N)}}{G_n^{(N)}}\right) g_n \\ &= \left\{\frac{f_n}{g_n} \times \left(G_n^{(N)} - g_n\right) + \left(f_n - F_n^{(N)}\right)\right\} \times \frac{g_n}{G_n^{(N)}} \\ &= \left\{I_n^{(1)} \times I_n^{(2)} + I_n^{(3)}\right\} \times I_n^{(4)}. \end{aligned}$$

By Assumption 3 in Section 4, we have

$$|I_n^{(1)}| = \left|\frac{f_n}{g_n}\right| \leq \|\!|q|\!\|.$$

Denote the positive integer L determined in Proposition 9 by L_9. Then we have

$$|I_n^{(2)}| = |G_n^{(N)} - g_n| \leq \frac{6}{N\pi}\left(\gamma^{N+|n|} + \gamma^{N-|n|}\right) \quad \text{for } n \text{ with } |n| \leq N/2$$

if $N \geq L_9$. Due to Proposition 11, we have

$$|I_n^{(3)}| = |F_n^{(N)} - f_n| \leq \frac{6\|\!|q|\!\|}{N\pi}\left(\gamma^{N+|n|} + \gamma^{N-|n|}\right) \quad \text{for } n \text{ with } |n| \leq N/2$$

if $N \geq L_9$. Denote the positive integer L determined in Proposition 14 by L_{14}. Then we have

$$|I_n^{(4)}| = \left|\frac{g_n}{G_n^{(N)}}\right| \leq 25 \quad \text{for } n \text{ with } |n| \leq N/2$$

if $N \geq L_{14}$. Let $N_I = \max(L_9, L_{14})$. Then the above 4 estimates yield the following estimate.

$$|I_n| \leq (|I_n^{(1)}| \times |I_n^{(2)}| + |I_n^{(3)}|) \times |I_n^{(4)}| \leq \frac{300\|\!|q|\!\|}{N\pi}\left(\gamma^{N+|n|} + \gamma^{N-|n|}\right) \quad \text{for } n \text{ with } |n| \leq N/2$$

if $N \geq N_I$.

The summation of $\gamma^{N-|n|} + \gamma^{N+|n|}$ with respect to $n \in [-N/2, N/2]$ is estimated, in both cases of even N and odd N, as follows.

$$\sum_{-N/2 \leq n \leq N/2} (\gamma^{N-|n|} + \gamma^{N+|n|}) < \frac{2\gamma^{N/2}}{1-\gamma}.$$

Therefore we have

$$|I| \leq \sum_{-N/2 \leq n \leq N/2} |I_n| < \frac{600\|\!|q|\!\|}{N\pi(1-\gamma)}\gamma^{N/2} \quad \text{for } N \geq N_I.$$

It should be noted that the definitions of L_9 and L_{14} imply that N_I depends on κ and γ. □

7.2. Estimation of the term II

The term II_n defined in (20) of Section 6 is represented as follows.

$$II_n = \left(\frac{f_n}{g_n} - \frac{F_n^{(N)}}{G_n^{(N)}}\right) \times g_n = \left(II_n^{(1)} - II_n^{(2)}\right) \times II_n^{(3)}.$$

By Assumption 3, we have

$$|II_n^{(1)}| = \left|\frac{f_n}{g_n}\right| \leq \|\!|q|\!\|.$$

Denote L determined in Proposition 15 by L_{15}. Then we have

$$|II_n^{(2)}| = \left|\frac{F_n^{(N)}}{G_n^{(N)}}\right| \leq 49\|\!|q|\!\| \quad \text{for } n > N/2$$

if $N \geq L_{15}$. Denote L determined in Proposition 7 by L_7. Then we have

$$|II_n^{(3)}| = |g_n| \leq \frac{3}{2n\pi}\gamma^n \quad \text{for } n > N/2$$

if $N \geq L_7$. Let $N_{II} = \max(L_{15}, L_7)$. The above 3 estimates yield

$$|II_n| \leq \left(|II_n^{(1)}| + |II_n^{(2)}|\right) \times |II^{(3)}| \leq \frac{75|||q|||}{\pi}\frac{\gamma^n}{n} \quad \text{for } n > N/2$$

if $N \geq N_{II}$. The summation of γ^n/n with respect to $n \in (N/2, \infty)$ is calculated as follows.

$$\sum_{n>N/2} \frac{\gamma^n}{n} < \sum_{n>N/2} \frac{\gamma^n}{N/2} = \frac{2}{N}\sum_{n>N/2}\gamma^n.$$

Since we have, in both cases of even N and odd N,

$$\sum_{n>N/2}\gamma^n \leq \frac{\gamma^{N/2+1/2}}{1-\gamma},$$

we obtain

$$\sum_{n>N/2}\frac{\gamma^n}{n} < \frac{2\gamma^{N/2+1/2}}{N(1-\gamma)}.$$

Therefore we have

$$|II| \leq \sum_{n>N/2}|II_n| < \frac{75|||q|||}{\pi} \times \frac{2\gamma^{N/2+1/2}}{N(1-\gamma)} < \frac{150|||q|||}{N\pi(1-\gamma)}\gamma^{N/2} \quad \text{for } N \geq N_{II}.$$

It should be noted that the definitions of L_{15} and L_7 imply that N_{II} depends on κ and γ. □

7.3. Estimation of the term III

In the same manner as in the previous subsection, we have

$$|III| \leq \sum_{n<-N/2}|III_n| < \frac{150|||q|||}{N\pi(1-\gamma)}\gamma^{N/2} \quad \text{for } N \geq N_{II}.$$

7.4. Completion of the proof of the main theorem

Let $N_2 = \max(N_I, N_{II})$. Then Sections 7.1–7.3 yield

$$\begin{aligned}|I + II + III| &\leq |I| + |II| + |III| \\ &< \frac{600|||q|||}{N\pi(1-\gamma)}\gamma^{N/2} + \frac{150|||q|||}{N\pi(1-\gamma)}\gamma^{N/2} \times 2 = \frac{900|||q|||}{\pi(1-\gamma)}\frac{\gamma^{N/2}}{N}\end{aligned}$$

if $N \geq N_2$. □

8. Numerical tests

8.1. Behavior of numerical errors

We obtain closed analytical formulae calculating approximate solutions $u^{(N)}$ of $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ in forms of finite numbers of arithmetic operations except for the evaluation of cylindrical functions. The truncation error of approximate solutions is well estimated through Theorem 4.

The rounding error might pollute the convergent rate of the truncation error especially in the case of high frequency problems, since the kernel function becomes more and more oscillatory as κ does larger and larger.

The first numerical test concerns the above mentioned conflict between truncation error and rounding error. To see the exact situation, free use of multiple-precision arithmetic is instrumental in the test.

Table 1
Parameters for numerical estimator.

	κ	γ	$N = 2^n$	NN	Digit
Left column	1, 10, 100, 1000	$0.1 \le \gamma \le 0.9$	$1 \le n \le 10$	2048	30
Left column	100, 1000	$0.1 \le \gamma \le 0.9$	$11 \le n \le 13$	16384	30
Right column	1, 10, 100, 1000	$0.1 \le \gamma \le 0.9$	$1 \le n \le 10$	2048	3200
Right column	100, 1000	$0.1 \le \gamma \le 0.9$	$11 \le n \le 13$	16384	3200

8.1.1. Boundary data

The following Dirichlet boundary data f is employed on Γ_a.

$$f = \mathrm{e}^{\mathrm{i}\kappa\cos\theta} \quad \text{with } \theta \in [0, 2\pi].$$

Letting $t = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$ in the formula (1) on p. 14 of Watson [10], we have

$$f = \sum_{n=-\infty}^{\infty} \mathrm{i}^n J_n(\kappa)\mathrm{e}^{\mathrm{i}n\theta}.$$

Let $f_n = \mathrm{i}^n J_n(\kappa)$. Due to the formulae (5), (12) and (13), we have the following asymptotic behavior of $|f_n|$ as $n \to \pm\infty$.

$$|f_n| = \left|\frac{f_n}{g_n}\right| |g_n| = \left|\frac{J_n(\kappa)}{H_n^{(1)}(\kappa)J_n(\gamma\kappa)}\right| |g_n| \sim \sqrt{\frac{\pi|n|}{2}}\left(\frac{\mathrm{e}\kappa}{2\gamma|n|}\right)^{|n|} |g_n|.$$

Hence, under Assumption 2 there is a positive number C such that

$$|f_n| \le C|g_n| \quad \text{for } n \in \mathbb{Z}.$$

Therefore Assumption 3 holds, and Theorem 4 can be applied to this case.

8.1.2. Numerical estimator of error

Let NN be the number of evaluation points on Γ_a. Each evaluation point $\tilde{\boldsymbol{a}}_j$ is defined as follows.

$$\tilde{\boldsymbol{a}}_j = \boldsymbol{a}(\tilde{\theta}_j), \quad \tilde{\theta}_j = \frac{2\pi j}{NN} \quad \text{for } 0 \le j \le NN - 1.$$

Let N be the number of collocation points. Due to the formulae (3) and (4), Q_j, $0 \le j \le N-1$ are computed by Fast Fourier Transform. Values of approximate solutions are computed with Q_j by the first formula of $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$.

The following formula is employed for the numerical estimator of error.

$$E^{(NN)}(N) = \max_{0\le j\le NN-1} \left|u(\tilde{\boldsymbol{a}}_j) - u^{(N)}(\tilde{\boldsymbol{a}}_j)\right|. \tag{21}$$

We employ $NN = 2048$ for $2 \le N \le 1024$, and $NN = 16384$ for $2048 \le N \le 8192$.

8.1.3. Behavior of numerical estimator

The results of computation are given as two columns of graphs in Fig. 1. The left column corresponds to 30 decimal digit arithmetic, and the right one to 3200 decimal digit arithmetic, respectively. In each column, four graphs correspond to $\kappa = 1$, 10, 100 and 1000, in descending order respectively. In each graph, five polygonal lines correspond to $\gamma = 0.1$, 0.3, 0.5, 0.7 and 0.9, respectively. And, the abscissa axis means the number of collocation points, N, and the ordinate axis means the common logarithm of errors, $\log_{10} E^{(NN)}(N)$. It is to be noted that N is bounded by 1024 $(= 2^{10})$ in the cases of $\kappa = 1$ and 10, while it is extended to 8192 $(= 2^{13})$ in the cases of $\kappa = 100$ and 1000. Other values of parameters are listed in Table 1.

In the cases of $\kappa = 1$ and 10, the behavior of the numerical estimator with 3200 decimal digit arithmetic reflects the phenomenon of exponential decay of errors. But, in the cases of $\kappa = 100$ and 1000, the behavior does not yet reflect exponential decay of errors completely.

We remark that N should be taken greater than 1024 to see the exponential decay of the error estimator even if the number of digits employed in computation is 3200 in the cases of $\kappa = 100$ and 1000. Further we remark that the lowest part of the right column of Fig. 1 indicates that 3200 digits is still insufficient in the case of $\kappa = 1000$ with $\gamma = 0.1$ and probably with $\gamma = 0.3$, in order to observe the exponential decay of the numerical estimator of error.

8.1.4. A guide to practical computation

In the case of large κ, say $\kappa = 1000$, the numerical estimator does not decrease exponentially in general if the number of decimal digits of arithmetic is insufficient. This tendency is significantly remarkable for small γ. The pollution by rounding error is dominant in the computation with an insufficient number of digits.

We have found, however, that, even if the case of $\kappa = 1000$, our estimator of error guarantees the accuracy of the computed values within 10 decimal digits when we employ 30 decimal digit arithmetic for $\gamma = 0.9$ with $N = 4096$ or more.

Regarding this finding as a guide, we have decided values of parameters for the second test.

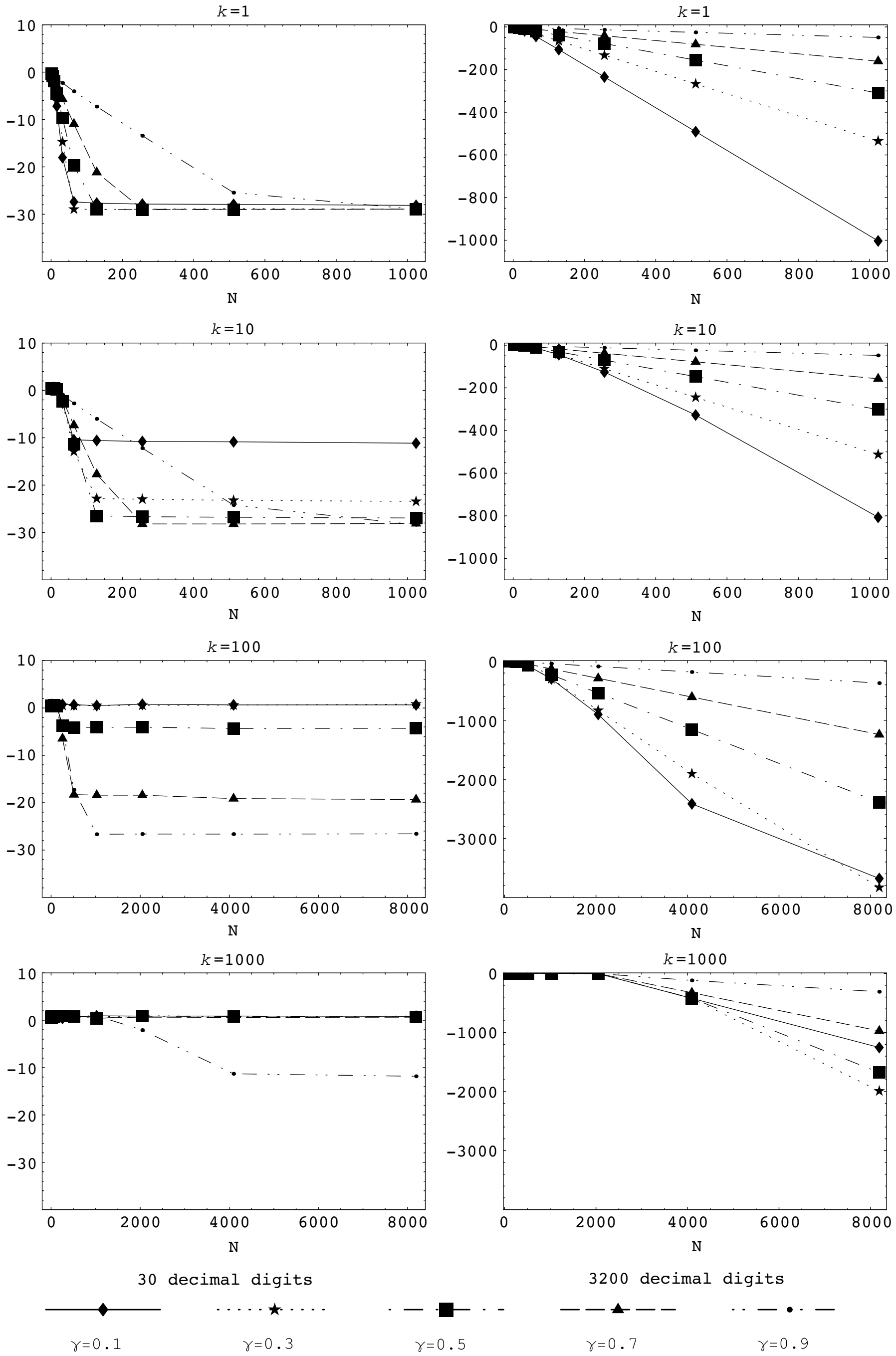


Fig. 1. Behavior of errors on Γ_a with common logarithmic scaling ordinate.

8.2. *Visualization of the scattering phenomena around a circular disc*

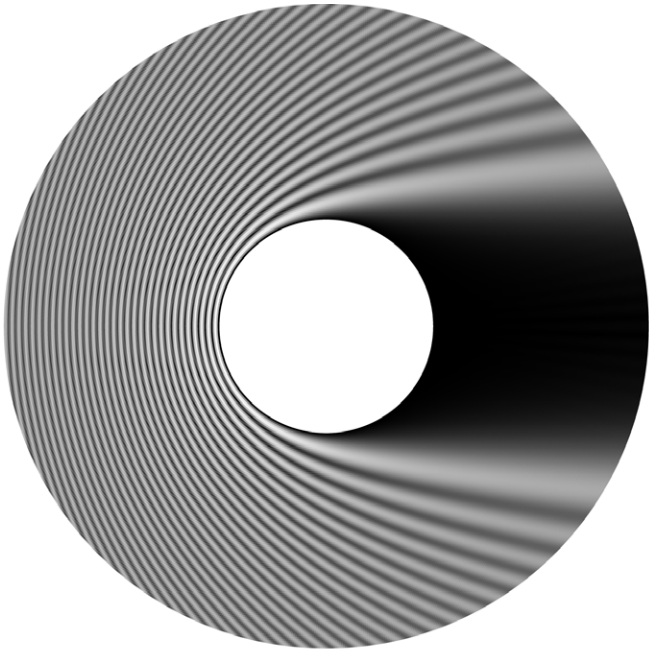
8.2.1. *Profiles of absolute values of total waves*

Let u_i be the incident wave with the form of plane wave along the direction of x axis. Namely we set

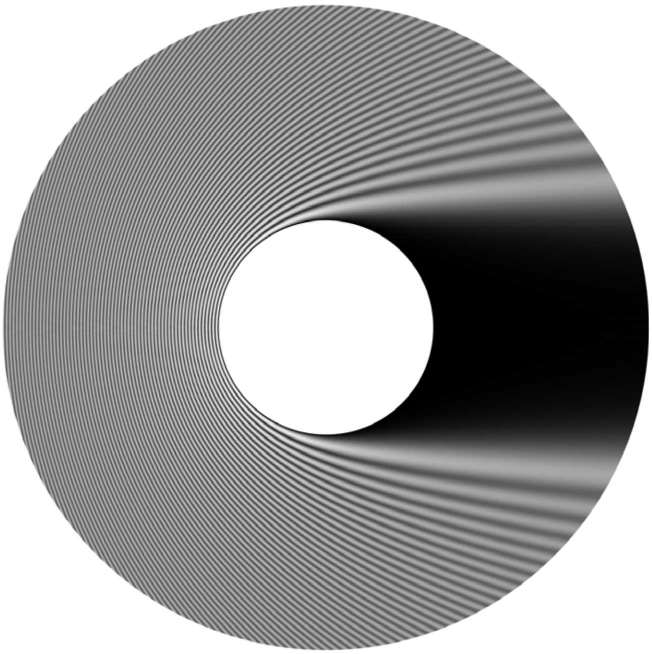
$$u_i(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}kx}.$$

Let f be the boundary value of $-u_i$ restricted on Γ_a. Therefore we have

$$f(\boldsymbol{a}(\theta)) = -\mathrm{e}^{\mathrm{i}\kappa\cos\theta}.$$



$\kappa = 50$



$\kappa = 100$

Fig. 2. Profiles of absolute values of total waves by grayscale plotting.

The solution u of ($\mathrm{E_f}$) with the above f is denoted by u_s, and the solution $u^{(N)}$ of ($\mathrm{E_f^{(N)}}$) by $u_s^{(N)}$. As usual the function u_s is said to be the scattering wave. Define the total wave u, and its approximation $u^{(N)}$ through

$$u = u_i + u_s \quad \text{and} \quad u^{(N)} = u_i + u_s^{(N)}.$$

As was shown in Section 8.1.1, the density function of $q(\theta)$ for this Dirichlet data f has a finite value of $|||q|||$, if the kernel function g satisfies Assumption 2. Fig. 2 shows profiles of absolute values of total waves $u^{(N)}$ around the circle with $\kappa = 50$ in the upper case, and with $\kappa = 100$, in the lower case. In both cases, a is taken to equal 1 and $\gamma = 0.9$ is employed to

Table 2
Parameters of the total waves in Fig. 2.

	$\kappa\ (=k)$	a	u_i	f	δ	γ	N	L
Upper	50	1	e^{i50x}	$-e^{i50\cos\theta}$	$1 \le \delta \le 3$	0.9	1024	128
Lower	100	1	e^{i100x}	$-e^{i100\cos\theta}$	$1 \le \delta \le 3$	0.9	2048	256

obtain $u_s^{(N)}$ through 30 decimal digit arithmetic. The numbers, N, of collocation points is 1024 in the case of $\kappa = 50$, and 2048 in the case of $\kappa = 100$. The profiles are shown in the annular region where $1 \le \delta \le 3$. For each direction from the center of the circle to a collocation point, equi-distant L points are taken as evaluation points of $u^{(N)}(\boldsymbol{r})$ where $L = 128$ in the case of $\kappa = 50$, and $L = 256$ in the case of $\kappa = 100$, respectively. More precisely we consider the closed interval on the ray along the direction starting from the point with $r = 1$ to the point with $r = 3$. The interval is divided equally into L segments. All the end points of the segments except for the points with $r = 3$ are employed as evaluation points. These values of computational parameters and related items are listed in Table 2.

In our visualization procedure, the absolute values of total waves $u(\boldsymbol{r})$ are normalized into the range of the interval $[0, 1]$. As a matter of fact, in practical computation, $u^{(N)}(\boldsymbol{r})$ is employed as $u(\boldsymbol{r})$. The profiles in Fig. 2 are drawn in a way of grayscale plotting in order that the value $|u(\boldsymbol{r})| = 1$, and the value $|u(\boldsymbol{r})| = 0$, may correspond to white, and to black, respectively.

8.2.2. Observation of shadow area

Due to the property of grayscale plotting, we may roughly understand in Fig. 2 that the plotted points scaled as white ($|u(\boldsymbol{r})| = 1$) correspond to the points at which the total wave attains either the top or the bottom of the wave, namely its peak, and that the plotted points scaled as black ($|u(\boldsymbol{r})| = 0$) correspond to the points at which the total wave is in quiescent state, namely in the middle part of the wave.

Comparing the upper case with the lower one, we see that the total wave behaves almost identically in both cases with $\kappa = 50$ and with $\kappa = 100$ except for the wavelength, and that the wavelength of the former seems to be almost twice longer than that of the latter in places where the waves seem to behave periodically, as a natural consequence of the ratio of normalized numbers between the former and the latter.

In both cases, the total waves seemingly almost vanish behind the disc towards the positive direction of the x axis. Since we treat the case of a progressing incident wave along this direction, these areas of vanishing wave may be considered as shadow areas in the scattering phenomena. Among the huge amounts of literature we have noted that Morse and Feshbach investigated the shadow phenomenon behind a cylindrical obstacle in a 3-dimensional scattering problem using the solutions of the Helmholtz equation in p. 1380 of their classical text [12].

8.3. Software for computing

For multiple-precision computation, we have employed software libraries MPFR [13] and GMP [14]. The former is a library for floating point arithmetic with arbitrary precision, which is based on the latter. We have coded our routine for Bessel functions following the routine by Ooura [15], and our routine for a Fast Fourier Transform following the sample routine in p. 164 of Brigham [16].

For visualization of numerical data, we have used a mathematical software system Mathematica [17] and a library program "psbasic" by Mizushima [18]. The program "psbasic" has been essential in order to generate our postscript codes of the waves.

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References

[1] F.J. Sánchez-Sesma, E. Rosenblueth, Ground motion at canyons of arbitrary shape under incident SH waves, Earthq. Eng. Struct. Dyn. 7 (1979) 441–450.
[2] F.J. Sánchez-Sesma, A boundary method applied to elastic scattering problems, Arch. Mech. 33 (1981) 167–179.
[3] T. Ushijima, FEM–FSM combined method for 2D exterior Laplace and Helmholtz problems, in: T. Chan, T. Kako, H. Kawarada, O. Pironneau (Eds.), Domain Decomposition Methods in Science and Engineering, Proceedings of 12th International Conference of Domain Decomposition Methods, Chiba, Japan, 25–29 October, 1999, 2001, pp. 223–230. DDM.org, Printed in Japan.
[4] T. Ushijima, FEM–CSM combined method for planar exterior Laplace problems, Japan J. Indust. Appl. Math. 18 (2001) 359–382.
[5] T. Ushijima, Equi-distant collocation method for periodic functions with kernel expression, in: Z.-C. Shi, H. Kawarada (Eds.), Proceedings of Fifth China–Japan Joint Seminar on Numerical Mathematics, Shanghai, 21–25 August, 2000, Science Press, Beijing, New York, 2002, pp. 220–226.

[6] T. Ushijima, F. Chiba, A fundamental solution method for the reduced wave problem in a domain exterior to a disc, J. Comput. Appl. Math. 152 (2002) 545–557.
[7] T. Ushijima, F. Chiba, Error estimate for a fundamental solution method applied to reduced wave problems in a domain exterior to a disc, J. Comput. Appl. Math. 159 (2003) 137–148.
[8] F. Chiba, T. Ushijima, A fundamental solution method with multiple-precision computation applied to reduced wave Neumann problems in the exterior region of a disc, Trans. Japan Soc. Ind. Appl. Math. 15 (2005) 361–384 (in Japanese).
[9] M. Katsurada, H. Okamoto, A Mathematical study of charge simulation method I, J. Fac. Sci. Univ. Tokyo Sect. IA. Math. 35 (1988) 507–518.
[10] G.N. Watson, A Treatise on the Theory of Bessel Functions, second edition, Cambridge University Press, Cambridge, 1966.
[11] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, ninth printing, Dover Publications, New York, 1972.
[12] P.M. Morse, H. Feshbach, Method of Theoretical Physics Part II, McGraw-Hill, New York, 1953.
[13] The MPFR Team, MPFR The Multiple Precision Floating-Point Reliable Library, edition 2.2.0, INRIA Lorraine/LORIA, Nancy 2005. http://www.mpfr.org/.
[14] Free Software Foundation, GNU MP The GNU Multiple Precision Arithmetic Library, edition 4.2.1, 2006. http://gmplib.org/.
[15] T. Ooura, Ooura's mathematical software packages, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 2001. http://www.kurims.kyoto-u.ac.jp/~ooura/index.html.
[16] E.O. Brigham, The Fast Fourier Transform, Prentice-Hall, New Jersey, 1974.
[17] S. Wolfram, The Mathematica Book, fifth edition, Wolfram Media Inc., Illinois, 2003.
[18] J. Mizushima, Postscript manual, Department of Mechanical Engineering, Graduate School of Engineering, Doshisha University, Kyoto, 2001 (in Japanese). http://www1.doshisha.ac.jp/~jmizushi/ps.html.