



Existence of positive solutions of nonlinear m -point BVP for an increasing homeomorphism and positive homomorphism on time scales[☆]

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ABSTRACT

In this paper, we will consider the following multipoint boundary value problem for the following second-order dynamic equations on time scales

$$(\phi(u^\Delta))^\nabla + a(t)f(t, u(t)) = 0, \quad t \in (0, T),$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)),$$

where $\phi : R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. By using fixed point theorems, we obtain an existence theorem of positive solutions for the above boundary value problem, which includes and improve some related results in the relevant literature. As an application, an example to demonstrate our results is given.

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1. Introduction

A time scale \mathbf{T} is a nonempty closed subset of R . We make the blanket assumption that $0, T$ are points in \mathbf{T} . By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is $(0, T) \cap \mathbf{T}$.

In this paper, we will be concerned with the existence of positive solutions for the following dynamic equations on time scales:

$$(\phi(u^\Delta))^\nabla + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \quad (1.1)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)), \quad (1.2)$$

where $\phi : R \rightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$.

A projection $\phi : R \rightarrow R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:

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- (i) if $x \leq y$, then $\phi(x) \leq \phi(y)$, $\forall x, y \in R$;
- (ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
- (iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in R_+ = [0, +\infty)$.

We will assume that the following conditions are satisfied throughout this paper:

- (H₁) $0 < \xi_1 < \dots < \xi_{m-2} < \rho(T)$, $a_i, b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$;
- (H₂) $a(t) \in C_{ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (\xi_{m-2}, T)$, such that $a(t_0) > 0$;
- (H₃) $f \in C([0, T] \times [0, +\infty), [0, +\infty))$. (The Δ -derivative and the ∇ -derivative in (1.1), (1.2) and the C_{ld} space in (H₂) are defined in Section 2.)

For the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results in recent years, see [1–6,27] and the references therein. Recently, much attention has been paid to the existence of positive solutions for second-order nonlinear boundary value problems on time scales, for examples, see [7–9,6,10,29] and the references therein. At the same time, multipoint nonlinear boundary value problems with p -Laplacian operators on time scales have also been studied extensively in the literature, for details, see [1,3,4,11,12,6,28] and the references therein. But to the best of our knowledge, few people considered the second-order dynamic equations of increasing homeomorphism and positive homomorphism on time scales.

The present work is motivated by the recent papers [13–15]. In [15], Yang and Xiao studied the existence of multiple positive solutions for the following multipoint BVP:

$$(\phi(x'(t)))' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \quad (1.3)$$

$$x(0) = \sum_{i=1}^{n-2} \alpha_i x(\xi_i), \quad \phi(x'(1)) = \sum_{i=1}^{n-2} \beta_i \phi(x'(\xi_i)), \quad (1.4)$$

where $\phi : R \rightarrow R$ is an odd, increasing homeomorphism from R to R . By using the fixed point theorems, they obtained new results on the existence of at least three positive solutions of the above boundary value problem.

The rest of the paper is arranged as follows. We state some basic time scale definitions and prove several preliminary results in Section 2, Section 3 is devoted to establish the key conditions in Theorem 3.1 to show the existence of positive solutions of the BVP (1.1) and (1.2), the main tool being a recent new fixed point theorem in cone [6,10]. At the end of the paper, we will give two examples which illustrate that our work is true. We also point out that when $T = R$, $p = 2$, (1.1) and (1.2) becomes a boundary value problem of differential equations and is just the problem considered in [16]. Our main results include and extend the main results of [13,14,16,17].

2. Preliminaries and some lemmas

For convenience, we list the following definitions which can be found in [18–21].

Definition 2.1. A time scale T is a nonempty closed subset of real numbers R . For $t < \sup T$ and $r > \inf T$, define the forward jump operator σ and backward jump operator ρ , respectively, by

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in T \mid \tau > t\} \in T, \\ \rho(r) &= \sup\{\tau \in T \mid \tau < r\} \in T \end{aligned}$$

for all $t, r \in T$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If T has a right scattered minimum m , define $T_k = T - \{m\}$; otherwise set $T_k = T$. If T has a left scattered maximum M , define $T^k = T - \{M\}$; otherwise set $T^k = T$.

Definition 2.2. For $f : T \rightarrow R$ and $t \in T^k$, the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$.

For $f : T \rightarrow R$ and $t \in T_k$, the nabla derivative of f at t is the number $f^\nabla(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon |\rho(t) - s|,$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e. ld -continuous), if f is continuous at each left-dense point in T and its right-sided limit exists at each right-dense point in T .

Definition 2.4. If $G^\Delta = f(t)$, then we define the delta integral by

$$\int_a^b f(t) \Delta t = G(b) - G(a).$$

If $F^\nabla(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t) \nabla t = F(b) - F(a).$$

To prove the main results in this paper, we will employ several lemmas. especially [Lemma 2.1](#) is based on the following linear BVP

$$(\phi(u^\Delta))^\nabla + h(t) = 0, \quad t \in (0, T), \quad (2.1)$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)). \quad (2.2)$$

Lemma 2.1 (See [22]). If $\sum_{i=1}^{m-2} a_i \neq 1$ and $\sum_{i=1}^{m-2} b_i \neq 1$, then for $h \in C_{ld}[0, T]$ the BVP (2.1) and (2.2) have the unique solution

$$u(t) = \int_0^t \phi^{-1} \left(\int_s^T h(\tau) \nabla \tau - A \right) \Delta s + B, \quad (2.3)$$

where

$$A = - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i}, \quad B = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T h(\tau) \nabla \tau - A \right) \Delta s}{1 - \sum_{i=1}^{m-2} a_i}.$$

Let

$$E = \left\{ u \in C_{ld}([0, T], R) \mid u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \phi(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi(u^\Delta(\xi_i)) \right\},$$

where $C_{ld}([0, T], R)$ is the set of all ld-continuous functions from $[0, T]$ to R , and let the norm on E be the maximum norm. Then $(E, \|\cdot\|)$ is a Banach space. We define two cones by

$$P = \{u : u \in E, u(t) \geq 0, t \in [0, T]\},$$

and

$$P' = \{u : u \in E, u(t) \text{ is concave, nonnegative and increasing on } [0, T]\}.$$

Lemma 2.2. If $u \in P'$, then for $t \in [0, T]$, we have $\inf_{t \in [0, T]} u(t) \geq \gamma \|u\|$, where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i \xi_i}{\left(1 - \sum_{i=1}^{m-2} a_i\right) T + \sum_{i=1}^{m-2} a_i \xi_i}, \quad \|u\| = \max_{t \in [0, T]} |u(t)|.$$

Proof. If $u \in P'$, then $u(t)$ is concave down on $(0, T)$.

For each $i \in \{1, 2, \dots, m-2\}$, we have

$$\frac{u(T) - u(0)}{T - 0} \geq \frac{u(T) - u(\xi_i)}{T - \xi_i},$$

i.e.,

$$Tu(\xi_i) - \xi_i u(T) \geq (T - \xi_i)u(0),$$

so that

$$T \sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i \xi_i u(T) \geq \sum_{i=1}^{m-2} a_i (T - \xi_i) u(0).$$

With the condition $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we have

$$u(0) \geq \frac{\sum_{i=1}^{m-2} a_i \xi_i}{T - \sum_{i=1}^{m-2} a_i (T - \xi_i)} u(T) = \frac{\sum_{i=1}^{m-2} a_i \xi_i}{\left(1 - \sum_{i=1}^{m-2} a_i\right) T + \sum_{i=1}^{m-2} a_i \xi_i} u(T).$$

This completes the proof. \square

Let

$$K = \{u | u \in E, u(t) \text{ is nonnegative and increasing on } [0, T], u(t) \geq \gamma \|u\|, t \in [0, T]\},$$

where γ is the same as in Lemma 2.2.

From Lemma 2.1, it is easy to see that the BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u solves the equation

$$u(t) = \int_0^t \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B},$$

where

$$\tilde{A} = - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) f(\tau, u(\tau)) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i}, \quad (2.4)$$

$$\tilde{B} = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s}{1 - \sum_{i=1}^{m-2} a_i}. \quad (2.5)$$

We define the operator $F : P \rightarrow E$ as follows

$$(Fu)(t) = \int_0^t \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B},$$

where \tilde{A} and \tilde{B} are defined by (2.4) and (2.5).

It is obvious that K is a cone in E . From the expression of $(Fu)(t)$, we can easily get $(Fu)(t) \in P'$, by Lemma 2.2, $F(K) \subset K$. So by applying the Arzela–Ascoli theorem on time scales [23], we can obtain that $F(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [24], it is easy to prove that F is continuous. Hence, $F : K \rightarrow K$ is completely continuous.

The approach is mainly based on the following fixed point theorem, they can be found in [25,26].

Let E be a real Banach space and P be a cone in E . $\rho : P \rightarrow R$ is said to be a convex functional on P if $\rho(tx + (1-t)y) \leq t\rho(x) + (1-t)\rho(y)$ for all $x, y \in P$ and $t \in [0, 1]$.

Lemma 2.3 (See [25,26]). Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ is completely continuous and $\rho : P \rightarrow [0, +\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \neq \theta$. If one of the two conditions is satisfied:

- (H₄) $\rho(Ax) \leq \rho(x)$, $\forall x \in P \cap \partial\Omega_1$ and $\inf_{x \in P \cap \partial\Omega_2} \rho(x) > 0$, $\rho(Ax) \geq \rho(x)$, $\forall x \in P \cap \partial\Omega_2$,
- (H₅) $\inf_{x \in P \cap \partial\Omega_1} \rho(x) > 0$, $\rho(Ax) \geq \rho(x)$, $\forall x \in P \cap \partial\Omega_1$ and $\rho(Ax) \leq \rho(x)$, $\forall x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Remark 2.1 (See [25,26]). Obviously, $\rho(x) = \|x\|$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \neq \theta$. Moreover $\inf_{x \in P \cap \partial\Omega_1} \|x\| > 0$ and $\inf_{x \in P \cap \partial\Omega_2} \|x\| > 0$ since $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$.

To employ the above lemmas, we define $\rho(u) = \max_{t \in [0, T]} u(t) = \|u\|$, then $\rho : K \rightarrow [0, +\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(u) > 0$ for $u \neq 0$.

Now, for convenience, we introduce the following notations. Let

$$\begin{aligned} \varphi(s) &= \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right), \\ m &= \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\}^{-1}. \end{aligned} \quad (2.6)$$

3. Existence theorems of positive solutions

Theorem 3.1. Assume that (H_1) , (H_2) and (H_3) hold, and if there exist constants a and b with $0 < b < a$ satisfying one of the two conditions:

(H_6) $a \geq \frac{b}{\gamma^2}$, $f(t, u) \geq \phi(am)$, for $t \in [0, T]$, $a\gamma \leq u \leq a$ and $f(t, u) \leq \phi(bm)$ for $t \in [0, T]$, $u \leq \frac{b}{\gamma}$,

(H_7) $f(t, u) \leq \phi(am)$, for $t \in [0, T]$, $u \leq \frac{a}{\gamma}$ and $f(t, u) \geq \phi(bm)$, for $t \in [0, T]$, $b\gamma \leq u \leq b$,

then (1.1) and (1.2) have at least a positive solution.

Proof. Set

$$\Omega_1 = \{u \in C_{ld}([0, T], R) | \rho(u) < b\}, \quad \Omega_2 = \{u \in C_{ld}([0, T], R) | \rho(u) < a\}.$$

It is clear that Ω_1 and Ω_2 are open sets with $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$.

Suppose (H_6) is satisfied. If $u \in K \cap \partial\Omega_1$, then $\rho(u) = b$ and for $t \in [0, T]$, $\|u\| \leq \frac{b}{\gamma}$. Therefore,

$$\begin{aligned} \int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} &= \int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) f(\tau, u(\tau)) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \\ &\leq \phi(bm) \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right], \end{aligned}$$

so that

$$\begin{aligned} \varphi(s) &= \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \\ &\leq bm\phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right]. \end{aligned}$$

And thus, by (2.6), we have

$$\begin{aligned} \rho(Fu) &= \max_{0 \leq t \leq T} \left[\int_0^t \varphi(s) \Delta s + \tilde{B} \right] = \int_0^T \varphi(s) \Delta s + \tilde{B} \\ &= \int_0^T \varphi(s) \Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s) \Delta s}{1 - \sum_{i=1}^{m-2} a_i} \end{aligned}$$

$$\begin{aligned}
&\leq bm \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right. \\
&\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\} \\
&= b = \rho(u).
\end{aligned}$$

If $u \in K \cap \partial \Omega_2$, then $\rho(u) = a$ and for $t \in [0, T]$,

$$a\gamma = \gamma \|u\| \leq u(t) \leq \rho(u) = a.$$

Therefore,

$$\begin{aligned}
\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} &= \int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) f(\tau, u(\tau)) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&\geq \phi(am) \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right],
\end{aligned}$$

so that

$$\begin{aligned}
\varphi(s) &= \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \\
&\geq am \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right].
\end{aligned}$$

And thus, it follows that

$$\begin{aligned}
\rho(Fu) &= \max_{0 \leq t \leq T} \left[\int_0^t \varphi(s) \Delta s + \tilde{B} \right] = \int_0^T \varphi(s) \Delta s + \tilde{B} \\
&= \int_0^T \varphi(s) \Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s) \Delta s}{1 - \sum_{i=1}^{m-2} a_i} \\
&\geq am \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right. \\
&\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\} \\
&= a = \rho(u).
\end{aligned}$$

Thus, by Lemma 2.3, condition (H₄) is satisfied, (1.1) and (1.2) have at least one positive solution.

In the following, when (H_7) is satisfied, we check condition (H_5) in Lemma 2.3. If $u \in K \cap \partial\Omega_1$, then $\rho(u) = b$ and for $t \in [0, T]$,

$$b\gamma = \gamma\|u\| \leq u(t) \leq \rho(u) = a.$$

Therefore,

$$\begin{aligned} \rho(Fu) &= \max_{0 \leq t \leq T} \left[\int_0^t \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B} \right] \\ &= \int_0^T \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B} \\ &= \int_0^T \varphi(s) \Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s) \Delta s}{1 - \sum_{i=1}^{m-2} a_i} \\ &\geq bm \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\} \\ &= b = \rho(u). \end{aligned}$$

If $u \in K \cap \partial\Omega_2$, then $\rho(u) = a$ and for $t \in [0, T]$, we have

$$\|u\| \leq \frac{a}{\gamma}.$$

Therefore,

$$\begin{aligned} \rho(Fu) &= \max_{0 \leq t \leq T} \left[\int_0^t \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B} \right] \\ &= \int_0^T \phi^{-1} \left(\int_s^T a(\tau) f(\tau, u(\tau)) \nabla \tau - \tilde{A} \right) \Delta s + \tilde{B} \\ &= \int_0^T \varphi(s) \Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s) \Delta s}{1 - \sum_{i=1}^{m-2} a_i} \\ &\leq am \left\{ \int_0^T \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right. \\ &\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1 - \sum_{i=1}^{m-2} a_i} \int_0^{\xi_i} \phi^{-1} \left[\int_s^T a(\tau) \nabla \tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau) \nabla \tau}{1 - \sum_{i=1}^{m-2} b_i} \right] \Delta s \right\} \\ &= a = \rho(u). \end{aligned}$$

Thus, by Lemma 2.3, condition (H_5) is satisfied, so (1.1) and (1.2) have at least one positive solution. \square

4. Some examples

In this section, we present a simple example to explain our results.

Example 4.1. Let $\mathbf{T} = \{(\frac{1}{2})^n : n \in N\} \cup \{1\}$, $T = 1$. Consider the following BVP on time scales

$$(\phi(u^\Delta))^\nabla + f(t, u(t)) = 0, \quad t \in (0, T), \quad (4.1)$$

$$u(0) = \frac{1}{4}u\left(\frac{1}{3}\right), \quad \phi(u^\Delta(T)) = \frac{1}{2}\phi\left(u^\Delta\left(\frac{1}{3}\right)\right), \quad (4.2)$$

where

$$\phi(u) = \begin{cases} u^3, & u \leq 0, \\ u, & u > 0, \end{cases} \quad f(t, u) = (1+t)\left(\frac{u}{20}\right)^{20}, \quad (t, u) \in [0, 1] \times [0, +\infty).$$

It is easy to check that $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous. In this case, $a(t) \equiv 1$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{3}$, it follows from a direct calculation that

$$\begin{aligned} m &= \left\{ \int_0^T \phi^{-1} \left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1} \right] \Delta s + \frac{a_1}{1-a_1} \int_0^{\xi_1} \phi^{-1} \left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1} \right] \Delta s \right\}^{-1} \\ &= \left[\int_0^1 \left(1-s + \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}} \right)^{\frac{1}{2}} ds + \frac{\frac{1}{4}}{1-\frac{1}{4}} \int_0^{\frac{1}{3}} \left(1-s + \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}} \right)^{\frac{1}{2}} ds \right]^{-1} \\ &= \left[\int_0^1 \left(\frac{5}{3} - s \right)^{\frac{1}{2}} ds + \frac{1}{3} \int_0^{\frac{1}{3}} \left(\frac{5}{3} - s \right)^{\frac{1}{2}} ds \right]^{-1} \approx 0.8281, \\ \gamma &= \frac{a_1 \xi_1}{(1-a_1)T + a_1 \xi_1} = \frac{\frac{1}{4} \cdot \frac{1}{3}}{(1-\frac{1}{4}) \cdot 1 + \frac{1}{4} \cdot \frac{1}{3}} = \frac{1}{10}. \end{aligned}$$

Choose $a = 400$, $b = 1$, it is easy to check that $400 = a \geq \frac{b}{\gamma^2} = 100$,

$$\phi(am) = \phi(400 \times 0.8281) = (400 \times 0.8281) = 331.24,$$

$$\phi(bm) = \phi(0.8281) = 0.8281,$$

and f satisfies that

$$f(t, u) \geq \left(\frac{u}{20}\right)^{20} \geq \left(\frac{40}{20}\right)^{20} = 2^{20} \geq 331.24 = \phi(am), \quad \text{for } 40 \leq u \leq 400,$$

$$f(t, u) \leq 2 \times \left(\frac{u}{20}\right)^{20} \leq 2 \times \left(\frac{10}{20}\right)^{20} = 2 \times \left(\frac{1}{2}\right)^{20} \leq 0.6857 = \phi(bm), \quad \text{for } u \leq 10,$$

thus, condition (H_6) in Theorem 3.1 is satisfied, the BVP (4.1) and (4.2) have at least one positive solution.

If we take

$$f(t, u) = (1+t) \cdot \frac{u}{5000} + 1, \quad (t, u) \in [0, 1] \times [0, +\infty).$$

By computation, we can get

$$f(t, u) \leq 2 \times \frac{4000}{5000} + 1 \leq 331.24 = \phi(am), \quad \text{for } u \leq 4000,$$

$$f(t, u) \geq \frac{1}{5000} + 1 \geq 0.8281 = \phi(bm), \quad \text{for } b\gamma = \frac{1}{10} \leq u \leq 1 = b,$$

thus, condition (H_7) in Theorem 3.1 is satisfied, the BVP (4.1) and (4.2) has also at least one positive solution.

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