Journal of Computational and Applied Mathematics 234 (2010) 3326–3331

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On a new notion of the solution to an ill-posed problem

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ARTICLE INFO

Article history:
Received 3 January 2010
Received in revised form 21 April 2010

MSC:
47A52
65F22
65J20

Keywords:
Ill-posed problems
Regularizer
Stable solution of ill-posed problems

ABSTRACT

A new understanding of the notion of the stable solution to ill-posed problems is proposed. The new notion is more realistic than the old one and better fits the practical computational needs. A method for constructing stable solutions in the new sense is proposed and justified. The basic point is: in the traditional definition of the stable solution to an ill-posed problem $Au = f$, where A is a linear or nonlinear operator in a Hilbert space H, it is assumed that the noisy data $\{f_\delta, \delta\}$ are given, $\|f - f_\delta\| \le \delta$, and a stable solution $u_\delta := R_\delta f_\delta$ is defined by the relation $\lim_{\delta\to 0} \|R_\delta f_\delta - y\| = 0$, where y solves the equation $Au = f$, i.e., $Ay = f$. In this definition y and f are unknown. Any $f \in B(f_\delta, \delta)$ can be the exact data, where $B(f_\delta, \delta) := \{f : \|f - f_\delta\| \le \delta\}$.

The new notion of the stable solution excludes the unknown y and f from the definition of the solution. The solution is defined only in terms of the noisy data, noise level, and an a priori information about a compactum to which the solution belongs.

1. Introduction

Let

$$Au = f, \tag{1.1}$$

where $A : H \to H$ is a linear closed operator, densely defined in a Hilbert space H. Problem (1.1) is called ill-posed if A is not a homeomorphism of H onto H, that is, either Eq. (1.1) does not have a solution, or the solution is non-unique, or the solution does not depend on f continuously. Let us assume that (1.1) has a solution, possibly non-unique. Let $N(A)$ be the null space of A, and y be the unique normal solution to (1.1), i.e., $y \perp N(A)$. Given noisy data f_δ, $\|f_\delta - f\| \le \delta$, one wants to construct a stable approximation $u_\delta := R_\delta f_\delta$ of the solution y, $\|u_\delta - y\| \to 0$ as $\delta \to 0$.

Traditionally (see, e.g., [1,2]) one calls a family of operators R_h a regularizer for problem (1.1) (with not necessarily linear operator A) if

(a) $R_h A(u) \to u$ as $h \to 0$ for any $u \in D(A)$,
(b) $R_h f_\delta$ is defined for any $f_\delta \in H$ and there exists $h = h(\delta) \to 0$ as $\delta \to 0$ such that

$$\|R_{h(\delta)} f_\delta - y\| \to 0 \quad \text{as } \delta \to 0. \tag{$*$}$$

In this definition y is fixed and ($*$) must hold for any $f_\delta \in B(f, \delta) := \{f_\delta : \|f_\delta - f\| \le \delta\}$.

In practice one does not know the solution y and the exact data f. The only available information is a family f_δ and some *a priori information about f or about the solution y*. This a priori information often consists of the knowledge that $y \in \mathcal{K}$, where $\mathcal{K}$ is a compactum in H. Thus

$$y \in S_\delta := \{v : \|A(v) - f_\delta\| \le \delta, v \in \mathcal{K}\}.$$

We assume that the operator A is known exactly, and we always assume that $f_\delta \in B(f, \delta)$, where $f = A(y)$.

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doi:10.1016/j.cam.2010.04.032

Definition. We call a family of operators $R(\delta)$ a regularizer if

$$\sup_{v\in S_\delta} \|R(\delta)f_\delta - v\| \le \eta(\delta) \to 0 \quad \text{as } \delta \to 0. \tag{1.2}$$

There is a *crucial difference* between our new Definition (1.2) and the standard definition $(*)$:

In $(*)$ y is fixed, while in (1.2) v is an arbitrary element of S_δ and the supremum of the norm in (1.2) over all such v must tend to zero as $\delta \to 0$.

The new definition is more realistic and better fits computational needs because not only the solution y to (1.1) satisfies the inequality $\|Ay - f_\delta\| \le \delta$, but any $v \in S_\delta$ satisfies this inequality $\|Av - f_\delta\| \le \delta$, $v \in \mathcal{K}$. The data f_δ may correspond to any $f = Av$, where $v \in S_\delta$, and not only to $f = Ay$, where y is a solution of Eq. (1.1). Therefore it is more natural to use definition (1.2) than the traditional definition $(*)$.

Our goal is to illustrate the practical difference between these two definitions, and to construct regularizer in the new sense (1.2) for problem (1.1) with an arbitrary, not necessarily bounded, linear operator A, which is closed and densely defined in H. This is done in Section 2.

In Section 1 this is done for a class of Eq. (1.1) with nonlinear operators $A : X \to Y$, where X and Y are Banach spaces. In this case we make two assumptions, (A1) and (A2):

(A1) *$A : X \to Y$ is a closed, nonlinear, injective map, $f \in \mathcal{R}(A)$, where $\mathcal{R}(A)$ it is the range of A,*
and

(A2) *the functional ϕ has the following properties:*

$$\phi : D(\phi) \to [0, \infty), \qquad \phi(u) > 0 \quad \text{if } u \ne 0, \qquad D(\phi) \subseteq D(A),$$

the sets $\mathcal{K} = \mathcal{K}_c := \{v : \phi(v) \le c\}$ are compact in X for every $c = const > 0$,
and if $v_n \to v$, then $\phi(v) \le \liminf_{n\to\infty} \phi(v_n)$.

The last inequality holds if ϕ is *lower semicontinuous*. In Hilbert spaces and in reflexive Banach spaces norms are lower semicontinuous [3].

Let us give some examples of equations for which assumptions (A1) and (A2) are satisfied.

Example 1. A is a linear injective compact operator, $f \in \mathcal{R}(A)$, $\phi(v)$ is a norm on $X_1 \subset X$, where X_1 is densely embedded in X, the embedding $i : X_1 \to X$ is compact, and $\phi(v)$ is lower semicontinuous.

Example 2. A is a nonlinear injective continuous operator $f \in \mathcal{R}(A)$, A^{-1} is not continuous, ϕ is as in Example 1.

Example 3. A is linear, injective, densely defined, closed operator, $f \in \mathcal{R}(A)$, A^{-1} is unbounded, ϕ is as in Example 1, $X_1 \subseteq D(A)$.

Let us demonstrate by Example A that *a regularizer in the sense $(*)$ may be not a regularizer in the sense* (1.2).

In Example B a theoretical construction of a regularizer in the sense (1.2) is given for some Eq. (1.1) with nonlinear operators.

In Section 2 a novel theoretical construction of a regularizer in the sense (1.2) is given for a very wide class of Eq. (1.1) with linear operators A.

Example A. *Stable numerical differentiation.*

In this Example the results from [4,5], are used, see also Chapter 15 in the book [6], where the problem of stable numerical differentiation is discussed in detail. This Example is borrowed from [5].

Consider stable numerical differentiation of noisy data. The problem is:

$$Au := \int_0^x u(s)ds = f(x), \qquad f(0) = 0, \quad 0 \le x \le 1. \tag{1.3}$$

The data are the values f_δ and the constant M_a. Here f_δ are the "noisy" data, $\|f_\delta - f\| \le \delta$, where *the norm is* $L^\infty(0, 1)$ *norm*, and the constant M_a defines a compact $\mathcal{K}$. This compact $\mathcal{K}$ consists of the L^∞ functions which satisfy the inequality $\|u\|_a \le M_a$, $a \ge 0$,

$$\mathcal{K} := \{u : \|u\|_a \le M_a\}.$$

The norm

$$\|u\|_a := \sup_{\substack{x,y\in[0,1]\\ x\ne y}} \frac{|u(x) - u(y)|}{|x - y|^a} + \sup_{0\le x\le 1} |u(x)| \quad \text{if } 0 \le a \le 1,$$

$$\|u\|_a := \sup_{0\le x\le 1} (|u(x)| + |u'(x)|) + \sup_{\substack{x,y\in[0,1]\\ x\ne y}} \frac{|u'(x) - u'(y)|}{|x - y|^{a-1}}, \quad 1 < a \le 2.$$

If $a > 1$, then we define

$$R(\delta)f_\delta := \begin{cases} \dfrac{f_\delta(x+h(\delta)) - f_\delta(x-h(\delta))}{2h(\delta)}, & h(\delta) \le x \le 1-h(\delta), \\ \dfrac{f_\delta(x+h(\delta)) - f_\delta(x)}{h(\delta)}, & 0 \le x < h(\delta), \\ \dfrac{f_\delta(x) - f_\delta(x-h(\delta))}{h(\delta)}, & 1-h(\delta) < x \le 1, \end{cases} \tag{1.4}$$

where

$$h(\delta) = c_a \delta^{\frac{1}{a}}, \tag{1.5}$$

and c_a is a constant given explicitly (cf. [7]).

We prove that (1.4) is a regularizer for (1.3) in the sense (1.2), and

$$\mathcal{K} := \{v : \|v\|_a \le M_a, a > 1\}.$$

In this example we do not use lower semicontinuity of the norm $\phi(v)$ and do not define ϕ.

Let

$$S_{\delta,a} := \{v : \|Av - f_\delta\| \le \delta, \|v\|_a \le M_a\}.$$

To prove that (1.4) and (1.5) is a regularizer in the sense (1.2) we use the following estimate

$$\begin{aligned} \sup_{v \in S_{\delta,a}} \|R(\delta)f_\delta - v\| &\le \sup_{v \in S_{\delta,a}} \{\|R(\delta)(f_\delta - Av)\| + \|R(\delta)Av - v\|\} \le \frac{\delta}{h(\delta)} + M_a h^{a-1}(\delta) \\ &\le c(a)\delta^{1-\frac{1}{a}} := \eta(\delta) \to 0 \quad \text{as } \delta \to 0, \end{aligned} \tag{1.6}$$

where $c(a) > 0$ is a constant which can be calculated explicitly, $c(a) = \frac{1}{c_a} + M_a c_a^{a-1}$, and c_a is the constant defined in (1.5).

Therefore, it follows from (1.6) that formulas (1.4) and (1.5) yield a regularizer in the sense (1.2) for the problem of stable numerical differentiation.

If $a = 1$ and $M_1 < \infty$, then we can prove the following result.

Claim. *There is no regularizer for problem (1.3) in the sense (1.2) even if the regularizer is sought in the set of all operators, including nonlinear ones.*

More precisely, it is proved in [6], pp. 197–235, where the stable numerical differentiation problem is discussed in detail, that

$$\inf_{R(\delta)} \sup_{v \in S_{\delta,1}} \|R(\delta)f_\delta - v\| \ge c > 0,$$

where $S_{\delta,1} = S_{\delta,a}|_{a=1}$, $c > 0$ is a constant independent of δ, and *the infimum is taken over all operators $R(\delta)$ acting from $L^\infty(0, 1)$ into $L^\infty(0, 1)$, including nonlinear ones*.

On the other hand, if $a = 1$ and $M_1 < \infty$, then a regularizer in the sense $(*)$ does exist, but the rate of convergence in $(*)$ may be as slow as one wishes, if $u(x)$ is chosen suitably (see [7] or [6]).

Let us compare the new definition of the regularizer with the standard one.

It is proved in Example A that if and only if $a > 1$ the regularizer in the new sense does exist, and explicit form of this regularizer and the error estimate are given. This error estimate is valid for the regularizer in the usual sense, because the new regularizer, if it exists, is also a regularizer in the usual sense. On the other hand, when $a = 1$, then the regularizer in the new sense does not exist, and the regularizer in the usual sense, although exists, but its convergence rate can be as slow as one wishes for a suitable data. Therefore, one may say that in this case the usual regularizer does not yield a solution computable from the numerical analysis point of view.

Example B. Construction of a regularizer in the sense (1.2) for some nonlinear equations.

Assuming (A1) and (A2), let us construct a regularizer for (1.1) in the sense (1.2). We use the ideas from [5,8]. Let $A(u) = f$.

Define $F_\delta(v) := \|Av - f_\delta\| + \delta\phi(v)$ and consider the minimization problem of finding the infimum $m(\delta)$ of the functional $F_\delta(v)$ on a set S_δ:

$$m(\delta) := \inf_{v \in S_\delta} F_\delta(v), \qquad S_\delta := \{v : \|Av - f_\delta\| \le \delta, \phi(v) \le c\}. \tag{1.7}$$

Here

$$\mathcal{K} = \mathcal{K}_c := \{v : \phi(v) \le c\}$$

is a compact set in X by the Assumption (A2). The constant $c > 0$ can be chosen arbitrary large and fixed at the beginning of the argument, and then one can choose a smaller constant c_1, specified below. Since

$$F_\delta(u) = \delta + \delta\phi(u) := c_1\delta, \qquad c_1 := 1 + \phi(u),$$

where $A(u) = f$, one concludes that

$$m(\delta) \le c_1\delta. \tag{1.8}$$

Let v_j be a minimizing sequence for the functional $F_\delta(v)$. If j is sufficiently large, then

$$F_\delta(v_j) \le 2m(\delta) \le 2c_1\delta,$$

and

$$\phi(v_j) \le 2c_1.$$

By Assumption (A2), as $j \to \infty$, one can select a convergent subsequence, denoted again v_j, and obtain

$$v_j \to v_\delta, \qquad \phi(v_\delta) \le 2c_1. \tag{1.9}$$

Take $\delta = \delta_m \to 0$ and denote $v_{\delta_m} := w_m$. Then (1.9) and Assumption (A2) imply the existence of a subsequence, denoted again w_m, such that:

$$w_m \to w, \qquad A(w_m) \to A(w), \qquad \|A(w) - f\| = 0. \tag{1.10}$$

Thus $A(w) = f$. Since A is injective by Assumption (A1), it follows that $w = u$, where u is the unique solution to the equation $A(u) = f$.

Define now $R(\delta)f_\delta$ by the formula

$$R(\delta)f_\delta := v_\delta,$$

where v_δ is defined in (1.9).

Theorem 1.1. *$R(\delta)$ is a regularizer for problem (1.1) in the sense (1.2).*

Proof. Assume the contrary:

$$\sup_{v \in S_\delta} \|R(\delta)f_\delta - v\| = \sup_{v \in S_\delta} \|v_\delta - v\| \ge \gamma > 0, \tag{1.11}$$

where $\gamma > 0$ is a constant independent of δ. Since $\phi(v_\delta) \le 2c_1$ by (1.9), and $\phi(v) \le c$ by (1.7), one can choose convergent in X sequences

$$w_m := v_{\delta_m} \to \tilde{w} \quad \text{as } \delta_m \to 0,$$

and

$$v_m \to \tilde{v},$$

such that

$$\|w_m - v_m\| \ge \frac{\gamma}{2}, \qquad \|\tilde{w} - \tilde{v}\| \ge \frac{\gamma}{2},$$

and

$$A(\tilde{w}) = f, \qquad A(\tilde{v}) = f.$$

By the injectivity of A it follows that $\tilde{w} = \tilde{v} = u$. This contradicts the inequality $\|\tilde{w} - \tilde{v}\| \ge \frac{\gamma}{2} > 0$. This contradiction proves the theorem.

The conclusions $A(\tilde{w}) = f$ and $A(\tilde{v}) = f$, that we have used above, follow from the inequalities

$$\|A(v_\delta) - f_\delta\| \le \delta, \qquad \|A(v) - f_\delta\| \le \delta, \qquad \|f - f_\delta\| \le \delta,$$

after passing to the limit $\delta \to 0$. In passing to the limit we have used the closedness of the operator A, which is a part of the assumption (A2). $\square$

2. Construction of a regularizer in the sense (1.2) for linear equations

If A is a linear closed densely defined in H operator, then $T = A^*A$ is a densely defined self-adjoint operator. Let $T_a := T + aI$, where $a = const > 0$. The operator $T_a^{-1}A^*$ is densely defined and closable. Its closure is a bounded operator, defined on all of H, and $\|T_a^{-1}A^*\| \leq \frac{1}{2\sqrt{a}}$. See [9–12] for details and other results. Let E_s be the resolution of the identity of the self-adjoint operator T, $d\rho := d(E_s y, y)$, and

$$\mathcal{K} := \left\{ y : \int_0^\infty s^{-2p} d\rho \leq k_p^2 \right\}, \quad p \in (0, 1), k_p > 0,$$

where p and k_p are constants.

Our basic result is:

Theorem 2.1. *The operator $R_\delta = T_{a(\delta)}^{-1}A^*$ is a regularizer for problem (1.1) in the sense (1.2) if $\lim_{\delta\to 0} \frac{\delta}{[a(\delta)]^{1/2}} = 0$ and $\lim_{\delta\to 0} a(\delta) = 0$. Moreover, if $a(\delta) = b_p \delta^{\frac{2}{2p+1}}$, then*

$$\sup_{y\in\mathcal{K},\ \|Ay-f_\delta\|\leq\delta} \|R(\delta)f_\delta - y\| \leq C_p \delta^{\frac{2p}{2p+1}}, \tag{2.1}$$

where

$$C_p = \frac{1}{2\sqrt{b_p}} + c_p k_p b_p^p, \qquad c_p = p^p(1-p)^{1-p}, \qquad b_p := (4pc_p k_p)^{-\frac{2}{2p+1}}.$$

The above choice of $a(\delta)$ is optimal in the sense that the right-hand side of (2.2) (see below) is minimal for this choice of $a(\delta)$.

Proof. Let

$$\epsilon := \sup_{y\in\mathcal{K},\|Ay-f_\delta\|\leq\delta} \|T_a^{-1}A^* f_\delta - y\| := \sup \|T_a^{-1}A^* f_\delta - y\|.$$

Then, with $Ay = f$, one has

$$\epsilon \leq \sup \|T_a^{-1}A^*(f_\delta - f)\| + \sup \|T_a^{-1}A^*Ay - y\| := J_1 + J_2,$$

where

$$J_1 \leq \frac{\delta}{2\sqrt{a}},$$

and

$$J_2^2 \leq \sup\{a^2\|T_a^{-1}y\|^2\} \leq \sup \int_0^\infty \frac{a^2}{(s+a)^2} d(E_s y, y).$$

Thus,

$$J_2^2 \leq \left(\max_{s\geq 0} \frac{as^p}{a+s} \right)^2 k_p^2 = c_p^2 k_p^2 a^{2p},$$

because $\max_{s\geq 0} \frac{as^p}{a+s}$ is attained at $s = \frac{pa}{1-p}$ and is equal to $c_p a^p$, where

$$c_p := p^p(1-p)^{1-p}, \qquad k_p^2 := \sup_{y\in\mathcal{K}} \int_0^\infty s^{-2p} d(E_s y, y).$$

Consequently,

$$J_2 \leq c_p k_p a^p,$$

and

$$\epsilon \leq \frac{\delta}{2\sqrt{a}} + c_p k_p a^p. \tag{2.2}$$

Minimizing the right-hand side of (2.2) with respect to $a > 0$, one obtains inequality (2.1).

The minimizer of the right-hand side of (2.2) is

$$a = a(\delta) = b_p \delta^{\frac{2}{2p+1}}, \qquad b_p := (4pc_p k_p)^{-\frac{2}{2p+1}},$$

and the minimum of the right-hand side of (2.2) is $C_p\delta^{\frac{2p}{2p+1}}$, where

$$C_p := \frac{1}{2\sqrt{b_p}} + c_p k_p b_p^p. \tag{2.3}$$

Theorem 2.1 is proved. $\square$

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