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Highlights (for review)

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- A general failure rate model $r(t) = a + bt^{\theta-1}$ has been studied when a and b are not necessarily independent.
- Bayes' estimators of the parameters a and b, under symmetric and asymmetric loss functions, have been obtained.
- The monotonicity of the estimators have been studied through simulated as well as real data sets.

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Parameter Estimates of General Failure Rate Model: A Bayesian Approach

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Abstract

The failure rate function plays an important role in studying the lifetime distributions in reliability theory and life testing models. A study of the general failure rate model $r(t) = a + bt^{\theta-1}$, under squared error loss function, taking a and b independent exponential random variables, has been performed in the literature. In this article, we consider a and b not necessarily independent. The estimates of the parameters a and b under squared error loss, linex loss and entropy loss functions are obtained, and further examined using simulated and real data sets.

Key Words and Phrases: Farlie-Gumbell-Morgenstern family of distributions, IFR and DFR classes, Markov Chain Monte Carlo simulation, Type-II censoring.
AMS 2010 Classifications: Primary 62N02; Secondary 62F15.

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1 Introduction

Failure is an unavoidable phenomenon with technological products and systems. Reliability is a measure of failure uncertainty. The failure rate function plays an important role in reliability theory. The failure rate function and the distribution function are equivalent in the sense that knowing one, other can be uniquely determined by the relationship

$$\overline{F}_X(t) = 1 - F_X(t) = e^{-\int_0^t r_X(u)du},$$

or equivalently,

$$f_X(t) = r_X(t)e^{-\int_0^t r_X(u)du},$$

where f_X, F_X, $\overline{F}_X$ and r_X are probability density function, distribution function, survival function and failure rate function, respectively.

In reliability, lifetime distributions are often specified by choosing a particular failure rate function. The constant failure rate characterizes exponential distribution. The linear failure rate (LFR) distribution arises often in reliability literature and is motivated by its application to human survival data, see, for instance, Kodlin (1967) and Carbone et al. (1967). Various distributional properties and applications of the LFR distribution to life testing and reliability studies have been described by Sen (2000) and the references therein. Estimation of the unknown parameters in lifetime distributions of the individual components belonging to a multi-component system is an interesting problem in reliability analysis. These estimators may be extremely useful in some ways, since they reflect the component reliability after being assembled into an operational system (cf. Usher and Hodgson, 1988). Such estimators can be used under appropriate conditions to predict the reliability of new configurations of the components of the system. The estimation of the LFR model $r_X(t) = a + bt$ with a and b nonnegative constants, using classical method, has been studied in the literature by Bain (1974), Shaked (1978), Sen and Bhattacharyya (1995) among others.

Ashour and Youssef (1991) have investigated the Bayesian estimators for the parameters of the LFR model based on Type-II censored samples. However, their derivation of the marginal posterior distributions seems to be erroneous (cf. Lin et al., 2006). Pandey et al. (1993) have discussed the same problem with a simpler assumption on the joint prior distribution. The Bayesian estimation and prediction problems for the LFR model under general progressively Type-II censored samples are considered by Lin et al. (2006). The estimation of LFR distribution based on records and inter-record times has been discussed in Lin et al. (2003). In life testing and reliability studies, LFR distributions are useful in modelling the life length of a system or component when failures occur at random, and also from ageing or wear-out. But this failure rate model cannot describe other failure rate models except the linearly increasing one. To overcome this difficulty and to accommodate more varieties of failure rate models, Sarhan (1999) has analysed a more general failure rate model of the form

$$r_X(t) = a + bt^{\theta-1}, \tag{1.1}$$

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for some nonnegative constants a, b and θ. The model (1.1) generalizes exponential distribution (when $b = 0$ or $\theta = 1$, $r_X(t) = a$ or $a + b$, which gives rise to an exponential distribution), Rayleigh distribution ($a = 0$, $\theta = 2$), Weibull distribution ($a = 0$) and IFR distribution ($\theta = 2$). For $0 \leqslant \theta < 1$, (1.1) gives DFR (decreasing in failure rate) distribution, $\theta = 1$ gives CFR (constant failure rate) distribution and for $\theta > 1$, it gives IFR (increasing in failure rate) distributions. The general failure rate model given in (1.1) is a useful model to specify the lifetime distributions in reliability theory and life testing.

Sarhan (1999) has obtained the Bayes' estimators of (a, b) under the squared error loss function taking a and b independent exponential random variables with known parameters, and θ a known constant. But, a and b may not always be independent. Also, several common situations may arise when over-estimation is more serious than under-estimation and vice versa. In that case the loss function cannot be symmetric (e.g., squared error loss). Here we consider the case when a and b are not necessarily independent and the data are Type-II censored. Let θ be known and let the prior belief of the experimenter is that (a, b) have joint probability density function given by

$$h(a,b) = f(a)g(b) + \rho f(a)g(b)[1 - 2F(a)][1 - 2G(b)], \quad -1 \leqslant \rho \leqslant 1, \tag{1.2}$$

where f and g are the marginal probability density functions of a and b, and F and G are the distribution functions corresponding to f and g, respectively. The probability density function given in (1.2) is the well known *Farlie-Gumbel-Morgenstern* bivariate density (cf. Farlie, 1960). We assume that the value of ρ in (1.2) is known. If ρ is unknown but fixed, one may either estimate it from the data or get it from the past experience; if ρ is random, one may think of a prior distribution of ρ based on some past experience. One such case may be the distribution of ρ as uniform over $(-1, 1)$, which is considered in Remark 2.2. Clearly, when $\rho = 0$, we get the result of Sarhan (1999). It is to be mentioned here that parameter estimation of the general failure rate model (1.1) using masked data is considered in Sarhan (2004). Let us consider data on live births. As is well known, the failure rate at the time of birth (i.e. time t=0) is always very high, and decreases rapidly with time. If we consider the data during the time interval $[0, t_0)$ for some specified t_0 after which the life of the child could be considered more or less constant, then a Weibull distribution (with shape parameter less than unity) can be fitted to such data set on live births. But with the advancement of medical science the rate of still births is ever changing and can be considered to be a random variable. This may motivate one to model a failure time data as given in (1.1) with (a, b) random variables.

The paper is arranged as follows. In Section 2, we formulate the problem. The Bayes' estimators of the parameters a and b under different loss functions *viz.* squared error loss function, linex loss function and entropy loss function are obtained in Section 3. Section 4 is devoted to simulation results along with an application to a real data set. Finally, Section 5 concludes the present study along with possibility of further work.

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2 Formulation of the problem

In the present context, we use the following assumptions:

Assumption 2.1 *A fixed number of units, say n, are put on test and the data collected on the life of the units are Type-II censored. That is, the test is terminated once r (pre-specified number) units fail. The failure times of the first r units are known. Let these be* $t_1, t_2, \ldots, t_r$.

Assumption 2.2 *No two units fail together. This means,* $t_1 < t_2 < \ldots < t_r$.

Assumption 2.3 *The marginal distributions of a and b are exponential with known means* $1/\lambda_1$ *and* $1/\lambda_2$, *respectively.*

Assumption 2.4 *Failure times of the units are statistically independent.*

Remark 2.1 *In practice, if* λ_1 *and* λ_2 *are not known, they can either be estimated from the data or may be taken as random variables having certain distributions. Estimates obtained by this kind of analysis is known as 'Hierarchical Bayes estimates'.*

Under Assumption 2.3, (1.2) becomes

$$h(a,b) = \lambda_1\lambda_2 e^{-\lambda_1 a-\lambda_2 b} + \lambda_1\lambda_2\rho\left(2e^{-2\lambda_1 a} - e^{-\lambda_1 a}\right)\left(2e^{-2\lambda_2 b} - e^{-\lambda_2 b}\right).$$

Given a and b, the survival function corresponding to (1.1) is given by

$$\overline{F}_X(t|a,b) = \exp\left[-\left(at + bt^\theta/\theta\right)\right],\ t \geqslant 0. \tag{2.3}$$

If T denotes the random variable having failure rate function given by (1.1), then the joint probability density function of (a,b) and T is given by

$$\begin{aligned} g_1(t,a,b) &= f_X(t|a,b)h(a,b) \\ &= \left(a + bt^{\theta-1}\right)\exp\left[-\left(at + bt^\theta/\theta\right)\right]\left[\lambda_1\lambda_2 e^{-\lambda_1 a-\lambda_2 b}\right. \\ &\quad \left.+\lambda_1\lambda_2\rho\left(2e^{-2\lambda_1 a} - e^{-\lambda_1 a}\right)\left(2e^{-2\lambda_2 b} - e^{-\lambda_2 b}\right)\right], \end{aligned} \tag{2.4}$$

where $f_X(t|a,b)$ is the probability density function corresponding to (2.3). This now gives the marginal probability density function of T as

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$$
\begin{aligned}
f_T(t) &= \int_0^\infty \int_0^\infty g_1(t,a,b)da\ db \\
&= \lambda_1\lambda_2 \int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-\lambda_1 a-\lambda_2 b}da\ db \\
&\quad +\lambda_1\lambda_2\rho\left[4\int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-2\lambda_1 a-2\lambda_2 b}da\ db\right. \\
&\quad -2\int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-\lambda_1 a-2\lambda_2 b}da\ db \\
&\quad -2\int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-2\lambda_1 a-\lambda_2 b}da\ db \\
&\quad \left.+\int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-\lambda_1 a-\lambda_2 b}da\ db\right] \\
&= \lambda_1\lambda_2 I_1 + \lambda_1\lambda_2\rho\left[4I_2-2I_3-2I_4+I_1\right], \text{ say,}
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-\lambda_1 a-\lambda_2 b}da\ db \\
&= \frac{\theta}{(t+\lambda_1)^2(t^\theta+\theta\lambda_2)}+\frac{\theta^2 t^{\theta-1}}{(t+\lambda_1)(t^\theta+\theta\lambda_2)^2}, \\
I_2 &= \int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-2\lambda_1 a-2\lambda_2 b}da\ db \\
&= \frac{\theta}{(t+2\lambda_1)^2(t^\theta+2\theta\lambda_2)}+\frac{\theta^2 t^{\theta-1}}{(t+2\lambda_1)(t^\theta+2\theta\lambda_2)^2}, \\
I_3 &= \int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-\lambda_1 a-2\lambda_2 b}da\ db \\
&= \frac{\theta}{(t+\lambda_1)^2(t^\theta+2\theta\lambda_2)}+\frac{\theta^2 t^{\theta-1}}{(t+\lambda_1)(t^\theta+2\theta\lambda_2)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^\infty \int_0^\infty (a+bt^{\theta-1})e^{-(at+bt^\theta/\theta)}e^{-2\lambda_1 a-\lambda_2 b}da\ db \\
&= \frac{\theta}{(t+2\lambda_1)^2(t^\theta+\theta\lambda_2)}+\frac{\theta^2 t^{\theta-1}}{(t+2\lambda_1)(t^\theta+\theta\lambda_2)^2}.
\end{aligned}
$$

Thus, we hav

$$
\begin{aligned}
f_T(t) &= \frac{\lambda_1\lambda_2\theta\left\{(\theta+1)t^\theta+\theta\lambda_1 t^{\theta-1}+\theta\lambda_2\right\}}{(\lambda_1+t)^2(t^\theta+\theta\lambda_2)^2} \\
&\quad +\frac{\rho\lambda_1\lambda_2\theta t^\theta}{(2\lambda_2\theta+t^\theta)(\lambda_2\theta+t^\theta)(2\lambda_1+t)(\lambda_1+t)} \\
&\quad \left[\frac{t^2-2\lambda_1^2}{(2\lambda_1+t)(\lambda_1+t)}+\frac{\theta\left(t^{2\theta}-2\lambda_2^2\theta^2\right)}{(2\lambda_2\theta+t^\theta)(\lambda_2\theta+t^\theta)}\right]. \qquad (2.5)
\end{aligned}
$$

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Remark 2.2 *If it is assumed that ρ is unknown with a uniform prior distribution in $(-1,1)$, then the probability density function of T becomes $f_T(t)=\lambda_1\lambda_2 I_1$, which is same as (2.5) with $\rho=0$.*

3 Bayes' estimators of a and b

Once the values of $t_1,t_2,\ldots,t_r$ and that of n are known, the likelihood function can be written as (cf. Lawless, 1982)

$$L(\mathbf{t}|a,b)=\Pi_{i=1}^{r} f_X(t_i|a,b)\left[\overline{F}_X(t_r|a,b)\right]^{n-r}, \tag{3.6}$$

where $\mathbf{t}=(t_1,t_2,\ldots t_r)$. On using (2.3), (3.6) reduces to

$$\begin{aligned} L(\mathbf{t}|a,b) &= \left[\Pi_{i=1}^{r}\left(a+bt_i^{\theta-1}\right)\right] e^{-aS_1-bS_2} \\ &= \left[\sum_{j=0}^{r} a^{r-j}b^j M_j(\mathbf{t})\right] e^{-aS_1-\frac{b}{\theta}S_2}, \end{aligned}$$

where $S_1=\sum_{i=1}^{r} t_i+(n-r)t_r$, $S_2=\sum_{i=1}^{r} t_i^{\theta}+(n-r)t_r^{\theta}$ and $M_j(\mathbf{t})=\sum_{1\leqslant i_1<\ldots<i_j\leqslant r} t_{i_1}^{\theta-1}t_{i_2}^{\theta-1}\ldots t_{i_j}^{\theta-1}$, for $j=1,2,\ldots,r$ with $M_0(\mathbf{t})=1$. A similar kind of expression may be obtained in Sarhan (1999). Now, the joint distribution of $\mathbf{t}$, a and b is

$$\begin{aligned} f^*(a,b,\mathbf{t}) &= L(\mathbf{t}|a,b)h(a,b) \\ &= \sum_{j=0}^{r} a^{r-j}b^j M_j(\mathbf{t}) e^{-aS_1-\frac{b}{\theta}S_2}\left[\lambda_1\lambda_2 e^{-\lambda_1 a-\lambda_2 b}\right. \\ &\quad \left.+\lambda_1\lambda_2\rho\left\{4e^{-\lambda_1 a-\lambda_2 b}-2e^{-\lambda_1 a-2\lambda_2 b}-2e^{-2\lambda_1 a-\lambda_2 b}+e^{-\lambda_1 a-\lambda_2 b}\right\}\right] \\ &= \lambda_1\lambda_2\sum_{j=0}^{r} a^{r-j}b^j M_j(\mathbf{t})\left[e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)}\right. \\ &\quad +\rho\left\{4e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)}-2e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)}\right. \\ &\quad \left.\left.-2e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)}+e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)}\right\}\right], \end{aligned}$$

which gives the marginal density function of $\mathbf{t}$ as

$$\begin{aligned} f_1^*(\mathbf{t}) &= \int_0^\infty\int_0^\infty f^*(a,b,\mathbf{t})\,da\,db \\ &= \int_0^\infty\int_0^\infty \lambda_1\lambda_2\sum_{j=0}^{r} a^{r-j}b^j M_j(\mathbf{t})\left[e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)}\right. \\ &\quad +\rho\left\{4e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)}-2e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)}\right. \\ &\quad \left.\left.-2e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)}+e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)}\right\}\right]da\,db \\ &= \lambda_1\lambda_2\left[I_5+\rho\left\{4I_6-2I_7-2I_8+I_5\right\}\right], \text{ say,} \end{aligned}$$

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where

$$\begin{aligned} I_5 &= \int_0^\infty \int_0^\infty \sum_{j=0}^{r} a^{r-j} b^j M_j(\mathbf{t}) e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)} da\ db \\ &= \sum_{j=0}^{r} M_j(\mathbf{t}) \int_0^\infty a^{r-j} e^{-a(S_1+\lambda_1)} da \int_0^\infty b^j e^{-b(\frac{S_2}{\theta}+\lambda_2)} db \\ &= \sum_{j=0}^{r} M_j(\mathbf{t}) \frac{\Gamma(r-j+1)}{(S_1+\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(\frac{S_2}{\theta}+\lambda_2)^{j+1}} \\ &= \Phi(1,1,1,1), \end{aligned}$$

$$\begin{aligned} I_6 &= \int_0^\infty \int_0^\infty \sum_{j=0}^{r} a^{r-j} b^j M_j(\mathbf{t}) e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)} da\ db \\ &= \sum_{j=0}^{r} M_j(\mathbf{t}) \frac{\Gamma(r-j+1)}{(S_1+2\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(\frac{S_2}{\theta}+2\lambda_2)^{j+1}}, \\ &= \Phi(1,1,2,2), \end{aligned}$$

$$\begin{aligned} I_7 &= \int_0^\infty \int_0^\infty \sum_{j=0}^{r} a^{r-j} b^j M_j(\mathbf{t}) e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)} da\ db \\ &= \sum_{j=0}^{r} M_j(\mathbf{t}) \frac{\Gamma(r-j+1)}{(S_1+\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(\frac{S_2}{\theta}+2\lambda_2)^{j+1}} \\ &= \Phi(1,1,1,2) \end{aligned}$$

and

$$\begin{aligned} I_8 &= \int_0^\infty \int_0^\infty \sum_{j=0}^{r} a^{r-j} b^j M_j(\mathbf{t}) e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)} da\ db \\ &= \sum_{j=0}^{r} M_j(\mathbf{t}) \frac{\Gamma(r-j+1)}{(S_1+2\lambda_1)^{r-j+1}} \frac{\Gamma(j+1)}{(\frac{S_2}{\theta}+\lambda_2)^{j+1}}, \\ &= \Phi(1,1,2,1) \end{aligned}$$

so that

$$\begin{aligned} f_1^*(\mathbf{t}) &= \lambda_1\lambda_2\theta[\Phi(1,1,1,1) + \rho\{4\Phi(1,1,2,2) - 2\Phi(1,1,1,2) - 2\Phi(1,1,2,1) + \Phi(1,1,1,1)\}] \\ &= K^{-1}, \text{ say,} \end{aligned}$$

where

$$\Phi(l,m,p,q) = \sum_{j=0}^{r} M_j(\mathbf{t}) \frac{\Gamma(r-j+l)}{a_{1p}^{r-j+l}} . \frac{\Gamma(j+m)}{a_{2q}^{j+m}},$$

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$a_{1p} = S_1 + p\lambda_1$ and $a_{2q} = \frac{S_2}{\theta} + q\lambda_2$.

Hence the posterior joint probability density function of (a, b) is given by

$$\begin{aligned} f_2(a,b|\mathbf{t}) &= \frac{f^*(a,b,\mathbf{t})}{f_1^*(\mathbf{t})} \\ &= K\lambda_1\lambda_2 \sum_{j=0}^{r} a^{r-j} b^j M_j(\mathbf{t}) \Big[e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)} \\ &\quad +\rho \Big\{ 4e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)} - 2e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+2\lambda_2)} \\ &\quad - 2e^{-a(S_1+2\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)} + e^{-a(S_1+\lambda_1)-b(\frac{S_2}{\theta}+\lambda_2)} \Big\} \Big]. \end{aligned}$$

3.1 Estimation under squared error loss function

Let us consider a loss function of the form

$$L_1^*(\psi, \hat{\psi}) = k_1(a - \hat{a})^2 + k_2(b - \hat{b})^2, \tag{3.7}$$

where $k_1,\ k_2 > 0,\ \psi = (a, b)$, and $\hat{\psi} = (\hat{a}, \hat{b})$ is the estimator of ψ. It is well known that under the loss function of the form (3.7), Bayes' estimator of a parameter is its posterior mean. Thus, Bayes' estimators of a and b are given by

$$\hat{a}_{BS} = E_{f_2}(a) = \int_0^\infty \int_0^\infty a f_2(a,b|\mathbf{t}) da\ db$$

and

$$\hat{b}_{BS} = E_{f_2}(b) = \int_0^\infty \int_0^\infty b f_2(a,b|\mathbf{t}) da\ db.$$

These, after simplification, reduce respectively to

$$\hat{a}_{BS} = K\lambda_1\lambda_2[\Phi(2,1,1,1) + \rho\{4\Phi(2,1,2,2) - 2\Phi(2,1,1,2) - 2\Phi(2,1,2,1) + \Phi(2,1,1,1)\}]$$

and

$$\hat{b}_{BS} = K\lambda_1\lambda_2[\Phi(1,2,1,1) + \rho\{4\Phi(1,2,2,2) - 2\Phi(1,2,1,2) - 2\Phi(1,2,2,1) + \Phi(1,2,1,1)\}].$$

3.2 Estimation under linex loss function

Here we consider a loss function of the form

$$L_2(\psi, \hat{\psi}) = l_1 \left[e^{c_1(\hat{a}-a)} - c_1(\hat{a} - a) - 1 \right] + l_2 \left[e^{c_2(\hat{b}-b)} - c_2(\hat{b} - b) - 1 \right],$$

where $c_1,\ c_2,\ l_1,\ l_2$ are constants, and ψ and $\hat{\psi}$ are as defined earlier.

The Bayes' estimators of a and b are then given by

$$\begin{aligned} \hat{a}_{BL} &= -\frac{1}{c_1} \ln E_{f_2}\left(e^{-c_1 a}|\mathbf{t}\right) \\ &= -\frac{1}{c_1} \ln \left[\int_0^\infty \int_0^\infty e^{-c_1 a} f_2(a,b|\mathbf{t}) da\ db \right] \end{aligned}$$

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and

$$\begin{aligned}\hat{b}_{BL} &= -\frac{1}{c_2}\ln E_{f_2}\left(e^{-c_2 b}|\mathbf{t}\right)\\ &= -\frac{1}{c_2}\ln\left[\int_0^\infty\int_0^\infty e^{-c_2 b} f_2(a,b|\mathbf{t})da\ db\right],\end{aligned}$$

respectively. These, after simplification, reduce respectively to

$$\hat{a}_{BL} = -\frac{1}{c_1}\ln\{K\lambda_1\lambda_2[\Phi^*(1,1,1,1)+\rho\{4\Phi^*(1,1,2,2)-2\Phi^*(1,1,2,1)-2\Phi^*(1,1,1,2)+\Phi^*(1,1,1,1)\}]\}$$

and

$$\hat{b}_{BL} = -\frac{1}{c_2}\ln\{K\lambda_1\lambda_2[\Phi^{**}(1,1,1,1)+\rho\{4\Phi^{**}(1,1,2,2)-2\Phi^{**}(1,1,2,1)-2\Phi^{**}(1,1,1,2)+\Phi^{**}(1,1,1,1)\}]\},$$

where $\Phi^* = \Phi$ with a_{1p} replaced by $a^*_{1p} = c_1 + S_1 + p\lambda_1$ and $\Phi^{**} = \Phi$ with a_{2q} replaced by $a^*_{2q} = c_2 + \frac{S_2}{\theta} + q\lambda_2$.

Remark 3.1 *One may wonder how to get some idea of the values of c_1 and c_2. As mentioned in introduction that linex loss function is used when the over-estimation (under-estimation) is more serious than under-estimation (over-estimation). So, depending on the seriousness of the problem, one may decide on the values of c_1 and c_2 and the selection of the values is clearly case-dependent and should be handled with care based on either the past experience or the gravity of the problem due to over-estimation/under-estimation.*

3.3 Estimation under entropy loss function

Here we consider a loss function of the form

$$L^*_3(\psi,\hat{\psi}) = m_1\left[\frac{\hat{a}}{a} - \ln\frac{\hat{a}}{a} - 1\right] + m_2\left[\frac{\hat{b}}{b} - \ln\frac{\hat{b}}{b} - 1\right],$$

where m_1 and m_2 are constants, and ψ and $\hat{\psi}$ are as defined earlier.

The Bayes' estimators of a and b are then given by

$$\hat{a}_{BE} = \frac{1}{E_{f_2}\left(\frac{1}{a}|\mathbf{t}\right)}$$

and

$$\hat{b}_{BE} = \frac{1}{E_{f_2}\left(\frac{1}{b}|\mathbf{t}\right)}$$

respectively, which, after simplification, reduce respectively to

$$\hat{a}_{BE} = [K\lambda_1\lambda_2\{\Phi(0,1,1,1)+\rho(4\Phi(0,1,2,2)-2\Phi(0,1,2,1)-2\Phi(0,1,1,2)+\Phi(0,1,1,1))\}]^{-1}$$

and

$$\hat{b}_{BE} = [K\lambda_1\lambda_2\{\Phi(1,0,1,1)+\rho(4\Phi(1,0,2,2)-2\Phi(1,0,2,1)-2\Phi(1,0,1,2)+\Phi(1,0,1,1))\}]^{-1}.$$

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4 Simulation study and analysis of a real data set

Once the Bayes' estimators of the parameters a and b are obtained, it might be of interest to know the behaviour of the estimators of the parameters of the underlying model. This is done in this section through a simulation study as detailed below. We further illustrate the performance of the estimators by analysing a real data set.

4.1 Simulation study

To study the performance of the estimators, we have carried out a series of 1000 simulations. In the tables given in Appendix-A, the values of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$ are given for different values of n, r, r/n, θ, λ_1, λ_2, ρ, c_1 and c_2. Keeping the other parameters fixed, the values of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$, for different values of n, r, r/n, θ, ρ, λ_1 and λ_2, are given in Tables 2-10 respectively. Note that here we have simulated $\hat{a}_{BE}$ for more number of values of r to get a clear picture. Those values are reported at the bottom portion of Table 3 where n is taken to be 100. It is to be mentioned here that $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$ are independent of c_1 and c_2, whereas $\hat{a}_{BL}$ and $\hat{b}_{BL}$ vary with c_1 and c_2 respectively. Table 11 shows the values of $\hat{a}_{BL}$ and $\hat{b}_{BL}$ for different values of c_1 and c_2 respectively, taking the other parameters fixed.

Table 1 summarizes the findings of the simulation results. In this table, the conclusions about the monotonicity of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$ have been made on the basis of the tabulated values. Once the values show non-monotone behaviour, we have simulated for more number of values, and the specific non-monotonic behaviour has been noted in Table 1. Graphical representations of such estimators in support of Table 1 are also given in Appendix-B (see Figures 1-4).

To compute these values, following methodology has been adopted and the simulation works have been done using R-Software.

(1) Markov Chain Monte Carlo (MCMC) method (using Metropolis-Hastings algorithm) has been adopted to generate samples from $f_T(t)$, the target distribution, given in (2.5). We take proposal distribution as exponential with rate $(\lambda_1+\lambda_2)/2$. In the simulation process, we take 5000 burn-in observations. Then, a sample of size n is chosen and out of these observations, Type II censored sample $(t_1, t_2, \ldots, t_r)$ is collected censoring at the point r. Such repeated samples of size 1000 are collected and required calculations made. Between two consecutive samples, 100 observations are discarded to minimize the dependency.

(2) For $j = 1, 2, \ldots, r$, the values of $M_j(\mathbf{t})$ are calculated from the values of $t_1, t_2, \ldots, t_r$.

(3) Taking some fixed values of n, r, θ, λ_1, λ_2, ρ, c_1 and c_2, the values of $\Phi(l, m, p, q)$, $\Phi^*(l, m, p, q)$ and $\Phi^{**}(l, m, p, q)$ are calculated for different values of l, m, p and q, which give the values of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$.

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(4) Averages are calculated from these 1000 values of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$, which give their final values. Variances of these $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$ are also calculated and it is clear from Table 12 that all are decreasing functions of n, as expected.

(5) Empirical Bayes' risk for estimating a and b under different loss functions are also calculated using the definition

$$R_i^E = \frac{1}{1000}\sum_{j=1}^{1000} L_i^*(\overline{\psi}, \hat{\psi}_j), \text{ where } \overline{\psi} = \frac{1}{1000}\sum_{j=1}^{1000} \hat{\psi}_j,$$

$i = 1, 2, 3$. Here $i = 1$ gives R_{BS}^E, $i = 2$ gives R_{BL}^E, and $i = 3$ gives R_{BE}^E. Accordingly, we define R_{BS}^E, R_{BL}^E and R_{BE}^E as empirical Bayes' risk under squared error loss function, linex loss function and entropy loss function respectively. Without any loss of generality, we choose each of k_1, k_2, l_1, l_2 and m_1, m_2 as unity. Taking fixed values of n, r, θ, λ_1, λ_2, c_1 and c_2, as in Table 8, the values of R_{BS}^E, R_{BL}^E and R_{BE}^E are obtained under 1000 repetitions for different values of ρ and reported in Table 13. It is observed that R_{BS}^E is decreasing in ρ. It is also observed that the value of R_{BS}^E is less for any $\rho > 0$ than that when a and b are taken as independent, i.e., $\rho = 0$. Similarly, we can find some ρ for which R_{BL}^E and R_{BE}^E are less than the risk for $\rho = 0$, which exhibits the necessity of taking a and b not independent.

Table 1: Conclusions

Parameters	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
n	RC	increasing	increasing	increasing	RC	increasing
r	decreasing	decreasing	BT	decreasing	RC	decreasing
r/n	decreasing	decreasing	RC	decreasing	RC	decreasing
θ	RC	increasing	UBT	UBT	RC	increasing
ρ	decreasing	decreasing	RC	decreasing	RC	decreasing
λ_1	decreasing	increasing	decreasing	increasing	decreasing	BT
λ_2	BT	decreasing	increasing	decreasing	BT	decreasing
c_1	unchanged	unchanged	decreasing	unchanged	unchanged	unchanged
c_2	unchanged	unchanged	unchanged	decreasing	unchanged	unchanged

RC: Rollercoaster; BT: Bathtub shaped; UBT: Upside down bathtub shaped.

In the above table, the first column represents the different parameters when others are kept fixed. To be more specific, in the intersection on n and $\hat{b}_{BS}$ 'increasing' is written. This means that $\hat{b}_{BS}$ is increasing in n when other parameters are kept fixed. A similar interpretation follows for the other cells of the table.

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4.2 Analysis of a real data set

Heuristically, we obtain the Bayes' estimate of the parameters a and b when their marginals are exponential with known means, and the joint distribution is a member of the Farlie-Gumbell-Morgenstern bivariate family with known shape parameter ρ. However, in practice, if these quantities are not known then they can be estimated from the given data. Here we consider a real data example in which the means of a and b are unknown as well as the pivotal parameter ρ is also unknown. The data set is taken from McGilchrist and Aisbett (1991) which consists of the recurrence time of infection for kidney patients using portable dialysis equipment. The infection occurs at the point of infection of the catheter, and when it occurs, the catheter has to be removed, the infection cleared up and then the catheter reinserted. Recurrence times are times from infection until next infection. For each patient two such recurrence times are given, namely, X_1= first recurrence time (FRT) and X_2= second recurrence time (SRT). For illustrative purpose, we consider the infection data set of 23 patients which was analysed by Mirhosseini et al. (2015). The data set is reproduced below:

Patient No.	1	2	3	4	5	6	7	8	9	10	11	12
X_1 :	8	22	447	30	24	7	511	53	15	7	96	185
X_2 :	16	28	318	12	15	9	30	196	154	333	38	177
Patient No.	13	14	15	16	17	18	19	20	21	22	23	
X_1 :	292	152	13	12	152	34	2	130	27	152	119	
X_2 :	114	562	66	40	156	30	25	26	58	30	8	

Firstly, exponential distribution with mean $1/\lambda_i$ $(i = 1, 2)$ is fitted to X_1 and X_2 respectively. To verify whether the exponential distribution is well-fitted to the marginal data, we compute the Kolmogorov-Smirnov (K-S) distance of FRT and SRT data, respectively. The K-S distance between the fitted distribution and the empirical distribution of X_1 is 0.25243 and the associated p-value is 0.1149. Similarly, the K-S distance and the associated p-value for X_2 are obtained as 0.18481 and 0.4193, respectively. Therefore, the exponential distribution can be used to fit the marginals reasonably well for the above data set. We obtain $\widehat{\lambda}_1 = 0.009311741$, $\widehat{\lambda}_2 = 0.008611007$ with Pearson's correlation coefficient between X_1 and X_2 as 0.191. The values are consistent with the values reported in Mirhosseini et al. (2015). It is to be mentioned here that, based on the data, the estimated value of ρ is $\hat{\rho} = 0.764$. Now we obtain the estimates of a and b under the three loss functions. The Tables 14-15 provide the estimated values of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$ for different values of r/n and θ, respectively. It is observed that $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{b}_{BL}$ and $\hat{b}_{LE}$ are decreasing while $\hat{a}_{BL}$ and $\hat{a}_{BE}$ are non-monotone in view of r/n. Similarly, $\hat{b}_{BS}$ and $\hat{b}_{BE}$ are increasing while others are non-monotone with respect to θ.

Therefore, the monotonic behaviour (in terms of r/n and θ) of the estimators as observed for simulated data is validated by the kidney infection data set as well. To verify the specific non-monotone behaviour as noted in Table 1 for simulated data, we estimate a and b for more number of values of r/n and θ which are not reported here.

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5 Conclusions

In reliability theory and life testing, the general failure rate model given in (1.1) is a relevant model to specify the lifetime distributions. Sarhan (1999) studied Bayes' estimation of the parameters $(a,\ b)$ using Type-II censored data under the squared error loss function, when a and b are assumed to be independent random variables. But, several common situations may occur when a and b are not necessarily independent. Further, in-discriminate use of squared error loss function is not appropriate particularly in those cases where the losses are not symmetric. In this article we investigate the Bayes' estimators of (a, b) using both symmetric as well as asymmetric loss functions taking a and b not necessarily independent. In model (1.1) we have assumed (a, b) to have Farlie-Gumbell-Morgenstern bivariate density with known shape parameter (dependent) ρ, while the marginals are exponential with known means $1/\lambda_1$ and $1/\lambda_2$, respectively. We have chosen basically three types of loss functions - symmetric (squared error loss) and asymmetric (linex loss and entropy loss) for better comprehension of Bayesian analysis. It is worth mentioning that, from Bayesian perspective, choice of loss function is an essential part in the estimation and prediction problems. Here two asymmetric loss functions have been chosen for their distinctive features - linex loss function has different shapes depending on c_1 and c_2, whereas entropy loss function has no change in shape. Since sometimes under-estimation/over-estimation of unknown parameters is more serious compared to over-estimation/under-estimation, one may decide on the constant c_1 and c_2 in the loss function based on the experience and/or gravity of the problem. One may refer to Zellner (1986) for a related problem in dam theory, and Varian (1975) for a related real estate problem. It may be possible to choose a number of loss functions but these three are chosen as representatives. One may choose any other asymmetric loss function depending on whether the overestimate or underestimate of parameters of interest is more serious in nature.

We carried out a simulation study to illustrate the properties of the estimators by using simulated and real data sets. At first, we demonstrate the performance of the estimators using simulated data. To this aim, the estimated values of $\hat{a}_{BS}$, $\hat{b}_{BS}$, $\hat{a}_{BL}$, $\hat{b}_{BL}$, $\hat{a}_{BE}$ and $\hat{b}_{BE}$ are obtained for different values of n, r, r/n, θ, λ_1, λ_2, ρ, c_1 and c_2. The performance of the estimators is also assessed on the basis of their variances, and the simulation results show that the variances decrease with increasing sample size. We have further examined the empirical Bayes' risk under the three loss functions and the necessity of taking a and b not independent has also been verified. Later on, for illustrative purpose, one real data set is also analysed. The parameters a and b are estimated based on the 'Kidney Infection' data following Farlie-Gumbell-Morgenstern distribution with unknown parameter ρ as well as unknown means of the marginals. It has been observed that the monotonicity of the estimators (with respect to r/n and θ) as obtained for simulated data is also preserved for this data set.

Once the estimates of a and b are obtained, estimates of the reliability function and the

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failure rate function can be computed using the relationships

$$\hat{\overline{F}}_X(t) = \exp\left[-\left(\hat{a}t + \hat{b}t^{\theta}/\theta\right)\right]$$

and

$$\hat{r}(t) = \hat{a} + \hat{b}t^{\theta-1}.$$

It is to be mentioned here that $\hat{\overline{F}}_X(t)$ and $\hat{r}(t)$ as obtained are not the Bayes' estimators of $F_X(t)$ and $r(t)$. For different values of t, the estimates of these two reliability measures can be calculated.

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Appendix-A[1]

Table 2: Estimates for different values of n ($r = 15,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

n	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
20	2.428505	2.470713	0.9571143	0.3315961	1.841[illegible]96	[illegible].108504
30	2.47891	2.809811	0.9602245	0.3504222	1.869038	1.24303
50	2.570068	3.526479	1.031021	0.3728826	1.[illegible]08104	1.544343
80	2.702363	3.833807	1.106235	0.3769[illegible]9	2.144[illegible]07	1.685688
90	2.651358	3.922087	1.117268	0.3771751	2.[illegible]123916	1.721286
100	2.668780	3.99165	1.149213	0.37[illegible]9229	2.[illegible]61693	1.764691

Table 3: Estimates for different values of r ($n = 50,\ \theta = [illegible]5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

r	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{b}_{BE}$
10	2.889158	3.872732	1.046[illegible]90	0.3765202	1.92015
15	2.588836	3.44[illegible]6	1.0[illegible]5380	0.3690954	1.511899
20	2.476374	3.041744	1.073117	0.3585487	1.254549
25	2.303392	2.494415	1.114468	0.3398418	1.003384
35	2.153625	1.5750[illegible]9	1.260905	0.2871969	0.6200885
45	2.056847	0.870086[illegible]	1.416823	0.2131185	0.3430589

r	5	[illegible]	15	20	25
$\hat{a}_{BE}$	2.76[illegible]692	2.413989	2.11847	2.017851	2.010617
r	30	35	40	45	50
$\hat{a}_{BE}$	[illegible]963753	1.991297	1.970913	2.041148	2.019716
r	5[illegible]	60	65	70	75
$\hat{a}_{BE}$	2.00211	2.005149	2.060117	2.058064	2.024053
	80	85	90	95	100
$\hat{a}_{BE}$	2.047208	1.992579	1.953117	1.900956	1.787256

[1] The algorithms used in the simulation and some additional simulation results in support of Table 1 for estimates which show non-monotone behaviour are placed on the website: ftp://210.212.53.189:4777 (user id: ftpadmin, password: rgipt@7898)

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Table 4: Estimates for different values of r/n ($n = 20,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

r/n	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
0.2	4.119059	4.166242	0.9955019	0.3820005	2.9[illegible]6978	2.[illegible]63668
0.3	3.687986	3.936842	1.010329	0.3778412	2.71[illegible]3	[illegible].300496
0.5	2.785889	3.239922	0.9199196	0.3623994	2.027800	1.573956
0.7	2.533489	2.584523	0.9568928	0.3353231	1.[illegible]01728	1.172435
0.9	2.282652	1.81435	1.025075	0.28969[illegible]	1.80761[illegible]	0.8027084

Table 5: Estimates for different values of r/n ($n = 50,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

r/n	$\hat{a}_{BS}$	$\hat{b}_{BS}$	[illegible]	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
0.1	3.612631	4.362042	1.082.87	0.3831901	2.679921	2.756424
0.2	2.758742	3.876915	[illegible]	0.3772475	2.091837	1.894838
0.3	2.543543	3.437434	1.02[illegible]774	0.3689237	1.958643	1.502838
0.5	2.360520	2.565028	1.109332	0.339517	1.900828	1.022597
0.7	2.164543	1.61[illegible]554	[illegible].265770	0.2893923	1.871797	0.6363879
0.9	2.033174	0.8[illegible]26358	1.442860	0.2144492	1.885202	0.3263893

Table 6: Estimates for different values of r/n ($n = 100,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

r/n	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
0.[illegible]	[illegible].931569	4.273523	1.169112	0.3804953	2.346933	2.171221
0.2	2.55[illegible]382	3.688619	1.163707	0.3736742	2.082102	1.512133
0.3	2.[illegible]6782	2.983645	1.217592	0.3572311	2.002086	1.135464
0.5	2.263834	1.907634	1.406253	0.3155373	2.019610	0.697279
0.7	2.196331	1.016472	1.659570	0.2451656	2.080462	0.3648891
0.9	1.996416	0.3898868	1.760295	0.1496071	1.950117	0.1425884

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Table 7: Estimates for different values of θ ($n = 50,\ r = 15,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

θ	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
1.2	2.53878	2.698973	0.7468843	0.3592404	1.4[illegible]4473	[illegible]118551
1.3	2.498865	2.978176	0.8089464	0.3638382	1.619551	1.247953
1.5	2.612741	3.455287	1.043905	0.368266[illegible]	2.[illegible]2360[illegible]	1.511676
1.7	2.655687	3.853542	1.295282	0.3755059	2.2507[illegible]3	1.815087
2.0	2.617994	4.128254	1.586975	0.3744949	2.357008	2.203706
2.5	2.649079	4.335498	1.803262	0.3745662	2.[illegible]48854	2.838067
3.0	2.569748	4.389099	1.803354	0.3730[illegible]	[illegible].400356	3.306292
3.5	2.531871	4.358319	1.802254	0.371[illegible]263	2.370501	3.552629
4.0	2.546643	4.336425	1.814499	0.3[illegible]115	2.386205	3.736094
4.5	2.504744	4.322051	1.792483	0.[illegible]70061	2.347539	3.79589
5.0	2.474104	4.302419	1.776[illegible]	0.3694796	2.319187	3.846737
7.0	2.445197	4.281886	1.76103[illegible]	0.3689414	2.292343	3.894736
10.0	2.466696	4.282187	1.[illegible]5499	0.3689829	2.312524	3.904676

Table 8: Estimates for differen[illegible] values o[illegible] ($n = 50,\ r = 15,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ c_1 = 5,\ c_2 = 10$).

ρ	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
-1.0	3.[illegible]8291[illegible]	5.472054	1.067603	0.4478139	3.009227	2.636043
-0.8	2.993178	5.280714	1.038365	0.4372742	2.833177	2.469603
-0.5	2.87[illegible]378	4.904747	1.039639	0.4222705	2.763539	2.234952
-0.1	2.[illegible]47384	4.355944	1.021720	0.4007587	2.598687	1.927057
0	2.70[illegible]45	4.114485	1.057667	0.3925779	2.736950	1.81324
.1	2.687283	3.867259	1.045536	0.3841437	2.6258	1.699069
0.2	2.[illegible]80645	3.844324	1.056197	0.3826008	2.664627	1.689498
0.5	2.675599	3.460527	1.073620	0.3685548	2.652695	1.527531
0.8	2.556038	3.089788	1.117351	0.3578624	2.623406	1.379521
1.0	2.53029	2.902948	1.166945	0.3519428	2.651116	1.306628

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Table 9: Estimates for different values of λ_1 ($n = 50,\ r = 15,\ \theta = 1.5,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

λ_1	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
0.05	4.05366	3.646147	1.435573	0.3633854	3.147897	1.70214
0.1	2.443404	3.356118	0.9997084	0.368385	1.876178	1.464352
0.2	1.656511	3.069585	0.761078	0.3770261	1.136215	1.360706
0.4	1.156696	2.816409	0.5889041	0.3915007	0.711273	1.323718
0.7	0.893846	2.762379	0.486466	0.4145108	0.5056124	1.411568
1.0	0.7624006	2.843336	0.4256194	0.4462848	0.4065848	1.580224
1.5	0.6312949	3.169195	0.3620832	0.4954608	0.3158953	1.916876
2.0	0.535375	3.588683	0.3153288	0.5511547	0.2584508	2.311455

Table 10: Estimates for different values of λ_2 ($n = 50,\ r = 15,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

λ_2	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
0.05	3.292485	10.68008	0.9125832	0.5066623	2.151261	4.835694
0.1	2.926236	5.977912	0.9462215	0.4347056	2.061211	2.609884
0.2	2.596785	3.441812	1.032488	0.3696412	2.017182	1.512799
0.4	2.485133	1.934206	1.192755	0.3050227	2.088596	0.8874038
0.7	2.507843	1.179368	1.427955	0.2527033	2.221971	0.5899946
1.0	2.621771	0.8550314	1.557053	0.2211348	2.378278	0.4535783
1.5	2.847125	0.6358815	1.835657	0.1876845	2.630853	0.3536515
2.0	3.064059	0.4427358	2.00267	0.164858	2.849913	0.2923578

Table 11: Estimates for different values of c_1 and c_2 ($n = 50,\ r = 15,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5$).

c_1	5	10	20	40	70	100
$\hat{b}_{BL}$	1.043313	0.6219532	0.3624302	0.1955869	0.1200798	0.1998313
c_2	5	10	20	40	70	100
$\hat{b}_{BE}$	0.2280337	0.1650221	0.1116809	0.07163291	0.0484698	0.03745599

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Table 12: Estimation of variances for different values of n ($r = 15,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ \rho = 0.5,\ c_1 = 5,\ c_2 = 10$).

n	$v\hat{a}_{BS}$	$v\hat{b}_{BS}$	$v\hat{a}_{BL}$	$v\hat{b}_{BL}$	$v\hat{a}_{BE}$	$v\hat{b}_{BE}$
20	2.786046	2.801144	0.1687665	0.007764694	2.[illegible]8183	0.5741148
30	2.457465	2.236923	0.1677134	0.004167229	2.[illegible]51588	[illegible]4579437
50	1.888847	1.857697	0.1675985	0.002163288	1.79[illegible]55	0.3609909
80	1.476776	1.527406	0.1652423	0.00141158[illegible]	1.45[illegible]42	0.2941667
90	1.435796	1.490229	0.1620072	0.00125574[illegible]	[illegible].4245[illegible]9	0.2921165
100	1.398129	1.306582	0.1686950	0.001065[illegible]2	1.[illegible]988	0.2654663

Table 13: Estimation of Risks for different values of [illegible] ($r = 15,\ \theta = 1.5,\ \lambda_1 = 0.1,\ \lambda_2 = 0.2,\ c_1 = 5,\ c_2 = 10$).

ρ	R^E_{BS}	R^E_{BL}	R^E_{BE}
-1.0	5.3789[illegible]	37.[illegible]374	0.2712732
-0.8	5.012635	[illegible]08[illegible]1	0.2837661
-0.5	4.726565	[illegible]5.09191	0.2942592
-0.2	4.690[illegible]37	27.32544	0.3198501
-0.1	4.579706	[illegible]2.56944	0.3070936
0.0	[illegible]220078	123.9033	0.3094569
0.1	4.137[illegible]00	21.84179	0.2806128
0.2	4.097[illegible]72	48.70084	0.2945915
[illegible].5	3.62[illegible]3694	36.62481	0.3146239
0.8	3.[illegible]61719	22.28436	0.3165357
1.[illegible]	[illegible].206366	78.3708	0.3034506

Table 14: Estimat[illegible] for the 'Kidney Infection' data for different values of r/n ($n = 23,\ \theta = 1.5,\ \lambda_1 = 0.009[illegible]2,\ \lambda_2 = 0.008611,\ \rho = 0.764,\ c_1 = 5,\ c_2 = 10$).

r/n	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
[illegible].22	[illegible].21449	82.77009	1.549067	0.6753049	23.17771	52.04123
0.44	23.3819	65.48655	1.509363	0.654526	18.06407	32.80404
0.65	18.58674	48.12896	1.557377	0.6160147	16.09956	21.56096
[illegible].87	17.16175	33.82378	1.873447	0.5630958	13.39893	14.85073
[illegible].96	15.79212	27.0784	2.02349	0.524512	13.6263	11.77327

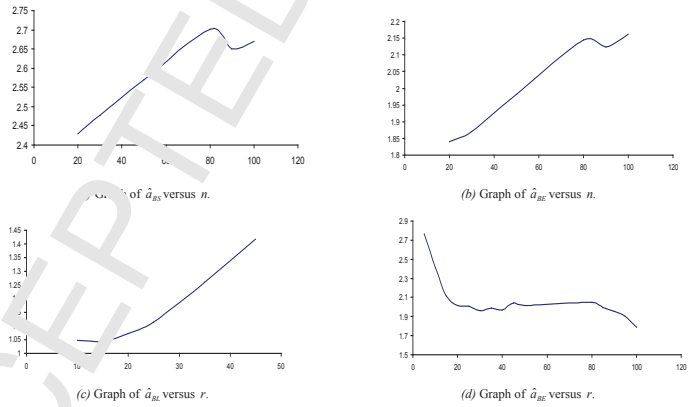
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Table 15: Estimates for the 'Kidney Infection' data for different values of θ ($n = 23,\ r = 15,\ \lambda_1 = 0.009312,\ \lambda_2 = 0.008611,\ \rho = 0.764,\ c_1 = 5, c_2 = 10$).

θ	$\hat{a}_{BS}$	$\hat{b}_{BS}$	$\hat{a}_{BL}$	$\hat{b}_{BL}$	$\hat{a}_{BE}$	$\hat{b}_{BE}$
1.2	22.61644	37.54734	1.155764	0.6051412	12.89[illegible]21	[illegible].14056
1.3	21.95622	43.1277	1.269933	0.6138438	14.36[illegible]83	18.60451
1.5	19.00188	48.34794	1.573204	0.6171839	[illegible].8913	21.7655
1.7	17.20973	55.83206	1.97864	0.6296[illegible]4	14.[illegible]29	26.70984
2.0	15.85381	59.67962	2.877858	0.6318392	[illegible].33879	32.78902
2.5	13.33766	64.93261	3.872856	0.641[illegible]459	12.34355	42.10837
3.0	11.94322	69.13956	4.078722	0.6[illegible]78208	11.1271	48.93992
5.0	9.958795	77.14183	3.860621	0.660503[illegible]	9.325226	62.52561
10.0	8.593999	78.70673	3.558454	[illegible].6586046	8.055473	68.26047

Appendix-B

(Graphical representations of estimates showing non-monotone behavior in support of Table 1)



(a) Graph of $\hat{a}_{BS}$ versus n.

(b) Graph of $\hat{a}_{BE}$ versus n.

(c) Graph of $\hat{a}_{BL}$ versus r.

(d) Graph of $\hat{a}_{BE}$ versus r.

Figure 1: Graphical representations of some estimators versus n and r.

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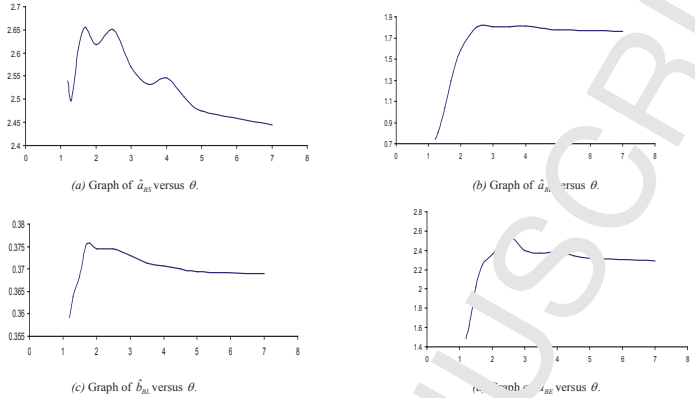


Figure 2: Graphical representations of some estimators versus θ.

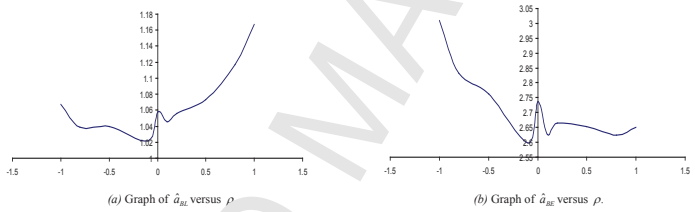


Figure 3: Graphical representation of $\hat{a}_{BL}$ and $\hat{a}_{BE}$ versus ρ.

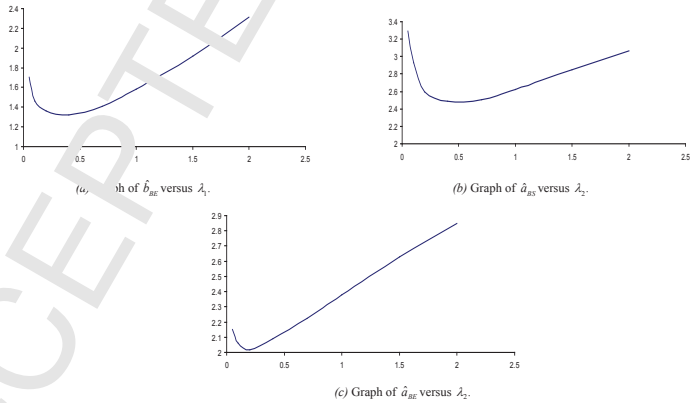


Figure 4: Graphical representations of some estimators versus λ_1 and λ_2.