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A note on an upper and a lower bound on sines between eigenspaces for regular Hermitian matrix pairs

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The main results of the paper are an upper and a lower bound for the Frobenius norm of the matrix $\sin\Theta$, of the sines of the canonical angles between unperturbed and perturbed eigenspaces of a regular generalized Hermitian eigenvalue problem $Ax = \lambda Bx$ where A and B are Hermitian $n\times n$ matrices, under a feasible non-Hermitian perturbation. As one application of the obtained bounds we present the corresponding upper and the lower bounds for eigenspaces of a matrix pair (A, B) obtained by a linearization of regular quadratic eigenvalue problem $\left(\lambda^2 M + \lambda D + K\right) u = 0$, where M is positive definite and D and K are semidefinite.

We also apply obtained upper and lower bounds to the important problem which considers the influence of adding a damping on mechanical systems. The new results show that for certain additional damping the upper bound can be too pessimistic, but the lower bound can reflect a behaviour of considered eigenspaces properly. The obtained results have been illustrated with several numerical examples.

1. Introduction

This paper considers a generalized Hermitian eigenvalue problem

$$Ax = \lambda Bx\,,$$

where A and B are $n \times n$ Hermitian matrices under a perturbation, $\widetilde{A} = A + \delta A$, $\widetilde{B} = B + \delta B$, and δA and δB do not need to be Hermitian, but structured and small enough to ensure that properties like regularity or semi-simple eigenvalues hold for both unperturbed pair (A, B) as well as for perturbed pair $(\widetilde{A}, \widetilde{B})$.

One of the contributions of this paper is a novel upper bound (we allow non-Hermitian perturbation) for the Frobenius norm of the $\sin\Theta$ matrix, where Θ denotes the matrix of the canonical angles between unperturbed eigenspace $\mathcal{X}_1 = \mathrm{span}(X_1)$ and corresponding perturbed eigenspace $\widetilde{\mathcal{X}}_1 = \mathrm{span}(\widetilde{X}_1)$, and the columns of X_1 and $\widetilde{X}_1$ are eigenvectors of the pairs (A, B) and $(\widetilde{A}, \widetilde{B})$, respectively.

But from our point of view, the most important contribution of the paper is the novel lower bound for the Frobenius norm of the $\sin\Theta$ matrix.

The problem of comparing unperturbed and perturbed eigenspaces $\mathcal{X}_1$ and $\widetilde{\mathcal{X}}_1$ has been widely studied and there is a vast amount of literature which contains different kinds of upper bounds on the sines of canonical angles between

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unperturbed and perturbed eigenspace. Here we will list just some of them which we find important and at the same time we apologize to the authors of omitted ones.

The classical results about $\sin\Theta$ can be found in [1,2] as well as in [3]. All these results belong to the so-called standard or absolute perturbation theory. On the other hand, the so-called relative perturbation results on $\sin\Theta$ can be found in [4–8] or [9].

But up to our knowledge, all these papers contain upper bounds, and there are not many lower bounds at all. One of the results considering the lower (as well as the upper) bounds on subspaces can be found in a recent paper of Cai and Zhang in [10]. There authors present upper and the lower bounds for the spectral and Frobenius $\sin\Theta$ distances between singular spaces. The presented lower bounds are within a constant factor of the corresponding upper bounds, which shows that obtained bounds are rate-optimal.

As we have mentioned, our bounds hold for regular pairs (A, B) and $(\widetilde{A}, \widetilde{B})$, that is for the pairs whose matrices can be singular, but they cannot share the same null-subspace. The behaviour of eigenspace of the singular pairs (A, B) under perturbation is, up to our knowledge, open problem. On the other hand the characterization of the singular pairs is very important and challenging problem. To confirm this we would like to point on the recent paper [11] where authors treat the problem of determining the nearest singular matrix pencil to a given regular matrix pencil $A + \lambda B$ (the distance to singularity), and the references therein, especially [12–14].

The new bounds from this paper are motivated by the results from [15] where one can find a lower and an upper bound for the Frobenius norm of the sines of canonical angles between unperturbed and perturbed eigenspaces of a simultaneously diagonalizable quadratic eigenvalue problem.

The new upper bound for $\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F$ has the same structure as most upper bounds for sines between angles of eigenspaces spanned by the columns of X_1 and $\widetilde{X}_1$, respectively. With some additional assumption on the matrix B (for more details see Section 2) it has the following form

$$\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \leq \frac{\|\widetilde{X}_1^{\dagger}\|\,\|X_2\|Err(\widetilde{X}_1, X_1)}{\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)},$$

where $Err(\widetilde{X}_1, X_1)$, roughly speaking, corresponding to residual like in the $\sin\Theta$ theorem from [1], and $\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)$ is a weighted gap function (for precise definition see (18)).

On the other hand the new lower bound has the following form

$$\frac{Err(\widetilde{X}_1, X_1)}{\|\widetilde{X}_1\|\,\|X_2^{\dagger}\|\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)} \leq \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F,$$

where $\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)$ denotes a weighted complete linkage clustering L_∞ distance (for precise definition see (17)).

In the second part of the paper, we will show how the new lower and upper bounds can be used to study the behaviour of the eigenspaces of the regular matrix pair (A, B) obtained by linearization of a regular quadratic eigenvalue problem (which represents the vibrating mechanical system)

$$\left(\lambda^2 M + \lambda D + K\right) u = 0,$$

where M is positive definite and D and K are Hermitian semidefinite. Especially we will present result about influence of the external or an additional damping on the mechanical system under consideration.

With a few simple examples we will illustrate a difference between the upper and lower bounds as well as a possible usage of them for measuring the influence of damping.

The paper is organized as follows. In Section 2, we present the main results of the paper, that is upper and a lower bound for $\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F$. In Section 3 we present an application of the bounds from the previous section on the eigenspaces of a pair obtained by a linearization of regular quadratic eigenvalue problem $\left(\lambda^2 M + \lambda D + K\right) u = 0$. The important problem of the damping influence on mechanical systems is studied in Section 3.1 and the obtained results are illustrated by a small numerical example.

2. Main result

In this section we will present our main result. Thus, let

$$Ax = \lambda Bx, \tag{1}$$

be the regular generalized Hermitian eigenvalue problem where A and B are $n \times n$ Hermitian matrices.

For the purpose of simplifying presentation of our main ideas we will assume that all eigenvalues of the pair (A, B) are semi-simple and that there exist non-singular matrices $X = \begin{bmatrix} x_1, \ldots, x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1, \ldots, y_n \end{bmatrix}$, whose columns are the right and left eigenvectors, respectively associated with λ_i, $i = 1, \ldots, n$.

Note, that both A and B can be singular, but due to the regularity assumption they cannot share the same null-subspace.

As it was mentioned in the introduction, the main result of this section has been motivated by the results from [15], on upper and lower bounds for the Frobenius norm of the sine of angles between unperturbed and perturbed subspaces of a simultaneously diagnosable Hermitian matrix triple M, D and K.

Before we state our main results, we will need the following considerations.

Let the columns of the matrices X and Y be the right and the left eigenvectors of (1) respectively. We can write

$$BX\Lambda - AX = 0\,, \qquad \Lambda Y^*B - Y^*A = 0\,, \tag{2}$$

$$\widetilde{B}\widetilde{X}\widetilde{\Lambda} - \widetilde{A}\widetilde{X} = 0\,, \qquad \widetilde{\Lambda}\widetilde{Y}^*\widetilde{B} - \widetilde{Y}^*\widetilde{A} = 0\,. \tag{3}$$

Note, that the regularity assumption implies that for $i = 1, \ldots, n$ either $y_i^*Ax_i \neq 0$ or $y_i^*Bx_i \neq 0$, that is $y_i^*Ax_i$ and $y_i^*Bx_i$ cannot be equal to zero in the same time.

Let us emphasize, that from (2) it follows that for i for which $y_i^*Bx_i \neq 0$, it holds

$$\lambda_i = \frac{y_i^*Ax_i}{y_i^*Bx_i}\,, \tag{4}$$

which includes zero eigenvalues, and the rest of them are infinite (those for which $y_i^*Bx_i = 0$ and $y_i^*Ax_i \neq 0$).

Given k, $1 \leq k < n$ let us decompose X and Y in a following way

$$X = \begin{bmatrix}X_1, X_2\end{bmatrix}, \; X_1 = \begin{bmatrix}x_1, \ldots, x_k\end{bmatrix}, \; X_2 = \begin{bmatrix}x_{k+1}, \ldots, x_n\end{bmatrix}, \tag{5}$$

$$Y = \begin{bmatrix}Y_1, Y_2\end{bmatrix}, \; Y_1 = \begin{bmatrix}y_1, \ldots, y_k\end{bmatrix}, \; Y_2 = \begin{bmatrix}y_{k+1}, \ldots, y_n\end{bmatrix}, \tag{6}$$

$$\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2), \; \Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_k), \; \Lambda_2 = \mathrm{diag}(\lambda_{k+1}, \ldots, \lambda_n), \tag{7}$$

and denote by $\mathcal{X}_1 = \mathrm{span}(X_1)$ and $\mathcal{X}_2 = \mathrm{span}(X_2)$, and similarly for their perturbed quantities.

The fact that we consider perturbation of the subspace spanned by the first k eigenvectors does not effect on generality of our results. Indeed, one can apply the same consideration on any eigenspace using corresponding permutation of the columns of the matrices X and Y (the right and the left eigenvectors) and associated eigenvalues.

In what follows, we present an expression for the norm of the $\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)$ matrix of the canonical angles between unperturbed and perturbed eigenspaces $\mathcal{X}_1$ and $\widetilde{\mathcal{X}}_1$, respectively, based on the similar consideration as in [15].

For that purpose let $X = QR$, be the QR decomposition of the matrix X, then

$$S \doteq X^{-*} = QR^{-*}\,,$$

and similarly hold for perturbed quantities.

Obviously the columns of X and S span the same subspaces, thus we denote

$$\begin{bmatrix}X_1, X_2\end{bmatrix} = [Q_1, Q_2]\begin{bmatrix}R_{11} & R_{12}\\ 0 & R_{22}\end{bmatrix}, \quad \begin{bmatrix}S_1, S_2\end{bmatrix} = [Q_1, Q_2]\begin{bmatrix}R_{11}^{-*} & 0\\ -R_{22}^{-*}R_{12}^{*}R_{11}^{-*} & R_{22}^{-*}\end{bmatrix}. \tag{8}$$

Recall, that $\mathcal{X}_1$ denotes the subspace spanned by the columns of X_1, and similarly for perturbed subspaces.

Note that $S_2 = Q_2R_{22}^{-*}$, and $\widetilde{X}_1 = \widetilde{Q}_1\widetilde{R}_{11}$. Using the standard result [3, Exercise 6. pg.36] we can write:

$$\sigma_{\min}(R_{22}^{-1})\sigma_{\min}(\widetilde{R}_{11})\|Q_2^*\widetilde{Q}_1\|_F \leq \|S_2^*\widetilde{X}_1\|_F \leq \sigma_{\max}(R_{22}^{-1})\sigma_{\max}(\widetilde{R}_{11})\|Q_2^*\widetilde{Q}_1\|_F,$$

which, using the fact that $\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F = \|Q_2^*\widetilde{Q}_1\|_F$, implies

$$\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \leq \frac{1}{\sigma_{\min}(R_{22}^{-1})\sigma_{\min}(\widetilde{R}_{11})}\|S_2^*\widetilde{X}_1\|_F\,, \tag{9}$$

$$\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \geq \frac{1}{\sigma_{\max}(R_{22}^{-1})\sigma_{\max}(\widetilde{R}_{11})}\|S_2^*\widetilde{X}_1\|_F\,. \tag{10}$$

Now we can state our main result.

Theorem 2.1. *Let (A, B) be a Hermitian regular pair. Let X, Y be non-singular matrices decomposed as in* (5) and (6), *whose columns are the right and left eigenvectors of the regular eigenvalue problem* (1), *respectively. Let δA and δB be perturbations such that $\widetilde{A} \doteq A + \delta A$ and $\widetilde{B} \doteq B + \delta B$. Further, let $X_1 = \mathrm{span}(X_1)$ and $\widetilde{\mathcal{X}}_1 = \mathrm{span}(\widetilde{X}_1)$ denote eigensubspces spanned by the columns of X_1 and $\widetilde{X}_1$, respectively. For δA and δB small enough the following inequalities hold*

$$\frac{\frac{Err(\widetilde{X}_1, X_1)}{\sigma_{\max}(R_{22}^{-1})\sigma_{\max}(\widetilde{R}_{11})}}{\max\limits_{\substack{1\leq j\leq k\\ k+1\leq i\leq n}} |(y_i^*Bx_i)\widetilde{\lambda}_j - (y_i^*Ax_i)|} \leq \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \leq \frac{\frac{Err(\widetilde{X}_1, X_1)}{\sigma_{\min}(R_{22}^{-1})\sigma_{\min}(\widetilde{R}_{11})}}{\min\limits_{\substack{1\leq j\leq k\\ k+1\leq i\leq n}} |(y_i^*Bx_i)\widetilde{\lambda}_j - (y_i^*Ax_i)|}\,, \tag{11}$$

where $\widetilde{R}_{11}$ and R_{22}^{-1} are defined as in (8) *and*

$$Err(\widetilde{X}_1, X_1) = \sqrt{\sum_{i=k+1}^{n}\sum_{j=1}^{k}\left|y_i^*\delta B\widetilde{x}_j\widetilde{\lambda}_j - y_i^*\delta A\widetilde{x}_j\right|^2}\,. \tag{12}$$

Proof. Note that (3) can be written as

$$B\widetilde{X}\widetilde{\Lambda} - A\widetilde{X} = -\delta B\widetilde{X}\widetilde{\Lambda} + \delta A\widetilde{X}\,.$$

Now, if we multiply the above equality with Y^* from the left we can write

$$Y^*BXX^{-1}\widetilde{X}\widetilde{\Lambda} - Y^*AXX^{-1}\widetilde{X} = -Y^*\delta B\widetilde{X}\widetilde{\Lambda} + Y^*\delta A\widetilde{X}\,. \tag{13}$$

Using the fact that Y^*AX and Y^*BX are diagonal matrices whose diagonal elements are $y_i^*Ax_i$ and $y_i^*Bx_i$, respectively and interpreting (13) entriewise, we get for all $i = k+1, \dots, n$ and $j = 1, \dots, k$

$$(y_i^*Bx_i)(X^{-1}\widetilde{X})_{ij}\widetilde{\lambda}_j - (y_i^*Ax_i)(X^{-1}\widetilde{X})_{ij} = -y_i^*\delta B\widetilde{x}_j\widetilde{\lambda}_j + y_i^*\delta A\widetilde{x}_j\,,$$

or

$$(X^{-1}\widetilde{X})_{ij} = \frac{-y_i^*\delta B\widetilde{x}_j\widetilde{\lambda}_j + y_i^*\delta A\widetilde{x}_j}{(y_i^*Bx_i)\widetilde{\lambda}_j - (y_i^*Ax_i)}\,, \quad i = k+1, \dots, n, \quad j = 1, \dots, k\,. \tag{14}$$

Recall, if δA and δB are small enough and structured such that perturbed pair $(\widetilde{A}, \widetilde{B})$ is regular with all semi-simple eigenvalues, then $y_i^*Ax_i$ and $y_i^*Bx_i$ cannot be simultaneously equal to zero, for all $i = 1, \dots, n$ and all entries from (14) are well define.

Now, using (8) one can see that for $i = k+1, \dots, n$ and $j = 1, \dots, k$

$$(S_2^*\widetilde{X}_1)_{i-k,j} = (X^{-1}\widetilde{X})_{ij}\,.$$

Thus, we have the following equality

$$\|S_2^*\widetilde{X}_1\|_F^2 = \sum_{i=k+1}^{n}\sum_{j=1}^{k} \frac{\left|y_i^*\delta B\widetilde{x}_j\widetilde{\lambda}_j - y_i^*\delta A\widetilde{x}_j\right|^2}{\left|(y_i^*Bx_i)\widetilde{\lambda}_j - (y_i^*Ax_i)\right|^2}. \tag{15}$$

Now (11) simply follows from (9) and (10) and taking max and min of denominator from (15). $\square$

Note, that from (12) it follows that

$$Err(\widetilde{X}_1, X_1) = \left\|Y_2^*\delta B\widetilde{X}_1\widetilde{\Lambda}_1 - Y_2^*\delta A\widetilde{X}_1\right\|_F\,. \tag{16}$$

In what follows we will present slightly weaker upper and lower bounds for $\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F$ with additional assumption that B is non-singular. But before that we introduce some additional notations.

If X_2 and Y_2 from (5) and (6) satisfy that $y_i^*Bx_i \neq 0$, for all $i = k+1, \dots, n$, one can observe that the denominators on the both sides of the bounds (11) look like certain gaps. To be more precise, using (4) we can write

$$\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2) = \max_{\substack{1\le j\le k\\ k+1\le i\le n}} |(y_i^*Bx_i)(\widetilde{\lambda}_j - \lambda_i)|\,, \tag{17}$$

$$\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2) = \min_{\substack{1\le j\le k\\ k+1\le i\le n}} |(y_i^*Bx_i)(\widetilde{\lambda}_j - \lambda_i)|\,. \tag{18}$$

The $\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)$ from (18) denotes a weighted absolute gap between eigenvalues (diagonal entries) of $\widetilde{\Lambda}_1$ and Λ_2 while $\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)$ from (17) denotes a weighted complete linkage clustering L_∞ distance (see for example [16]).

Further, note that from (8) it follows

$$\sigma_{\max}(\widetilde{R}_{11}) = \|\widetilde{X}_1\|, \quad \sigma_{\max}(R_{22}^{-1}) \le \|X_2^\dagger\|\,, \tag{19}$$

$$\frac{1}{\sigma_{\min}(\widetilde{R}_{11})} = \|\widetilde{X}_1^\dagger\|, \quad \frac{1}{\sigma_{\min}(R_{22}^{-1})} = \|R_{22}\| \le \|X_2\|\,. \tag{20}$$

Thus, the following corollary contains slightly weaker upper and lower bounds for $\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F$ which involves $\|\widetilde{X}_1\|$, $\|X_2^\dagger\|$, $\|\widetilde{X}_1^\dagger\|$ and $\|X_2\|$ and $\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)$ and $\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)$.

Corollary 2.1. *Let (A, B) be a Hermitian regular pair. Let X, Y and δA and δB be as in Theorem 2.1. Further, let $\mathcal{X}_1 = \mathrm{span}(X_1)$ and $\widetilde{\mathcal{X}}_1 = \mathrm{span}(\widetilde{X}_1)$ denote eigensubspces spanned by the columns of X_1 and $\widetilde{X}_1$, respectively. Then the following inequalities hold*

$$\frac{Err(\widetilde{X}_1, X_1)}{\|\widetilde{X}_1\|\|X_2^\dagger\|\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)} \le \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \le \frac{\|\widetilde{X}_1^\dagger\|\|X_2\|Err(\widetilde{X}_1, X_1)}{\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)}\,, \tag{21}$$

where $Err(\widetilde{X}_1, X_1)$ is defined as in (12).

Proof. The inequalities (21) simply follow from (11) upon using (19)–(20) and (17)–(18). $\square$

2.1. Illustrative example

Let us demonstrate some interesting properties of the lower and upper bounds (21).

It is well-known that the upper bound in (21) (and similar bounds which one can find in [3] or [17]) can be too pessimistic, especially if the matrix is close to diagonal and the gaps are small. For illustration, let $B = \widetilde{B} = I$ and let A be diagonal

$$A = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_4 \end{bmatrix}, \qquad \widetilde{A} = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda+\psi & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_4 \end{bmatrix},$$

where $\widetilde{A}$ is its perturbation. We are interested in a change of eigenspace which corresponds with eigenvalue λ, that is $\Lambda_1 = \mathrm{diag}(\lambda, \lambda)$ and $\Lambda_2 = \mathrm{diag}(\mu_1, \mu_2, \mu_3, \mu_4)$, while for perturbed spectrum one has $\widetilde{\Lambda}_2 = \mathrm{diag}(\mu_1, \mu_2, \lambda+\psi, \mu_4)$. The considered subspaces are spanned by the columns of matrices

$$X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{X}_1 = \begin{bmatrix} \mathcal{O}(1) & 0 \\ 0 & 1 \\ 0 & 0 \\ \mathcal{O}(\varepsilon) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and Y_2 contains last four columns of identity matrix. Thus,

$$\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F = \mathcal{O}(\varepsilon).$$

Note that if $\lambda \approx \mu_3$ and if $|\psi| \ll 1$, from (18) it follows that $\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2) = |\psi|$. This implies that the upper bound from (21) can be too pessimistic even for small ε.

On the other hand, from (18) it follows that without losing any generality if we assume that $\mu_3 > \lambda$ is such that $\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2) = |\lambda - \mu_3|$, which in case $|\lambda - \mu_3| \geq 1$ implies a plausible lower bound.

To make example more clear, let use $\varepsilon = 0.001$, $\psi = 10^{-6}$, and

$$\lambda = 2, \mu_1 = 3, \mu_2 = 4, \mu_3 = 2.001, \mu_4 = 10,$$
$$\widetilde{\lambda} = 2, \widetilde{\mu}_1 = 3, \widetilde{\mu}_2 = 4, \widetilde{\mu}_3 = 2.000001, \widetilde{\mu}_4 = 10.$$

Then we have

$$\|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F = 5 \cdot 10^{-4} = \frac{1}{2}\|Y_2^* \delta A \widetilde{X}_1\|_F,$$

where $Err(\widetilde{X}_1, X_1) = \|Y_2^* \delta A \widetilde{X}_1\|_F$. The lower and upper bound from (21) are

$$1.25 \cdot 10^{-4} \leq \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \leq 10^3.$$

2.2. Zero and infinite eigenvalues

Although, Theorem 2.1 holds even if $y_i^* B x_i = 0$, for $i = k+1, \ldots, n$, it cannot be applied on general regular pairs (A, B), and $(\widetilde{A}, \widetilde{B})$, since

$$\widetilde{\lambda}_i = \frac{\widetilde{y}_i^* \widetilde{A} \widetilde{x}_i}{\widetilde{y}_i^* \widetilde{B} \widetilde{x}_i},$$

holds only if $\widetilde{y}_i^* \widetilde{B} \widetilde{x}_i \neq 0$.

However, instead of considering regular generalized Hermitian eigenvalue problem (1), we can consider corresponding eigenvalue problem in the cross-product form

$$\beta A x = \alpha B x, \tag{22}$$

which allows to replace eigenvalues λ_i (including zero or infinite eigenvalues) with corresponding pairs $(\alpha_i, \beta_i) \neq (0, 0)$, for $i = 1, \ldots, n$.

This approach allows to treat zero as well as infinite eigenvalues simultaneously.

Let (A, B), and $(\widetilde{A}, \widetilde{B})$ be regular pairs. Using notation as in section [3, VI.1.2] we denote

$$\alpha_i = y_i^* A x_i, \; \beta_i = y_i^* B x_i, \; \widetilde{\alpha}_i = \widetilde{y}_i^* \widetilde{A} \widetilde{x}_i, \; \widetilde{\beta}_i = \widetilde{y}_i^* \widetilde{B} \widetilde{x}_i, \qquad i = 1, \ldots, n. \tag{23}$$

Now we can generalized the result from Theorem 2.1 on the Hermitian regular pair with zero or infinite eigenvalues. The following corollary holds.

Corollary 2.2. *Let (A, B) be a Hermitian regular pair. Let X, Y and δA and δB be as in Theorem 2.1. Further, let $X_1 = \operatorname{span}(X_1)$ and $\widetilde{\mathcal{X}}_1 = \operatorname{span}(\widetilde{X}_1)$ denote eigenspaces spanned by the columns of X_1 and $\widetilde{X}_1$, respectively. Then the following inequalities hold*

$$\frac{Er_{cp}(\widetilde{X}_1, X_1)}{\|\widetilde{X}_1\| \|X_2^\dagger\| \max_{\substack{1\le j\le k\\ k+1\le i\le n}} |\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|} \le \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \le \frac{\|\widetilde{X}_1^\dagger\| \|X_2\| Er_{cp}(\widetilde{X}_1, X_1)}{\min_{\substack{1\le j\le k\\ k+1\le i\le n}} |\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|}, \tag{24}$$

where

$$Er_{cp}(\widetilde{X}_1, X_1) = \sqrt{\sum_{i=k+1}^{n}\sum_{j=1}^{k} \left|\left(y_i^* \delta B \widetilde{x}_j\right)\widetilde{\alpha}_j - \left(y_i^* \delta A \widetilde{x}_j\right)\widetilde{\beta}_j\right|^2}\,.$$

Proof. The proof of both inequalities from (24) follows using the similar consideration as in the proof of Theorem 2.1 and Corollary 2.1.

Indeed, from (14) and (23) we have

$$(X^{-1}\widetilde{X})_{ij} = \frac{-y_i^*\delta B\widetilde{x}_j\widetilde{\alpha}_j + y_i^*\delta A\widetilde{x}_j\widetilde{\beta}_j}{\beta_i\widetilde{\alpha}_j - \alpha_i\widetilde{\beta}_j}, \quad i = k+1, \ldots, n, \quad j = 1, \ldots, k\,.$$

This implies

$$\|S_2^*\widetilde{X}_1\|_F^2 = \sum_{i=k+1}^{n}\sum_{j=1}^{k} \frac{\left|y_i^*\delta B\widetilde{x}_j\widetilde{\alpha}_j - y_i^*\delta A\widetilde{x}_j\widetilde{\beta}_j\right|^2}{\left|\beta_i\widetilde{\alpha}_j - \alpha_i\widetilde{\beta}_j\right|^2}.$$

Now (24) simply follows from (9)–(10) and (19)–(20) and taking max and min of denominator from the above equality. □

We would like to emphasize two interesting properties regarding the upper and the lower bounds (24).

The first is that both bounds from (24) hold for all pairs $(\alpha_i, \beta_i) \neq (0, 0)$, $i = 1, \ldots, n$, which include zero as well as infinite eigenvalues of the regular Hermitian pair (A, B) and its corresponding perturbed matrix pair.

The second property refers to the upper bound from (24). Note that in the denominator we have the minimum over all $1 \le j \le k$ and $k+1 \le i \le n$ of

$$|\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|\,.$$

The above expression is equal to the numerator of the chordal distance $\chi\left((\alpha_i, \beta_i), (\widetilde{\alpha}_j, \widetilde{\beta}_j)\right)$ (see [3, Definition 1.20 VI.1.4]) between pairs (α_i, β_i) and $(\widetilde{\alpha}_j, \widetilde{\beta}_j)$, which is defined as

$$\chi\left((\alpha_i, \beta_i), (\widetilde{\alpha}_j, \widetilde{\beta}_j)\right) = \frac{|\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|}{\sqrt{|\alpha_i|^2 + |\beta_i|^2}\sqrt{|\widetilde{\alpha}_i|^2 + |\widetilde{\beta}_i|^2}}\,.$$

Further note, if $\alpha_i \ll 1$ and $\beta_i = \mathcal{O}(1)$ or $\beta_i \ll 1$ and $\alpha_i = \mathcal{O}(1)$ and if similarly holds for perturbed $\widetilde{\alpha}_i$ and $\widetilde{\beta}_i$, then

$$\chi\left((\alpha_i, \beta_i), (\widetilde{\alpha}_j, \widetilde{\beta}_j)\right) \approx |\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|\,.$$

On the other hand if $\alpha_i \gg 1$ or $\beta_i \gg 1$ and similarly $\widetilde{\alpha}_i \gg 1$ or $\widetilde{\beta}_i \gg 1$, then

$$\chi\left((\alpha_i, \beta_i), (\widetilde{\alpha}_j, \widetilde{\beta}_j)\right) \le |\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|\,,$$

which makes the quantity from denominator in (24) (for large α or β) more suitable then possible usage of chordal distance, like

$$\min_{\substack{1\le j\le k\\ k+1\le i\le n}} \chi\left((\alpha_i, \beta_i), (\widetilde{\alpha}_j, \widetilde{\beta}_j)\right)\,.$$

Remark 2.1. As the general drawback of the gap functions

$$\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2) = \max_{\substack{1\le j\le k\\ k+1\le i\le n}} |(y_i^*Bx_i)(\widetilde{\lambda}_j - \lambda_i)|\,,$$

$$\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2) = \min_{\substack{1\le j\le k\\ k+1\le i\le n}} |(y_i^*Bx_i)(\widetilde{\lambda}_j - \lambda_i)|\,.$$

or

$$d_{cp} = \max_{\substack{1\le j\le k\\ k+1\le i\le n}} |\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|\,, g_{cp} = \min_{\substack{1\le j\le k\\ k+1\le i\le n}} |\widetilde{\alpha}_j\beta_i - \alpha_i\widetilde{\beta}_j|$$

can be considered as the fact that unperturbed and perturbed quantities appear simultaneously.

One way to avoid this drawback is to estimate $\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)$ and $\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)$ or d_{cp} and g_{cp} using appropriate eigenvalue perturbation bounds. For example the perturbation bound for eigenvalues of the regular Hermitian pair under non-Hermitian perturbation among other references one can find in [18, Section 4.].

3. Application to mechanical systems

The lower and upper bound (11) are given in the most general form, where both A and B can be singular. But as has been shown in [19,20] if A and B are obtained from a linearization of a regular quadratic eigenvalue problem, then the zero subspaces can be efficiently deflated. In this section we will consider the following QEP (obtained from the vibrating mechanical system)

$$\left(\lambda^2 M + \lambda D + K\right) u = 0\,, \tag{25}$$

where M is positive definite, D and $K = GG^*$ are semidefinite, with G having full column rank. The corresponding perturbed QEP is

$$\left(\widetilde{\lambda}^2(M + \delta M) + \widetilde{\lambda}(D + \delta D) + (K + \delta K)\right)\widetilde{u} = 0\,. \tag{26}$$

After deflation, the linearization of (25) is given by (see [19])

$$\left(\lambda \begin{bmatrix} -M & \\ & I \end{bmatrix} - \begin{bmatrix} D & G \\ G^* & 0 \end{bmatrix}\right) x = 0,$$

where all matrices are of appropriate dimensions depending on the dimension of deflated zero subspace. Here the eigenvector x is defined as

$$x = \begin{bmatrix} \lambda u \\ G^* u \end{bmatrix}.$$

In that sense, the corresponding perturbed eigenvalue problem can be written as

$$\left(\widetilde{\lambda} \begin{bmatrix} -(M + \delta M) & \\ & I \end{bmatrix} - \begin{bmatrix} D + \delta D & G + \delta G \\ (G + \delta G)^* & 0 \end{bmatrix}\right) \widetilde{x} = 0.$$

Let

$$A = \begin{bmatrix} D & G \\ G^* & 0 \end{bmatrix}, B = \begin{bmatrix} -M & \\ & I \end{bmatrix}, \delta A = \begin{bmatrix} \delta D & \delta G \\ \delta G^* & 0 \end{bmatrix}, \delta B = \begin{bmatrix} -\delta M & \\ & 0 \end{bmatrix}, \tag{27}$$

be the matrices from the linearization. Further, let n be the dimension of the matrix A (and B), that is let n be the sum of the dimension of M and the rank of G (more details can be found in [19]). Further, let $X = \begin{bmatrix} x_1, \ldots, x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1, \ldots, y_n \end{bmatrix}$, be non-singular matrices whose ith column is the right and left eigenvector, respectively, associated with the eigenvalue λ_i.

Then, we have the following corollary.

Corollary 3.1. *Let (A, B) and $(A + \delta A, B + \delta B)$ be Hermitian regular pairs as in (27). Let X, Y be non-singular matrices decomposed as in (5) and (6), whose columns are the right and the left eigenvectors of the QEP (25), respectively. Further, let $\mathcal{X}_1 = \mathrm{span}(X_1)$ and $\widetilde{\mathcal{X}}_1 = \mathrm{span}(\widetilde{X}_1)$ denote the eigenspaces spanned by the columns of X_1 and $\widetilde{X}_1$, respectively. Then the following bounds hold*

$$\frac{Err(\widetilde{X}_1, X_1)}{\|\widetilde{X}_1\|\|X_2^\dagger\|\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)} \leq \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \leq \frac{\|\widetilde{X}_1^\dagger\|\|X_2\|Err(\widetilde{X}_1, X_1)}{\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)}, \tag{28}$$

where $X = \begin{bmatrix} X_1, X_2 \end{bmatrix}$,

$$Err(\widetilde{X}_1, X_1) = \sqrt{\sum_{i=k+1}^{n}\sum_{j=1}^{k}\left|-\widetilde{\lambda}_j^2 (y_i)_1^* \delta M \widetilde{u}_j + \widetilde{\lambda}_j((y_i)_1^* \delta D \widetilde{u}_j + (y_i)_2^* \delta G^* \widetilde{u}_j) + (y_i)_1^* \delta G \widetilde{G}^* \widetilde{u}_j\right|^2},$$

$$\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2) = \max_{\substack{1 \leq j \leq k \\ k+1 \leq i \leq n}} |\lambda_i (y_i)_1^* M u_i - (y_i)_2^* G^* u_i||(\widetilde{\lambda}_j - \lambda_i)|\,, \tag{29}$$

$$\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2) = \min_{\substack{1 \leq j \leq k \\ k+1 \leq i \leq n}} |\lambda_i (y_i)_1^* M u_i - (y_i)_2^* G^* u_i||(\widetilde{\lambda}_j - \lambda_i)|\,, \tag{30}$$

and

$$y_i = \begin{bmatrix} (y_i)_1 \\ (y_i)_2 \end{bmatrix}, i = 1, \ldots, n\,.$$

Proof. Note that from (11) it follows that for the proof of (28) we need to plug in δA and δB from (27) into $Err(\widetilde{X}_1, X_1)$ from (11)

$$Err(\widetilde{X}_1, X_1) = \sqrt{\sum_{i=k+1}^{n}\sum_{j=1}^{k}\left|y_i^*\delta B\widetilde{x}_j\widetilde{\lambda}_j + y_i^*\delta A\widetilde{x}_j\right|^2}\,, \tag{31}$$

and calculate the corresponding $\mathrm{d}_\infty(\widetilde{\Lambda}_1, \Lambda_2)$ and $\mathrm{gap}_w(\widetilde{\Lambda}_1, \Lambda_2)$ from (11).

For that purpose, recall (27) and the structure of eigenvectors

$$x_i = \begin{bmatrix}\lambda_i u_i\\ G^* u_i\end{bmatrix}, \quad y_i = \begin{bmatrix}(y_i)_1\\ (y_i)_2\end{bmatrix}.$$

We have

$$y_i^* B x_i = -\lambda_i (y_i)_1^* M u_i + (y_i)_2^* G^* u_i.$$

Note that for the considered problem $y_i^* B x_i \neq 0$ for all i (since we have deflated infinite eigenvalues as was explained in [19]), which implies

$$\lambda_i = \frac{y_i^* A x_i}{y_i^* B x_i}\,.$$

Thus, both denominators in (11) can be written as

$$\max_{\substack{1\le j\le k\\ k+1\le i\le n}} |(y_i^* B x_i)\widetilde{\lambda}_j - (y_i^* A x_i)| = \max_{\substack{1\le j\le k\\ k+1\le i\le n}} |\lambda_i (y_i)_1^* M u_i - (y_i)_2^* G^* u_i||(\widetilde{\lambda}_j - \lambda_i)|\,, \tag{32}$$

$$\min_{\substack{1\le j\le k\\ k+1\le i\le n}} |(y_i^* B x_i)\widetilde{\lambda}_j - (y_i^* A x_i)| = \min_{\substack{1\le j\le k\\ k+1\le i\le n}} |\lambda_i (y_i)_1^* M u_i - (y_i)_2^* G^* u_i||(\widetilde{\lambda}_j - \lambda_i)|\,. \tag{33}$$

Further, since δA and δB from (27) are defined as

$$\delta A = \begin{bmatrix}\delta D & \delta G\\ \delta G^* & 0\end{bmatrix}, \quad \delta B = \begin{bmatrix}-\delta M & \\ & 0\end{bmatrix},$$

by a simple multiplication with the corresponding eigenvectors y_i and $\widetilde{x}_j$ we get

$$y_i^*\delta B\widetilde{x}_j = -\widetilde{\lambda}_j (y_i)_1^*\delta M\widetilde{u}_j$$
$$y_i^*\delta A\widetilde{x}_j = \widetilde{\lambda}_j\left((y_i)_1^*\delta D\widetilde{u}_j + (y_i)_2^*\delta G^*\widetilde{u}_j\right) + (y_i)_1^*\delta G\widetilde{G}^*\widetilde{u}_j\,.$$

Now, the lower and upper bound (28) simply follows by plugging the above equalities and (32) and (33) into (31) and using (29) and (30). □

3.1. Damping influence on the mechanical system

One of the important problems in studding a mechanical system is damping influence. This means that we are interested in the properties of the mechanical system given the matrix triple M, D and K only under the changing of the damping matrix D. Here we will assume that all three matrices M, D and K are real symmetric and M is positive definite. In that case, we have

$$\delta M = 0 \quad \delta K = 0,\ \text{that is } \delta G = 0\,.$$

If we introduce two matrices whose columns are eigenvectors u_i of the quadratic eigenvalue problems (25) and (26) as

$$\widetilde{U}_{(1,k)} = \left[\widetilde{u}_1, \ldots, \widetilde{u}_k\right] \text{ and } U_{(k+1,n)} = \left[u_{k+1}, \ldots, u_n\right], \tag{34}$$

respectively, then we have the following corollary.

Corollary 3.2. *Let (A, B), $X_1 = \mathrm{span}(X_1)$ and $\widetilde{\mathcal{X}}_1 = \mathrm{span}(\widetilde{X}_1)$ be as in Corollary 3.1 and $X = \left[X_1, X_2\right]$. If we consider the perturbation only of the damping matrix D, that is $\delta M = 0$, $\delta G = 0$ and $\delta D \neq 0$, then the following bounds hold*

$$\frac{Err_D(\widetilde{X}_1, X_1)}{\|\widetilde{X}_1\|\|X_2^\dagger\|\mathrm{rd}_\infty(\widetilde{\Lambda}_1, \Lambda_2)} \le \|\sin\Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F \le \frac{\|\widetilde{X}_1^\dagger\|\|X_2\|Err_D(\widetilde{X}_1, X_1)}{\mathrm{rg}_w(\widetilde{\Lambda}_1, \Lambda_2)}\,, \tag{35}$$

where

$$Err_D(\widetilde{X}_1, X_1) = \sqrt{\sum_{i=k+1}^{n}\sum_{j=1}^{k}\left|u_i^*\delta D\widetilde{u}_j\right|^2} = \|U_{(k+1,n)}^*\delta D\widetilde{U}_{(1,k)}\|_F\,,$$

and $\widetilde{U}_{(1,k)}$ *and* $U_{(k+1,n)}$ *are defined as in* (34) *and*

$$\mathrm{rd}_{\infty}(\widetilde{\Lambda}_1, \Lambda_2) = \max_{\substack{1\le j\le k\\ k+1\le i\le n}} \frac{|\lambda_i^2 u_i^* M u_i - u_i^* G G^* u_i||(\widetilde{\lambda}_j - \lambda_i)|}{|\lambda_i \widetilde{\lambda}_j|}, \tag{36}$$

$$\mathrm{rg}_{w}(\widetilde{\Lambda}_1, \Lambda_2) = \min_{\substack{1\le j\le k\\ k+1\le i\le n}} \frac{|\lambda_i^2 u_i^* M u_i - u_i^* G G^* u_i||(\widetilde{\lambda}_j - \lambda_i)|}{|\lambda_i \widetilde{\lambda}_j|}. \tag{37}$$

Proof. For the proof we will need to calculate the sum from (15). Using the facts that $\delta M = \delta K = 0$, that is $\delta G = 0$ and that all unperturbed eigenvalues and eigenvectors come in conjugate pairs (due to assumption that all three matrices M, D and K are real symmetric) we have

$$x_i = \begin{bmatrix} \lambda_i u_i \\ G^* u_i \end{bmatrix}, \qquad y_i \doteq \begin{bmatrix} (y_i)_1 \\ (y_i)_2 \end{bmatrix} = \begin{bmatrix} \overline{\lambda}_i \overline{u}_i \\ G^* \overline{u}_i \end{bmatrix}.$$

All these imply

$$\delta A = \begin{bmatrix} \delta D & 0 \\ 0 & 0 \end{bmatrix}, \delta B = \begin{bmatrix} 0 & \\ & 0 \end{bmatrix},$$

and

$$\frac{\left|y_i^* \delta B \widetilde{x}_j \widetilde{\lambda}_j - y_i^* \delta A \widetilde{x}_j\right|}{|(y_i^* B x_i)\widetilde{\lambda}_j - (y_i^* A x_i)|} = \frac{\left|(y_i)_1^* \delta D (\widetilde{x}_j)_1\right|}{|(y_i^* B x_i)\widetilde{\lambda}_j - (y_i^* A x_i)|} = \frac{\left|(y_i)_1^* \delta D (\widetilde{x}_j)_1\right|}{|(y_i^* B x_i)(\widetilde{\lambda}_j - \lambda_i)|},$$

where the last equality follows from the same consideration as one above (32).

Furthermore, since

$$y_i^* B x_i = -\lambda_i^2 u_i^* M u_i + u_i^* G G^* u_i,$$

it holds that

$$\frac{\left|y_i^* \delta B \widetilde{x}_j \widetilde{\lambda}_j - y_i^* \delta A \widetilde{x}_j\right|}{|(y_i^* B x_i)\widetilde{\lambda}_j - (y_i^* A x_i)|} = \frac{\left|\widetilde{\lambda}_j \lambda_i u_i^* \delta D \widetilde{u}_j\right|}{|(\lambda_i^2 u_i^* M u_i - u_i^* G G^* u_i)(\widetilde{\lambda}_j - \lambda_i)|}.$$

Now, (35) follows by inserting the above equality into (15) and proceeding as in Theorem 2.1. □

3.1.1. Illustrative example II

Similarly as in the previous illustrative example we will demonstrate some interesting properties of the lower and upper bound (35) on a mechanical system defined by three matrices M, C_0 and K under the influence of the external damping C_v.

Thus, let M, C_0, K and C_v be 5×5 real symmetric matrices defined as

$$M = \begin{bmatrix} 1.0275 & -0.0223 & 0.0374 & 0.0341 & 0.0696 \\ -0.0223 & 4.0380 & -0.0226 & 0.0323 & 0.0700 \\ 0.0374 & -0.0226 & 8.9661 & 0.0367 & -0.0434 \\ 0.0341 & 0.0323 & 0.0367 & 16.0137 & -0.0750 \\ 0.0696 & 0.0700 & -0.0434 & -0.0750 & 25.0787 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.0299 & -0.0208 & 0.0935 & 0.0341 & 0.7464 \\ -0.0208 & 4.0399 & -0.1542 & 0.0333 & 0.5840 \\ 0.0935 & -0.1542 & 35.8618 & -0.1277 & -0.3171 \\ 0.0341 & 0.0333 & -0.1277 & 16.0308 & 0.0143 \\ 0.7464 & 0.5840 & -0.3171 & 0.0143 & 225.7028 \end{bmatrix},$$

$$C_v = \begin{bmatrix} 1.0271 & -0.0224 & 0.0184 & -0.0038 & -0.0147 \\ -0.0224 & 4.0376 & 0.0210 & 0.0033 & 0.0059 \\ 0.0184 & 0.0210 & 0.0004 & -0.0001 & -0.0002 \\ -0.0038 & 0.0033 & -0.0001 & 0 & 0.0001 \\ -0.0147 & 0.0059 & -0.0002 & 0.0001 & 0.0002 \end{bmatrix}.$$

The damping C_0 is the classical Rayleigh damping defined as $C_0 = 0.0001 \cdot M + 0.0002 \cdot K$ (for more details see [21] or [22]). For this case the spectrum of the QEP $\left(\lambda^2 M + \lambda C_0 + K\right) u = 0$ is

$$\lambda_1 = -1.4598 \cdot 10^{-4} - i, \lambda_2 = -1.4598 \cdot 10^{-4} + i, \lambda_3 = -1.4851 \cdot 10^{-4} - i,$$

$$\lambda_4 = -1.4851 \cdot 10^{-4} + i, \lambda_5 = -1.4987 \cdot 10^{-4} - 1.0005i,$$

Table 1
Results of error bound (35) for $\mathcal{X}_1 = \text{span}([x_1, x_2])$ and $\delta D = vC_v$.

(35)	$v = 0.01$	$v = 0.1$	$v = 1$	$v = 2$
lower bound	$9.1836 \cdot 10^{-4}$	0.00918	0.0911	0.143
$\|\sin \Theta(\mathcal{X}_1, \widetilde{\mathcal{X}}_1)\|_F$	$7.3217 \cdot 10^{-3}$	0.0327	0.3311	0.950
upper bound	59.925	599.53	6091	74811

Table 2
Results of error bound (35) for $\mathcal{Z}_1 = \text{span}([x_8, x_{10}])$ and $\delta D = vC_v$.

(35)	$v = 0.01$	$v = 0.1$	$v = 1$	$v = 2$
lower bound	$3.2 \cdot 10^{-8}$	$3.2 \cdot 10^{-7}$	$2.76 \cdot 10^{-6}$	$4.27 \cdot 10^{-6}$
$\|\sin \Theta(\mathcal{Z}_1, \widetilde{\mathcal{Z}}_1)\|_F$	$1.96 \cdot 10^{-7}$	$1.95 \cdot 10^{-6}$	$1.65 \cdot 10^{-5}$	$2.44 \cdot 10^{-5}$
upper bound	$1.02 \cdot 10^{-6}$	$1.02 \cdot 10^{-5}$	$8.8 \cdot 10^{-5}$	$1.37 \cdot 10^{-4}$

$$\lambda_6 = -1.4987 \cdot 10^{-4} + 1.0005i\,, \lambda_7 = -2.5098 \cdot 10^{-4} - 2i\,, \lambda_8 = -2.5098 \cdot 10^{-4} + 2i\,,$$
$$\lambda_9 = -3.5093 \cdot 10^{-4} - 3i\,, \lambda_{10} = -3.5093 \cdot 10^{-4} + 3i\,.$$

The matrices, M and K are chosen such that the pair (M, K) has the spectrum $\{1, 1, 4, 1.001, 9\}$. Let assume that one is interested in adding the external damping, which in our notation is given as a perturbation, $\delta D \doteq vC_v$ where $v \geq 0$ is a real parameter.

In the first part of this illustrative example we consider the external damping (the perturbation) with the strongest influence on the first two eigenvalues $\{1, 1\}$. The best choice for such C_v will be to use a projector on the subspace spanned by the first two columns of the matrix $S = U^{-*}$, where U is a nonsingular eigenvector matrix of the pair (M, K).

That is, let $S = \left[s_1, s_2, \ldots, s_5\right] = U^{-*}$, where U is a nonsingular eigenvector matrix of the pair (M, K). Then, external damping C_v is defined as

$$C_v = s_1 s_1^* + s_2 s_2^*\,.$$

Thus, for the illustration of the influence of external damping matrix on perturbed QEP

$$\left(\lambda^2 M + \lambda(C_0 + vC_v) + K\right) u = 0\,,$$

we will vary the parameter v such that $v \in \{0.01, 0.1, 1, 2\}$.

The first table shows results obtained from (35) for the angles between

$$\mathcal{X}_1 = \text{span}\left(\left[x_1, x_2\right]\right)\,,\ \text{and}\ \widetilde{\mathcal{X}}_1 = \text{span}\left(\left[\widetilde{x}_1, \widetilde{x}_2\right]\right)\,,$$

where

$$x_i = \begin{bmatrix} \lambda_i u_i \\ G^* u_i \end{bmatrix}, \qquad \widetilde{x}_i = \begin{bmatrix} \widetilde{\lambda}_i \widetilde{u}_i \\ \widetilde{G}^* \widetilde{u}_i \end{bmatrix}, i = 1, 2.$$

The matrix G is defined by $K = GG^*$.

For different values of parameter v we have the results in Table 1.

In the second part of this example we consider results obtained from bound (35) for the angles between

$$\mathcal{Z}_1 = \text{span}\left(\left[x_8, x_{10}\right]\right)\,,\ \text{and}\ \widetilde{\mathcal{Z}}_1 = \text{span}\left(\left[\widetilde{x}_8, \widetilde{x}_{10}\right]\right)\,,$$

and for the same external damping C_v. The obtain results are shown in Table 2.

Note, as one can see from Table 1, the external damping C_v has the strongest influence on the eigenspace spanned by the first two eigenvectors x_1 and x_2. Table 1 illustrates two important properties of the bounds in (35). First, it shows that sometimes the upper bound can be too pessimistic, but the lower bound reflects a behaviour of considered eigenspaces and, second the both bounds depend almost linearly on parameter v of the external damping ($\delta D = vC_v$) as long as the spectra, $\widetilde{\Lambda}_1$ and Λ_2, remain separated enough.

On the other hand Table 2 illustrates that the external damping used in this example has almost no influence on some parts of the spectrum, especially it leaves the eigenspace spanned by the eigenvectors x_8 and x_{10} almost unchanged. This is caused by the property of the external damping C_v, whose null-space is "close" to the subspace spanned by the last two eigenvectors u_4 and u_5 of the pair (M, K), that is $\|C_v u_i\|_2 = \mathcal{O}(10^{-5})$, for $i = 4, 5$.

4. Conclusion

The main results of the paper are upper and lower bounds for the Frobenius norm of the $\sin \Theta$ matrix of the canonical angles between unperturbed and perturbed eigenspaces of a regular generalized Hermitian eigenvalue problem $Ax = \lambda Bx$, where A and B are $n \times n$ Hermitian matrices, under a feasible non-Hermitian perturbation. We present an application of obtained bounds on the eigenspaces of a pair (A, B) obtained by a linearization of regular quadratic eigenvalue problem $\left(\lambda^2 M + \lambda D + K\right) u = 0$.

In Section 3.1 we apply our upper and lower bounds to the important problem of measuring the influence of damping on mechanical systems. The obtained results show that sometimes the upper bound can be too pessimistic, but the lower bound reflects a behaviour of considered eigenspaces properly as well as that both bounds depend linearly on a damping parameter as long as considered spectra remain separated enough.

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