



S-orthogonality and construction of Gauss–Turán-type quadrature formulae¹

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Received 17 January 1997; received in revised form 2 April 1997

Dedicated to William B. Gragg on his 60th birthday

Abstract

Using the theory of s -orthogonality and reinterpreting it in terms of the standard orthogonal polynomials on the real line, we develop a method for constructing Gauss–Turán-type quadrature formulae. The determination of nodes and weights is very stable. For finding all weights, our method uses an upper triangular system of linear equations for the weights associated with each node. Numerical examples are included.

Keywords: Gauss–Turán-type quadratures; s -orthogonal polynomials; Nonnegative measure; Extremal polynomial; Weights; Nodes; Degree of precision; Stieltjes procedure

AMS classification: Primary 65D30; 65D32; secondary 41A55

1. Introduction

Let \mathcal{P}_m be the set of all algebraic polynomials of degree at most m . In 1950, Turán [21] studied numerical quadratures of the form

$$\int_{-1}^1 f(t) dt = \sum_{i=0}^{k-1} \sum_{v=1}^n A_{i,v} f^{(i)}(\tau_v) + R_{n,k}(f), \quad (1.1)$$

where

$$A_{i,v} = \int_{-1}^1 l_{v,i}(t) dt \quad (v = 1, \dots, n; i = 0, 1, \dots, k-1)$$

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¹ The work of the first author was supported in part by the National Science Foundation under grant DMS-9305430. The work of the second author was supported in part by the Serbian Scientific Foundation under grant 0401F.

and $l_{v,i}(t)$ are the fundamental polynomials of Hermite interpolation. The coefficients $A_{i,v}$ are Cotes numbers of higher order. Evidently, the formula (1.1) is exact if $f \in \mathcal{P}_{k_{n-1}}$ and the points $-1 \leq \tau_1 < \dots < \tau_n \leq 1$ are arbitrary.

For $k=1$ the formula (1.1), i.e.,

$$\int_{-1}^1 f(t) dt = \sum_{v=1}^n A_{0,v} f(\tau_v) + R_{n,1}(f),$$

can be exact for all polynomials of degree at most $2n-1$ if the nodes τ_v are the zeros of the Legendre polynomial P_n , and it is the well-known Gauss–Legendre quadrature rule.

Because of Gauss’s result it is natural to ask whether knots τ_v can be chosen so that the quadrature formula (1.1) will be exact for algebraic polynomials of degree not exceeding $(k+1)n-1$. Turán [21] showed that the answer is negative for $k=2$, and for $k=3$ it is positive. He proved that the knots τ_v should be chosen as the zeros of the monic polynomial $\pi_n^*(t) = t^n + \dots$ which minimizes the integral

$$\int_{-1}^1 [\pi_n(t)]^4 dt,$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$.

More generally, the answer is negative for even, and positive for odd k , and then τ_v are the zeros of the polynomial minimizing

$$\int_{-1}^1 [\pi_n(t)]^{k+1} dt.$$

When $k=1$, then π_n^* is the monic Legendre polynomial \hat{P}_n .

Because of the above, we put $k=2s+1$. It is also interesting to consider, instead of (1.1), more general Gauss–Turán-type quadrature formulae

$$\int_{\mathbb{R}} f(t) d\lambda(t) = \sum_{i=0}^{2s} \sum_{v=1}^n A_{i,v} f^{(i)}(\tau_v) + R_{n,2s}(f), \tag{1.2}$$

where $d\lambda(t)$ is a nonnegative measure on the real line \mathbb{R} , with compact or infinite support, for which all moments

$$\mu_k = \int_{\mathbb{R}} t^k d\lambda(t), \quad k=0, 1, \dots,$$

exist and are finite, and $\mu_0 > 0$. It is known that formula (1.2) is exact for all polynomials of degree at most $2(s+1)n-1$, i.e.,

$$R_{n,2s}(f) = 0 \quad \text{for } f \in \mathcal{P}_{2(s+1)n-1}.$$

The knots τ_v ($v=1, \dots, n$) in (1.2) are the zeros of the monic polynomial $\pi_n^s(t)$, which minimizes the integral

$$F(a_0, a_1, \dots, a_{n-1}) = \int_{\mathbb{R}} [\pi_n(t)]^{2s+2} d\lambda(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$. This minimization leads to the conditions

$$\int_{\mathbb{R}} [\pi_n(t)]^{2s+1} t^k d\lambda(t) = 0 \quad (k = 0, 1, \dots, n - 1). \tag{1.3}$$

Usually, instead of $\pi_n^s(t)$ we write $P_{s,n}(t)$.

The case $d\lambda(t) = w(t)dt$ on $[a, b]$ has been investigated by the Italian mathematicians Ossicini [15], Ghizzetti and Ossicini [7] and Guerra [9, 10]. It is known that there exists a unique $P_{s,n}(t) = \prod_{v=1}^n (t - \tau_v)$, whose zeros τ_v are real, distinct and located in the interior of the interval $[a, b]$. These polynomials are known as *s-orthogonal* (or *s-self associated*) *polynomials* in the interval $[a, b]$ with respect to the weight function w (for more details see [4, 15–17]). For $s = 0$ we have the standard case of orthogonal polynomials, and (1.2) then becomes the well-known Gauss–Christoffel formula.

An iterative process for computing the coefficients of *s-orthogonal* polynomials in a special case, when the interval $[a, b]$ is symmetric with respect to the origin and the weight function w is an even function, was proposed by Vincenti [24]. He applied his process to the Legendre case. When n and s increase, the process becomes numerically unstable.

At the Third Conference on Numerical Methods and Approximation Theory (Niš, 18–21 August, 1987) (see [13]) we presented a stable method for numerically constructing *s-orthogonal* polynomials and their zeros. It uses an iterative method with quadratic convergence based on a discretized Stieltjes procedure and the Newton–Kantorovič method. Since the proceedings of this conference may not be widely available, we recall this method in Section 2. In Section 3, we develop a numerical procedure for calculating the coefficients $A_{i,v}$ in (1.2). Some alternative methods were proposed by Stroud and Stancu [20] (see also [19]) and Milovanović and Spalević [14]. Remarks on the Chebyshev measure are made in Section 4. Finally, a few numerical examples are presented in Section 5.

2. Construction of *s-orthogonal* polynomials

The basic idea for our method to numerically construct *s-orthogonal* polynomials with respect to the measure $d\lambda(t)$ on the real line \mathbb{R} is a reinterpretation of the “*orthogonality conditions*” (1.3). For given n and s , we put $d\mu(t) = d\mu^{s,n}(t) = (\pi_n(t))^{2s} d\lambda(t)$. The conditions can then be written as

$$\int_{\mathbb{R}} \pi_k^{s,n}(t) t^v d\mu(t) = 0 \quad (v = 0, 1, \dots, k - 1),$$

where $\{\pi_k^{s,n}\}$ is a sequence of monic orthogonal polynomials with respect to the new measure $d\mu(t)$. Of course, $P_{s,n}(\cdot) = \pi_n^{s,n}(\cdot)$. As we can see, the polynomials $\pi_k^{s,n}$ ($k = 0, 1, \dots$) are implicitly defined, because the measure $d\mu(t)$ depends on $\pi_n^{s,n}(t)$. A general class of such polynomials was introduced and studied by Engels (cf. [2, pp. 214–226]).

We will write simply $\pi_k(\cdot)$ instead of $\pi_k^{s,n}(\cdot)$. These polynomials satisfy a three-term recurrence relation

$$\begin{aligned} \pi_{v+1}(t) &= (t - \alpha_v)\pi_v(t) - \beta_v\pi_{v-1}(t), \quad v = 0, 1, \dots, \\ \pi_{-1}(t) &= 0, \quad \pi_0(t) = 1, \end{aligned} \tag{2.1}$$

Table 1

n	$d\mu^{s,n}(t)$	Orthogonal polynomials			
0	$(\pi_0^{s,0}(t))^{2s} d\lambda(t)$	$\pi_0^{s,0}$			
1	$(\pi_1^{s,1}(t))^{2s} d\lambda(t)$	$\pi_0^{s,1}$	$\pi_1^{s,1}$		
2	$(\pi_2^{s,2}(t))^{2s} d\lambda(t)$	$\pi_0^{s,2}$	$\pi_1^{s,2}$	$\pi_2^{s,2}$	
3	$(\pi_3^{s,3}(t))^{2s} d\lambda(t)$	$\pi_0^{s,3}$	$\pi_1^{s,3}$	$\pi_2^{s,3}$	$\pi_3^{s,3}$
⋮					

where, because of orthogonality,

$$\alpha_v = \alpha_v(s, n) = \frac{(\pi_v, \pi_v)}{(\pi_v, \pi_v)} = \frac{\int_{\mathbb{R}} t \pi_v^2(t) d\mu(t)}{\int_{\mathbb{R}} \pi_v^2(t) d\mu(t)}, \tag{2.2}$$

$$\beta_v = \beta_v(s, n) = \frac{(\pi_v, \pi_v)}{(\pi_{v-1}, \pi_{v-1})} = \frac{\int_{\mathbb{R}} \pi_v^2(t) d\mu(t)}{\int_{\mathbb{R}} \pi_{v-1}^2(t) d\mu(t)},$$

and, by convention, $\beta_0 = \int_{\mathbb{R}} d\mu(t)$.

The coefficients α_v and β_v are the fundamental quantities in the constructive theory of orthogonal polynomials. They provide a compact way of representing orthogonal polynomials, requiring only a linear array of parameters. The coefficients of orthogonal polynomials, or their zeros, in contrast, need two-dimensional arrays.

Knowing the coefficients α_v and β_v ($v=0, 1, \dots, n-1$) gives us access to the first $n+1$ orthogonal polynomials $\pi_0, \pi_1, \dots, \pi_n$. Of course, for a given n , we are interested only in the last of them, i.e., $\pi_n \equiv \pi_n^{s,n}$. Thus, for $n=0, 1, \dots$, the diagonal (boxed) elements in Table 1 are our s -orthogonal polynomials $\pi_n^{s,n}$.

A stable procedure for finding the coefficients α_v and β_v is the discretized Stieltjes procedure, especially for infinite intervals of orthogonality (see [3–6]). Unfortunately, in our case this procedure cannot be applied directly, because the measure $d\mu(t)$ involves an unknown polynomial $\pi_n^{s,n}$. Consequently, we consider the system of nonlinear equations

$$\begin{aligned} f_0 &\equiv \beta_0 - \int_{\mathbb{R}} \pi_n^{2s}(t) d\lambda(t) = 0, \\ f_{2v+1} &\equiv \int_{\mathbb{R}} (\alpha_v - t) \pi_v^2(t) \pi_n^{2s}(t) d\lambda(t) = 0 \quad (v=0, 1, \dots, n-1), \\ f_{2v} &\equiv \int_{\mathbb{R}} (\beta_v \pi_{v-1}^2(t) - \pi_v^2(t)) \pi_n^{2s}(t) d\lambda(t) = 0 \quad (v=1, \dots, n-1), \end{aligned} \tag{2.3}$$

which follows from (2.2).

Let \mathbf{x} be a $(2n)$ -dimensional column vector with components $\alpha_0, \beta_0, \dots, \alpha_{n-1}, \beta_{n-1}$ and $\mathbf{f}(\mathbf{x})$ a $(2n)$ -dimensional vector with components $f_0, f_1, \dots, f_{2n-1}$, given by (2.3), in which $\pi_0, \pi_1, \dots, \pi_n$ are

thought of as being expressed in terms of the α 's and β 's via (2.1). If $W = W(\mathbf{x})$ is the corresponding Jacobian of $\mathbf{f}(\mathbf{x})$, then we can apply Newton–Kantorovič's method

$$\mathbf{x}^{[k+1]} = \mathbf{x}^{[k]} - W^{-1}(\mathbf{x}^{[k]})\mathbf{f}(\mathbf{x}^{[k]}) \quad (k = 0, 1, \dots) \tag{2.4}$$

for determining the coefficients of the recurrence relation (2.1). If a sufficiently good approximation $\mathbf{x}^{[0]}$ is chosen, the convergence of the method (2.4) is quadratic.

Notice that the elements of the Jacobian can be easily computed in the following manner.

First, we have to determine the partial derivatives $a_{v,i} = \partial\pi_v/\partial\alpha_i$ and $b_{v,i} = \partial\pi_v/\partial\beta_i$. Differentiating the recurrence relation (2.1) with respect to α_i and β_i , we obtain

$$a_{v+1,i} = (t - \alpha_v)a_{v,i} - \beta_v a_{v-1,i}, \quad b_{v+1,i} = (t - \alpha_v)b_{v,i} - \beta_v b_{v-1,i},$$

where

$$a_{v,i} = 0, \quad b_{v,i} = 0 \quad (v \leq i),$$

$$a_{i+1,i} = -\pi_i(t), \quad b_{i+1,i} = -\pi_{i-1}(t).$$

These relations are the same as those for π_v , but with other (delayed) initial values. The elements of the Jacobian are

$$\begin{aligned} \frac{\partial f_{2v+1}}{\partial \alpha_i} &= 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)[(\alpha_v - t)p_{v,i}(t) + \frac{1}{2}\delta_{v,i}\pi_v^2(t)\pi_n(t)] d\lambda(t), \\ \frac{\partial f_{2v+1}}{\partial \beta_i} &= 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)(\alpha_v - t)q_{v,i}(t) d\lambda(t), \\ \frac{\partial f_{2v}}{\partial \alpha_i} &= 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)(\beta_v p_{v-1,i}(t) - p_{v,i}(t)) d\lambda(t), \\ \frac{\partial f_{2v}}{\partial \beta_i} &= 2 \int_{\mathbb{R}} \pi_n^{2s-1}(t)[(\beta_v q_{v-1,i}(t) - q_{v,i}(t)) + \frac{1}{2}\delta_{v,i}\pi_{v-1}^2(t)\pi_n(t)] d\lambda(t), \end{aligned} \tag{2.5}$$

where

$$p_{v,i}(t) = \pi_v(t)(a_{v,i}\pi_n(t) + sa_{n,i}\pi_v(t)), \quad q_{v,i}(t) = \pi_v(t)(b_{v,i}\pi_n(t) + sb_{n,i}\pi_v(t)),$$

and $\delta_{v,i}$ is the Kronecker delta.

All of the above integrals in (2.3) and (2.5) can be computed exactly, except for rounding errors, by using a Gauss–Christoffel quadrature formula with respect to the measure $d\lambda(t)$,

$$\int_{\mathbb{R}} g(t) d\lambda(t) = \sum_{v=1}^N A_v^{(N)} g(\tau_v^{(N)}) + R_N(g), \tag{2.6}$$

taking $N = (s + 1)n$ knots. This formula is exact for all polynomials of degree at most $2N - 1 = 2(s + 1) \cdot n - 1 = 2(n - 1) + 2ns + 1$.

Thus, for all calculations we use only the fundamental three-term recurrence relation (2.1) for the orthogonal polynomials $\pi_k(\cdot; d\lambda)$ and the Gauss–Christoffel quadrature (2.6). As initial values $\alpha_v^{[0]} = \alpha_v^{[0]}(s, n)$ and $\beta_v^{[0]} = \beta_v^{[0]}(s, n)$ we take the values obtained for $n - 1$, i.e., $\alpha_v^{[0]} = \alpha_v(s, n - 1)$, $\beta_v^{[0]} = \beta_v(s, n - 1)$, $v \leq n - 2$. For $\alpha_{n-1}^{[0]}$ and $\beta_{n-1}^{[0]}$ we use corresponding extrapolated values.

In the case $n = 1$ we solve the equation

$$\phi(\alpha_0) = \phi(\alpha_0(s, 1)) = \int_{\mathbb{R}} (t - \alpha_0)^{2s+1} d\lambda(t) = 0,$$

and then determine

$$\beta_0 = \beta_0(s, 1) = \int_{\mathbb{R}} (t - \alpha_0)^{2s} d\lambda(t).$$

The zeros $\tau_v = \tau_v(s, n)$ ($v = 1, \dots, n$) of $\pi_n^{s,n}$, i.e., the nodes of the Gauss–Turán-type quadrature formula (1.2), we obtain very easily as eigenvalues of a (symmetric tridiagonal) Jacobi matrix J_n using the QR algorithm, namely,

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \mathbf{O} \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{n-1}} \\ \mathbf{O} & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix},$$

where $\alpha_v = \alpha_v(s, n)$, $\beta_v = \beta_v(s, n)$ ($v = 0, 1, \dots, n - 1$).

3. Calculation of coefficients

Let $\tau_v = \tau_v(s, n)$, $v = 1, \dots, n$, be the zeros of the s -orthogonal (monic) polynomial $\pi_n(t) (\equiv \pi_n^{s,n}(t))$. In order to find the coefficients $A_{i,v}$ in the Gauss–Turán-type quadrature formula

$$\int_{\mathbb{R}} f(t) d\lambda(t) = \sum_{i=0}^{2s} \sum_{v=1}^n A_{i,v} f^{(i)}(\tau_v) + R(f), \quad (3.1)$$

we define

$$\Omega_v(t) = \left(\frac{\pi_n(t)}{t - \tau_v} \right)^{2s+1} = \prod_{i \neq v} (t - \tau_i)^{2s+1} \quad (v = 1, \dots, n). \quad (3.2)$$

Then the coefficients $A_{i,v}$ can be expressed in the form (see [19])

$$A_{i,v} = \frac{1}{i!(2s-i)!} \left[D^{2s-i} \frac{1}{\Omega_v(t)} \int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1} - \pi_n(t)^{2s+1}}{x-t} d\lambda(x) \right]_{t=\tau_v},$$

where D is the differentiation operator. In particular, for $i = 2s$, we have

$$A_{2s,v} = \frac{1}{(2s)!(\pi_n'(\tau_v))^{2s+1}} \int_{\mathbb{R}} \frac{\pi_n(x)^{2s+1}}{x - \tau_v} d\lambda(x),$$

i.e.,

$$A_{2s,v} = \frac{B_v^{(s)}}{(2s)!(\pi_n'(\tau_v))^{2s}} \quad (v = 1, \dots, n),$$

where $B_v^{(s)}$ are the Christoffel numbers of the following Gaussian quadrature (with respect to the measure $d\mu(t) = \pi_n^{2s}(t) d\lambda(t)$):

$$\int_{\mathbb{R}} g(t) d\mu(t) = \sum_{v=1}^n B_v^{(s)} g(\tau_v) + R_n(g), \quad R_n(\mathcal{P}_{2n-1}) = 0. \tag{3.3}$$

Since $B_v^{(s)} > 0$, we conclude that $A_{2s,v} > 0$. The expressions for the other coefficients ($i < 2s$) become very complicated. For numerical calculation we could use a triangular system of linear equations obtained from the formula (3.1) by replacing f with the Newton polynomials: $1, t - \tau_1, \dots, (t - \tau_1)^{2s+1}, (t - \tau_1)^{2s+1}(t - \tau_2), \dots, (t - \tau_1)^{2s+1}(t - \tau_2)^{2s+1} \dots (t - \tau_n)^{2s}$ (cf. [4, Section 2.2.4]).

In this paper we take instead the polynomials

$$f_{k,v}(t) = (t - \tau_v)^k \Omega_v(t) = (t - \tau_v)^k \prod_{i \neq v} (t - \tau_i)^{2s+1}, \tag{3.4}$$

where $0 \leq k \leq 2s, 1 \leq v \leq n$.

Since the quadrature (3.1) is exact for all polynomials of degree at most $2(s + 1)n - 1$ and

$$\deg f_{k,v} = (n - 1)(2s + 1) + k \leq (2s + 1)n - 1,$$

we see that (3.1) is exact for the polynomials (3.4), i.e.,

$$R(f_{k,v}) = 0 \quad (0 \leq k \leq 2s, 1 \leq v \leq n).$$

Thus, we have

$$\sum_{i=0}^{2s} \sum_{j=1}^n A_{i,j} f_{k,v}^{(i)}(\tau_j) = \int_{\mathbb{R}} f_{k,v}(t) d\lambda(t),$$

i.e.,

$$\sum_{i=0}^{2s} A_{i,v} f_{k,v}^{(i)}(\tau_v) = \mu_{k,v}, \tag{3.5}$$

because for every $j \neq v$ we have $f_{k,v}^{(i)}(\tau_j) = 0$ when $0 \leq i \leq 2s$. Here, we have put

$$\mu_{k,v} = \int_{\mathbb{R}} f_{k,v}(t) d\lambda(t) = \int_{\mathbb{R}} (t - \tau_v)^k \prod_{i \neq v} (t - \tau_i)^{2s+1} d\lambda(t).$$

For each v , we have in (3.5) a system of $2s + 1$ linear equations in the same number of unknowns, $A_{i,v}, i = 0, 1, \dots, 2s$.

Using Leibniz's formula of differentiation, one easily proves the following auxiliary result.

Lemma 3.1. For the polynomials $f_{k,v}$ given by (3.4), we have

$$f_{k,v}^{(i)}(\tau_v) = \begin{cases} 0, & i < k, \\ i^{(k)} \Omega_v^{(i-k)}(\tau_v), & i \geq k, \end{cases}$$

where $i^{(k)} = i(i-1) \cdots (i-k+1)$ [with $0^{(0)} = 1$] and Ω_v is defined in (3.2).

Lemma 3.1 shows that each system of linear equations (3.5) is upper triangular. Thus, once all zeros of the s -orthogonal polynomial π_n , i.e., the nodes of the quadrature formula (3.1), are known, the determination of its weights $A_{i,v}$ is reduced to solving the n linear systems of $(2s+1)$ equations

$$\begin{bmatrix} f_{0,v}(\tau_v) & f'_{0,v}(\tau_v) & \cdots & f_{0,v}^{(2s)}(\tau_v) \\ & f'_{1,v}(\tau_v) & \cdots & f_{1,v}^{(2s)}(\tau_v) \\ & & \ddots & \\ & & & f_{2s,v}^{(2s)}(\tau_v) \end{bmatrix} \begin{bmatrix} A_{0,v} \\ A_{1,v} \\ \vdots \\ A_{2s,v} \end{bmatrix} = \begin{bmatrix} \mu_{0,v} \\ \mu_{1,v} \\ \vdots \\ \mu_{2s,v} \end{bmatrix}.$$

Put $a_{k,k+j} = f_{k-1,v}^{(k-1+j)}(\tau_v)$, so that the matrix of the system has elements $a_{l,j}$ ($1 \leq l, j \leq 2s+1$), with $a_{l,j} = 0$ for $j < l$. Then, by Lemma 3.1,

$$a_{l,j} = (j-1)^{(l-1)} \Omega_v^{(j-l)}(\tau_v) \quad (j \geq l; 1 \leq l, j \leq 2s+1). \tag{3.6}$$

Lemma 3.2. Let τ_1, \dots, τ_n be the zeros of the s -orthogonal polynomial π_n . For the elements $a_{l,j}$, defined by (3.6), the following relations hold:

$$\begin{aligned} a_{k,k} &= (k-1)! a_{1,1} \quad (1 \leq k \leq 2s+1), \\ a_{k,k+j} &= -(2s+1)(k+j-1)^{(k-1)} \sum_{l=1}^j u_l a_{l,j} \quad \left(\begin{matrix} 1 \leq k \leq 2s-j+1 \\ j = 1, \dots, 2s \end{matrix} \right), \end{aligned}$$

where

$$\begin{aligned} a_{1,1} &= \Omega_v(\tau_v) = [\pi'_n(\tau_v)]^{2s+1}, \\ u_l &= \sum_{i \neq v} (\tau_i - \tau_v)^{-l} \quad (l = 1, \dots, 2s). \end{aligned} \tag{3.7}$$

Proof. The first relation is an immediate consequence of the definition of $a_{k,k}$ and Lemma 3.1. To prove the second, define $v(t) = \sum_{i \neq v} (t - \tau_i)^{-1}$. Since $\Omega_v(t) = \prod_{i \neq v} (t - \tau_i)^{2s+1}$ we have that

$$\Omega'_v(t) = (2s+1)v(t)\Omega_v(t)$$

and

$$\begin{aligned} \Omega_v^{(j)}(t) &= \frac{d^{j-1}}{dt^{j-1}}(\Omega'_v(t)) = (2s+1) \frac{d^{j-1}}{dt^{j-1}}(v(t)\Omega_v(t)) \\ &= (2s+1) \sum_{l=0}^{j-1} \binom{j-1}{l} \Omega_v^{(j-1-l)}(t)v^{(l)}(t). \end{aligned}$$

Then, (3.6) becomes

$$a_{k,k+j} = (k + j - 1)^{(k-1)}(2s + 1) \sum_{l=1}^j \binom{j-1}{l-1} v^{(l-1)}(\tau_v) \Omega_v^{(j-l)}(\tau_v).$$

Since

$$v^{(l-1)}(\tau_v) = (-1)^{l-1} (l-1)! \sum_{i \neq v} (\tau_v - \tau_i)^{-l} = -(l-1)! u_l$$

and

$$\Omega_v^{(j-l)}(\tau_v) = \frac{a_{l,j}}{(j-1)^{(l-1)}} = \frac{(j-l)!}{(j-1)!} a_{l,j},$$

we get

$$a_{k,k+j} = -(2s + 1)(k + j - 1)^{(k-1)} \sum_{l=1}^j u_l a_{l,j}. \quad \square$$

Using the normalization

$$\hat{a}_{k,j} = \frac{a_{k,j}}{(j-1)! a_{1,1}} \quad (1 \leq k, j \leq 2s + 1), \tag{3.8}$$

and putting

$$b_k = (k-1) A_{k-1,v} \quad (1 \leq k \leq 2s + 1),$$

$$\hat{\mu}_{k,v} = \frac{\mu_{k,v}}{(\pi'_n(\tau_v))^{2s+1}} = \int_{\mathbb{R}} (t - \tau_v)^k \left(\prod_{i \neq v} \frac{t - \tau_i}{\tau_v - \tau_i} \right)^{2s+1} d\lambda(t), \tag{3.9}$$

we have the following result:

Theorem 3.3. For fixed v ($1 \leq v \leq n$), the coefficients $A_{i,v}$ in the Gauss–Turán-type quadrature formula (3.1) are given by

$$b_{2s+1} = (2s)! A_{2s,v} = \hat{\mu}_{2s,v},$$

$$b_k = (k-1)! A_{k-1,v} = \hat{\mu}_{k-1,v} - \sum_{j=k+1}^{2s+1} \hat{a}_{k,j} b_j \quad (k = 2s, \dots, 1), \tag{3.10}$$

where $\hat{\mu}_{k,v}$ are given by (3.9), and

$$\hat{a}_{k,k} = 1, \quad \hat{a}_{k,k+j} = -\frac{2s+1}{j} \sum_{l=1}^j u_l \hat{a}_{l,j} \quad (k = 1, \dots, 2s; j = 1, \dots, 2s - k + 1), \tag{3.11}$$

the u_l being defined by (3.7).

Proof. The relations (3.11) follow directly from Lemma 3.2 and the normalization (3.8).

The coefficients b_k ($1 \leq k \leq 2s + 1$) are obtained from the corresponding upper triangular system of equations $\hat{A}\hat{\mathbf{b}} = \mathbf{c}$, where

$$\hat{A} = [\hat{a}_{ij}], \quad \mathbf{b} = [b_1, \dots, b_{2s+1}]^T, \quad \mathbf{c} = [\hat{\mu}_{0,v}, \dots, \hat{\mu}_{2s,v}]^T. \quad \square$$

The normalized moments $\hat{\mu}_{k,v}$ can be computed exactly, except for rounding errors, by using the same Gauss–Christoffel formula as in the construction of s -orthogonal polynomials, i.e., (2.6) with $N = (s + 1)n$ knots.

4. Some remarks on the Chebyshev measure

In this section we discuss the particularly interesting case of the Chebyshev measure $d\lambda(t) = (1-t^2)^{-1/2} dt$. In 1930, Bernstein [1] showed that the monic Chebyshev polynomial $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^1 \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}} dt \quad (k \geq 0).$$

Thus, the Chebyshev–Turán formula

$$\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt = \sum_{i=0}^{2s} \sum_{v=1}^n A_{i,v} f^{(i)}(\tau_v) + R_n(f), \tag{4.1}$$

with $\tau_v = \cos((2v - 1)\pi/2n)$ ($v = 1, \dots, n$), is exact for all polynomials of degree at most $2(s + 1)n - 1$. Turán stated the problem of explicit determination of $A_{i,v}$ and its behavior as $n \rightarrow +\infty$ (see [22, Problem XXVI]). Some characterizations and solution for $s = 2$ were obtained by Micchelli and Rivlin [12], Riess [18], and Varma [23]. One simple answer to Turán’s question was given by Kis [11]. His result can be stated in the following form: *If g is an even trigonometric polynomial of degree at most $2(s + 1)n - 1$, then*

$$\int_0^\pi g(\theta) d\theta = \frac{\pi}{n(s!)^2} \sum_{j=0}^s \frac{S_j}{4^j n^{2j}} \sum_{v=1}^n g^{(2j)}\left(\frac{2v-1}{2n}\pi\right),$$

where S_{s-j} ($j = 0, 1, \dots, s$) denote the elementary symmetric polynomials with respect to the numbers $1^2, 2^2, \dots, s^2$, i.e.,

$$S_s = 1, \quad S_{s-1} = 1^2 + 2^2 + \dots + s^2, \dots, \quad S_0 = 1^2 \cdot 2^2 \cdot \dots \cdot s^2.$$

Consequently,

$$\int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{n(s!)^2} \sum_{j=0}^s \frac{S_j}{4^j n^{2j}} \sum_{v=1}^n [D^{2j} f(\cos \theta)]_{\theta=(2v-1/2n)\pi}.$$

Using the expansion

$$D^{2k} f(\cos \theta) = \sum_{i=1}^{2k} d_{k,i}(t) f^{(i)}(t), \quad t = \cos \theta, \quad k > 0,$$

where the functions $d_{k,j} \equiv d_{k,j}(t)$ are given recursively by

$$\begin{aligned} d_{k+1,1} &= (1-t^2)d_{k,1}'' - td_{k,1}', \\ d_{k+1,2} &= (1-t^2)d_{k,2}'' - td_{k,2}' + 2(1-t^2)d_{k,1}' - td_{k,1}, \\ d_{k+1,i} &= (1-t^2)d_{k,i}'' - td_{k,i}' + 2(1-t^2)d_{k,i-1}' - td_{k,i-1} + (1-t^2)d_{k,i-2}, \quad (i = 3, \dots, 2k), \\ d_{k+1,2k+1} &= 2(1-t^2)d_{k,2k}' - td_{k,2k} + (1-t^2)d_{k,2k-1}, \\ d_{k+1,2k+2} &= (1-t^2)d_{k,2k}, \end{aligned}$$

with $d_{1,1} = -t$ and $d_{1,2} = 1 - t^2$, we obtain (4.1). For example, when $s = 3$, we have

$$\begin{aligned} A_{0,v} &= \frac{\pi}{n}, \quad A_{1,v} = \frac{\pi\tau_v}{2304n^7}(784n^4 + 56n^2 - 1), \\ A_{2,v} &= \frac{\pi}{2304n^7}[(784n^4 - 392n^2 + 31)(1 - \tau_v^2) + 168n^2 - 15], \\ A_{3,v} &= -\frac{\pi\tau_v}{2304n^7}[(336n^2 - 89)(1 - \tau_v^2) + 15], \\ A_{4,v} &= \frac{\pi}{2304n^7}[(56n^2 - 65)(1 - \tau_v^2)^2 + 45(1 - \tau_v^2)], \\ A_{5,v} &= \frac{\pi\tau_v}{2304n^7}[674(1 - \tau_v^2)^2 - 240(1 - \tau_v^2)], \quad A_{6,v} = \frac{\pi}{2304n^7}(1 - \tau_v^2)^3. \end{aligned}$$

To conclude, we mention the corresponding formula (3.3) for the Chebyshev weight,

$$\int_{\mathbb{R}} g(t) \frac{\hat{T}_n^{2s}(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{4^{sn}n} \binom{2s}{s} \sum_{v=1}^n g(\tau_v) + R_n(g), \tag{4.2}$$

where $\tau_v = \cos(2v - 1)(\pi/2n)$ ($v = 1, \dots, n$). Note that all weights are equal, i.e., the formula (4.2) is one of Chebyshev type.

5. Numerical examples

Using the procedures outlined in Sections 2 and 3 for constructing s -orthogonal polynomials and calculating the coefficients in Gauss–Turán-type quadrature formulae, we prepared corresponding software with the following types of polynomials $\pi_n(\cdot; d\lambda)$ (identified by the integer ipoly):

- c ipoly - integer identifying the kind of polynomials:
- c
- c 0 = nonclassical polynomials with given coefficients
- c in the three-term recurrence relation
- c 1 = Legendre polynomials on $[-1, 1]$
- c 2 = Legendre polynomials on $[0, 1]$

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c 3 = Chebyshev polynomials of the first kind
c 4 = Chebyshev polynomials of the second kind
c 5 = Jacobi polynomials with parameters al=.5, be=-.5
c 6 = Jacobi polynomials with parameters al,be
c 7 = generalized Laguerre polynomials with parameter al
c 8 = Hermite polynomials
c 9 = generalized Gegenbauer polynomials with parameters al,be
c 10 = polynomials for the logistic weight
c      w(t)=e^{-t}/(1+e^{-t})^2 on the real line
c
c al,be - parameters for Jacobi, generalized Laguerre
c      and generalized Gegenbauer polynomials
c For ipoly=9, the weight function is given by
c w(x)=|x|^mu(1-x^2)^al, where be=(mu-1)/2.
    
```

All computations were done on the MICROVAX 3400 computer using VAX FORTRAN Ver. 5.3 in *D*- and *Q*-arithmetic, with machine precision $\approx 2.76 \times 10^{-17}$ and $\approx 1.93 \times 10^{-34}$, respectively. For example, taking $d\lambda(t) = e^{-t} dt$ on $(0, +\infty)$, for $s = 2$ and $n = 5$ we obtain the results in *D*-arithmetic shown in Table 2.

Finally, we give an example where it is preferable to use a formula of Turán type rather than the standard Gaussian formula,

$$\int_{\mathbb{R}} f(t) d\lambda(t) = \sum_{v=1}^n A_v f(t_v) + R_n(f) \tag{5.1}$$

for which $R_n(\mathcal{P}_{2n-1}) = 0$. The example is

$$I = \int_{-1}^1 e^t \sqrt{1-t^2} dt = 1.7754996892121809468785765372\dots$$

Here we have $f(t) = e^t$ and $d\lambda(t) = \sqrt{1-t^2} dt$ on $[-1, 1]$ (the Chebyshev measure of the second kind). Notice that $f^{(i)}(t) = f(t)$ for every $i \geq 0$.

The Gaussian formula (5.1) and the corresponding Gauss–Turán formula (3.1) give

$$I \approx I_n^G = \sum_{v=1}^n A_v e^{t_v} \tag{5.2}$$

and

$$I \approx I_{n,s}^T = \sum_{v=1}^n C_v^{(s)} e^{t_v}, \tag{5.3}$$

respectively, where $C_v^{(s)} = \sum_{i=0}^{2s} A_{i,v}$.

Table 3 shows the relative errors $|(I_{n,s}^T - I)/I_{n,s}^T|$ for $n = 1(1)5$ and $s = 0(1)5$. (Numbers in parentheses indicate decimal exponents and m.p. stands for machine precision.)

For $s = 0$ the quadrature formula (5.3) reduces to (5.2), i.e., $I_{n,0}^T \equiv I_n^G$. Notice that Turán’s formula (5.3) with n nodes has the same degree of exactness as the Gaussian formula with $(s + 1)n$ nodes, which explains its superior behavior in Table 3.

Table 2
Example. Laguerre case ($s = 2, n = 5$)

k	alpha (k)	beta (k)
0	0.206241261660323E+01	0.111900724691563E+17
1	0.817357215072019E+01	0.627220780166491E+01
2	0.143542025111386E+02	0.314187808183856E+02
3	0.206411614818251E+02	0.761775799352482E+02
4	0.268361238086797E+02	0.141467716850165E+03
	zero(1) = 0.511080817827157E+00	zero(2) = 0.365040485156886E+01
	$A(0, 1) = 0.831408096794173E+00$	$A(0, 2) = 0.167454288564437E+00$
	$A(1, 1) = 0.878844153076445E-01$	$A(1, 2) = -0.133418640886195E+00$
	$A(2, 1) = 0.777008304959738E-01$	$A(2, 2) = 0.101695158354974E+00$
	$A(3, 1) = 0.776770118733145E-02$	$A(3, 2) = -0.233384486558624E-01$
	$A(4, 1) = 0.124333607217694E-02$	$A(4, 2) = 0.920099700677729E-02$
	zero(3) = 0.100115534444780E+02	zero(4) = 0.204527761237753E+02
	$A(0, 3) = 0.113746188754331E-02$	$A(0, 4) = 0.152753792492066E-06$
	$A(1, 3) = -0.204892563320579E-02$	$A(1, 4) = -0.410956732811768E-06$
	$A(2, 3) = 0.191860247042219E-02$	$A(2, 4) = 0.484507006038965E-06$
	$A(3, 3) = -0.903002129075339E-03$	$A(3, 4) = -0.288211914479617E-06$
	$A(4, 3) = 0.265091858385108E-03$	$A(4, 4) = 0.791425834311650E-07$
	zero(5) = 0.374416573313175E+02	
	$A(0, 5) = 0.546801190168267E-13$	
	$A(1, 5) = -0.192133308928889E-12$	
	$A(2, 5) = 0.271424024484902E-12$	
	$A(3, 5) = -0.181974618995712E-12$	
	$A(4, 5) = 0.492724906167396E-13$	

Table 3
 Relative errors in quadrature sums $I_{n,s}^T$

n	$s = 0$	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
1	1.15(-1)	4.71(-3)	9.72(-5)	1.21(-6)	1.01(-8)	5.98(-11)
2	2.38(-3)	2.05(-7)	3.06(-12)	1.36(-17)	2.40(-23)	1.88(-29)
3	1.97(-5)	1.15(-12)	4.02(-21)	9.26(-31)	m.p.	m.p.
4	8.76(-8)	1.71(-18)	4.68(-31)	m.p.	m.p.	m.p.
5	2.43(-10)	9.40(-25)	m.p.	m.p.	m.p.	m.p.

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