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Numerical approach to differential equations of fractional order

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Abstract

In this paper, the variational iteration method and the Adomian decomposition method are implemented to give approximate solutions for linear and nonlinear systems of differential equations of fractional order. The two methods in applied mathematics can be used as alternative methods for obtaining analytic and approximate solutions for different types of differential equations. In these schemes, the solution takes the form of a convergent series with easily computable components. This paper presents a numerical comparison between the two methods for solving systems of fractional differential equations. Numerical results show that the two approaches are easy to implement and accurate when applied to differential equations of fractional order.

MSC: 65L05; 65L06; 26A33

Keywords: Variational iteration method; Lagrange multiplier; Adomian decomposition method; Fractional differential equations; Caputo fractional derivative

1. Introduction

The objective of the present paper is to extend the applications of the variational iteration method (VIM) and the Adomian decomposition method (ADM) to provide approximate solutions for the nonlinear system:

$$
\begin{aligned}
&D_*^{\alpha_1} x_1(t) = f_1(t, x_1, x_2, \ldots, x_n),\\
&D_*^{\alpha_2} x_2(t) = f_2(t, x_1, x_2, \ldots, x_n),\\
&\vdots\\
&D_*^{\alpha_n} x_n(t) = f_n(t, x_1, x_2, \ldots, x_n),
\end{aligned} \tag{1.1}
$$

where $D_*^{\alpha_i}$ is the derivative of x_i of order α_i in the sense of Caputo and $0 < \alpha_i \leqslant 1$, subject to the initial conditions

$$x_1(0) = c_1, \quad x_2(0) = c_2, \ldots, x_n(0) = c_n. \tag{1.2}$$

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Ordinary and partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Consequently, considerable attention has been given to the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest [5,8,12,17,30,31,33–35,45,47,51]. Most nonlinear fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques (see [7,11–16,46]) must be used. The decomposition method [1,3,4,9,49,52,53] and the VIM [2,6,18–29,38,50] are relatively new approaches to provide analytical approximations to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

Recently, the application of the two methods is successfully extended to obtain an analytical approximate solutions to linear and nonlinear differential equations of fractional order [10,22,36,37,39–44,48]. A comparison between the VIM and ADM for solving fractional differential equations is given in [40,41]. The fact that the VIM solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over ADM.

2. Basic definitions

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. A real function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(>\mu)$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)}\in C_\mu$, $m\in N$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$, of a function $f\in C_\mu$, $\mu \geqslant -1$, is defined as

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad \alpha>0, \quad x>0,$$

$$J^0 f(x)=f(x).$$

Properties of the operator J^α can be found in [33,35,45], we mention only the following:

For $f\in C_\mu$, $\mu\geqslant -1$, $\alpha, \beta \geqslant 0$ and $\gamma>-1$:

1. $J^\alpha J^\beta f(x)=J^{\alpha+\beta} f(x)$,
2. $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$,
3. $J^\alpha x^\gamma=(\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1))x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [8].

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x)=J^{m-\alpha}D^m f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,\mathrm{d}t, \tag{2.1}$$

for $m-1<\alpha\leqslant m$, $m\in\mathbb{N}$, $x>0$, $f\in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If* $m-1<\alpha\leqslant m$, $m\in N$ *and* $f\in C_\mu^m$, $\mu\geqslant -1$, *then*

$$D_*^\alpha J^\alpha f(x)=f(x),$$

and,

$$J^{\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0.$$

3. Decomposition method

The decomposition method requires that the system of nonlinear fractional differential equations (1.1) be expressed in the form

$$D_*^{\alpha_1} x_1(t) = \sum_{j=1}^{n} a_{1j}(t)x_j + f_1(t, x_1, x_2, \ldots, x_n) + g_1(t),$$

$$D_*^{\alpha_2} x_2(t) = \sum_{j=1}^{n} a_{2j}(t)x_j + f_2(t, x_1, x_2, \ldots, x_n) + g_2(t),$$

$$\vdots$$

$$D_*^{\alpha_n} x_n(t) = \sum_{j=1}^{n} a_{nj}(t)x_j + f_n(t, x_1, x_2, \ldots, x_n) + g_n(t), \tag{3.1}$$

where f_i is a nonlinear function and $0 < \alpha_i \leqslant 1$, for $i = 1, 2, \ldots, n$. Applying the fractional integral operator J^{α_i}, the inverse of the operator $D_*^{\alpha_i}$, to both sides of (3.1), we obtain

$$x_i(t) = x_i(0) + J^{\alpha_i} g_i(t) + J^{\alpha_i} \sum_{j=1}^{n} a_{ij}(t)x_j(t) + J^{\alpha_i} f_i(t, x_1(t), x_2(t), \ldots, x_n(t)). \tag{3.2}$$

The decomposition method suggests that the solution $x_i(t)$ be decomposed by the infinite series solution

$$x_i(t) = \sum_{k=0}^{\infty} x_i^k(t), \quad i = 1, 2, \ldots, n, \tag{3.3}$$

and the nonlinear function f_i in Eq. (3.1) is decomposed as follows:

$$f_i(t, x_1, x_2, \ldots, x_n) = \sum_{k=0}^{\infty} A_i^k(x_1^0, \ldots, x_1^k; x_2^0, \ldots, x_2^k; \ldots; x_n^0, \ldots, x_n^k), \quad i = 1, 2, \ldots, n, \tag{3.4}$$

where A_i^k are the so-called the Adomian polynomials. Substituting (3.3) and (3.4) into both sides of (3.2) gives

$$\sum_{k=0}^{\infty} x_i^k = x_i(0) + J^{\alpha_i} g_i(t) + J^{\alpha_i} \sum_{j=1}^{n} a_{ij}(t) \sum_{k=0}^{\infty} x_j^k(t) + J^{\alpha_i} \left(\sum_{k=0}^{\infty} A_i^k(x_1^0, \ldots, x_1^k; x_2^0, \ldots, x_2^k; \ldots; x_n^0, \ldots, x_n^k) \right). \tag{3.5}$$

From this equation, the iterates are determined by the following recursive way

$$x_i^0(t) = c_i + J^{\alpha_i} g_i(t), \quad i = 1, 2, \ldots, n,$$

$$x_i^{k+1}(t) = J^{\alpha_i} \sum_{j=1}^{n} a_{ij}(t)x_j^k(t) + J^{\alpha_i} A_j^k(x_1^0, \ldots, x_1^k; x_2^0, \ldots, x_2^k; \ldots; x_n^0, \ldots, x_n^k). \tag{3.6}$$

The Adomian polynomials A_i^k can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian [52]. The general form of formula for Adomian polynomials is

$$A_i^k = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} f_i\left(t, \sum_{k=0}^{\infty} \lambda^k x_1^k, \sum_{k=0}^{\infty} \lambda^k x_2^k, \ldots, \sum_{k=0}^{\infty} \lambda^k x_n^k\right)\right]_{\lambda=0}. \tag{3.7}$$

This formula is easy to compute by using Mathematica software or by setting a computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions.

Finally, we approximate the solution $x_i(t)$ by the truncated series

$$\phi_i^N(t) = \sum_{k=0}^{N-1} x_i^k(t), \tag{3.8}$$

where

$$\lim_{N\to\infty} x_i^N(t) = x_i(t). \tag{3.9}$$

However, in many cases the exact solution in a closed form may be obtained. Moreover, the decomposition series solutions generally converge very rapidly. The convergence of the decomposition series has been investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [1,9].

4. Variational iteration method

The principles of the VIM and its applicability for various kinds of differential equations are given in [2,6,18–29,38, 50]. To solve the system of nonlinear fractional differential equations (1.1) by means of the VIM, rewrite the system in the form

$$\begin{aligned}
D_*^{\alpha_1} x_1(t) &= f_1(x_1, x_2, \ldots, x_n) + g_1(t),\\
D_*^{\alpha_2} x_2(t) &= f_2(x_1, x_2, \ldots, x_n) + g_2(t),\\
&\vdots\\
D_*^{\alpha_n} x_n(t) &= f_n(x_1, x_2, \ldots, x_n) + g_n(t),
\end{aligned} \tag{4.1}$$

where $0 < \alpha_i \leqslant 1$, subject to the initial conditions

$$x_1(0) = c_1, \quad x_2(0) = c_2, \ldots, x_n(0) = c_n. \tag{4.2}$$

The correction functionals for the nonlinear system (4.1) can be approximately constructed as

$$\begin{aligned}
x_1^{k+1}(t) &= x_1^k(t) + \int_0^t \lambda_1 (x_1^{'k}(\tau) - f_1(\tilde{x}_1^k(\tau), \tilde{x}_2^k(\tau), \ldots, \tilde{x}_n^k(\tau)) - g_1(\tau))\,\mathrm{d}\tau,\\
x_2^{k+1}(t) &= x_2^k(t) + \int_0^t \lambda_2 (x_2^{'k}(\tau) - f_2(\tilde{x}_1^k(\tau), \tilde{x}_2^k(\tau), \ldots, \tilde{x}_n^k(\tau)) - g_2(\tau))\,\mathrm{d}\tau,\\
&\vdots\\
x_n^{k+1}(t) &= x_n^k(t) + \int_0^t \lambda_n (x_n^{'k}(\tau) - f_n(\tilde{x}_1^k(\tau), \tilde{x}_2^k(\tau), \ldots, \tilde{x}_n^k(\tau)) - g_n(\tau))\,\mathrm{d}\tau,
\end{aligned} \tag{4.3}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are general Lagrange multipliers [32], which can be identified optimally via variational theory [18,25,26,32], and $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ denote restricted variations. Making the above functionals stationary, we obtain the following stationary conditions

$$\lambda_i'(\tau)|_{\tau=t} = 0,$$

$$1 + \lambda_i(\tau)|_{\tau=t} = 0,$$

for $i = 1, 2, \ldots, n$. Therefore, the Lagrange multipliers can be easily identified as

$$\lambda_i = -1, \quad i = 1, 2, \ldots, n. \tag{4.4}$$

Substituting (4.4) into the correction functionals (4.3) results the following iteration formulas:

$$x_1^{k+1}(t) = x_1^k(t) - \int_0^t (D_*^{\alpha_1} x_1^k(\tau) - f_1(x_1^k(\tau), x_2^k(\tau), \ldots, x_n^k(\tau)) - g_1(\tau))\,d\tau,$$

$$x_2^{k+1}(t) = x_2^k(t) - \int_0^t (D_*^{\alpha_2} x_2^k(\tau) - f_2(x_1^k(\tau), x_2^k(\tau), \ldots, x_n^k(\tau)) - g_2(\tau))\,d\tau,$$

$$\vdots$$

$$x_n^{k+1}(t) = x_n^k(t) - \int_0^t (D_*^{\alpha_n} x_n^k(\tau) - f_n(x_1^k(\tau), x_2^k(\tau), \ldots, x_n^k(\tau)) - g_n(\tau))\,d\tau. \tag{4.5}$$

If we start with the initial approximations $x_1^0 = c_1, x_2^0 = c_2, \ldots, x_n^0 = c_n$, then the approximations $x_1^k, x_2^k, \ldots, x_n^k$ can be completely determined. Finally, we approximate the solution $x_i(t) = \lim_{k\to\infty} x_i^k(t)$ by the Nth term $x_i^N(t)$, for $i = 1, 2, \ldots, n$.

5. Linear systems of ordinary differential equations

In this section we apply the variational iteration method and the decomposition method on systems of linear differential equations off the form

$$x_1'(t) = \sum_{j=1}^{n} a_{1j}(t)x_j + g_1(t),$$

$$x_2'(t) = \sum_{j=1}^{n} a_{2j}(t)x_j + g_2(t),$$

$$\vdots$$

$$x_n'(t) = \sum_{j=1}^{n} a_{nj}(t)x_j + g_n(t). \tag{5.1}$$

If we set $\alpha_i = 1$, for $i = 1, 2, \ldots, n$, in the recurrence relation (3.6), then the Nth term approximate solution for the system of linear differential equations (5.1) using the decomposition method is given by

$$\phi_i^N(t) = \sum_{k=0}^{N-1} x_i^k(t), \tag{5.2}$$

where

$$x_i^0(t) = c_i + \int_0^t g_i(\tau)\,d\tau,$$

$$x_i^{k+1}(t) = \int_0^t \sum_{j=1}^{n} a_{ij}(\tau)x_j^k(\tau)\,d\tau. \tag{5.3}$$

In view of the iteration formulas (4.5), when $\alpha_i = 1$, for $i = 1, 2, \ldots, n$, the Nth term approximate solution for the system of linear differential equations (5.1) using the VIM is given by

$$x_i^{k+1}(t) = x_i^k(t) - \int_0^t \left(x_i'^k(\tau) - \sum_{j=1}^n a_{ij}(\tau) x_j^k(\tau) - g_i(\tau) \right) \mathrm{d}\tau. \tag{5.4}$$

Now, if we start with the initial approximation $x_i^0(t) = c_i + \int_0^t g_i(\tau)\,\mathrm{d}\tau$ then recursively, according to (5.4), we get the following approximations:

$$\begin{aligned}
x_i^1(t) &= c_i + \int_0^t g_i(\tau)\,\mathrm{d}\tau + \int_0^t \sum_{j=1}^n a_{ij}(\tau) x_j^0(\tau)\,\mathrm{d}\tau, \\
x_i^2(t) &= c_i + \int_0^t g_i(\tau)\,\mathrm{d}\tau + \int_0^t \sum_{j=1}^n a_{ij}(\tau) x_j^0(\tau)\,\mathrm{d}\tau + \int_0^t \sum_{j=1}^n a_{ij}(\tau) x_j^1(\tau)\,\mathrm{d}\tau, \\
&\vdots \\
x_i^N(t) &= c_i + \int_0^t g_i(\tau)\,\mathrm{d}\tau + \sum_{k=0}^{N-1} \int_0^t \sum_{j=1}^n a_{ij}(\tau) x_j^k(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{5.5}$$

It is clear that the Nth term approximate solution $x_i^N(t)$ for system (5.1) obtained using the VIM is the same approximate solution $\phi_i^N(t)$ obtained using the decomposition method. Therefore, if we start with the initial approximation $x_i^0(t) = c_i + \int_0^t g_i(\tau)\,\mathrm{d}\tau$ in the VIM, then the two methods produce the same approximate solution and they are equivalent for linear systems of ordinary differential equations.

6. Applications

To incorporate our discussion above, four special cases of the fractional system of differential equations (1.1) will be studied. In the first and second examples, we consider linear systems of ordinary and fractional differential equations, respectively, while in the third and fourth examples, we consider nonlinear systems of fractional differential equations. All the results are calculated by using the symbolic calculus software Mathematica.

Example 6.1. Consider the linear system of ordinary differential equations

$$\begin{aligned}
x'(t) &= y(t), \\
y'(t) &= 2x(t) - y(t),
\end{aligned} \tag{6.1}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = -1. \tag{6.2}$$

According to (5.3) or (5.4) the kth term approximate solutions using the decomposition method or the VIM for system (6.1) are given by

$$\begin{aligned}
x^k(t) &= 1 + \sum_{i=0}^{k-1} \int_0^t y^k(\tau)\,\mathrm{d}\tau, \\
y^k(t) &= -1 + \sum_{k=0}^{N-1} \int_0^t (2x^k(\tau) - y^k(\tau))\,\mathrm{d}\tau,
\end{aligned} \tag{6.3}$$

where $x^0(t) = 1$ and $y^0(t) = -1$.

Consequently, we obtain the following approximations

$$x^1 = 1 - t,$$
$$y^1 = -1 + 3t,$$
$$x^2 = 1 - t + \tfrac{3}{2}t^2,$$
$$y^2 = -1 + 3t - \tfrac{5}{2}t^2,$$
$$x^3 = 1 - t + \tfrac{3}{2}t^2 - \tfrac{5}{6}t^3,$$
$$y^3 = -1 + 3t - \tfrac{5}{2}t^2 + \tfrac{11}{6}t^3,$$
$$x^4 = 1 - t + \tfrac{3}{2}t^2 - \tfrac{5}{6}t^3 + \tfrac{11}{24}t^4,$$
$$y^4 = -1 + 3t - \tfrac{5}{2}t^2 + \tfrac{11}{6}t^3 - \tfrac{21}{24}t^4,$$
$$x^5 = 1 - t + \tfrac{3}{2}t^2 - \tfrac{5}{6}t^3 + \tfrac{11}{24}t^4 - \tfrac{21}{120}t^5,$$
$$y^5 = -1 + 3t - \tfrac{5}{2}t^2 + \tfrac{11}{6}t^3 - \tfrac{21}{24}t^4 + \tfrac{43}{120}t^5,$$
$$\vdots$$

and so on, in this manner the rest of components of the approximate solution for system (6.1) using the VIM and the decomposition method can be obtained.

The solution in series form is given by

$$\begin{aligned} x(t) &= 1 - t + \frac{3}{2}t^2 - \frac{5}{6}t^3 + \frac{11}{24}t^4 - \frac{21}{120}t^5 + \cdots, \\ &= \frac{2}{3}\left(1 - 2t + \frac{(-2t)^2}{2!} + \frac{(-2t)^3}{3!} + \frac{(-2t)^4}{4!} + \frac{(-2t)^5}{5!} + \cdots\right) \\ &\quad + \frac{1}{3}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right), \end{aligned} \tag{6.4}$$

$$\begin{aligned} y(t) &= -1 + 3t - \frac{5}{2}t^2 + \frac{11}{6}t^3 - \frac{21}{24}t^4 + \frac{43}{120}t^5 + \cdots, \\ &= -\frac{4}{3}\left(1 - 2t + \frac{(-2t)^2}{2!} + \frac{(-2t)^3}{3!} + \frac{(-2t)^4}{4!} + \frac{(-2t)^5}{5!} + \cdots\right) \\ &\quad + \frac{1}{3}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \cdots\right), \end{aligned} \tag{6.5}$$

which converges to the exact solution

$$\begin{aligned} x(t) &= \tfrac{2}{3}e^{-2t} + \tfrac{1}{3}e^{t}, \\ y(t) &= -\tfrac{4}{3}e^{-2t} + \tfrac{1}{3}e^{t}. \end{aligned} \tag{6.6}$$

Example 6.2. Consider the linear system of fractional differential equations

$$D_*^{\alpha_1} x(t) = x(t) + y(t),$$
$$D_*^{\alpha_2} y(t) = -x(t) + y(t), \tag{6.7}$$

subject to the initial conditions

$$x(0) = 0, \quad y(0) = 1. \tag{6.8}$$

According to the formulas (4.5), the iteration formulas for system (6.7) are given by

$$x^{k+1}(t) = x^k(t) - \int_0^t (D_*^{\alpha_1} x^k(\tau) - x^k(\tau) - y^k(\tau))\, d\tau,$$
$$y^{k+1}(t) = y^k(t) - \int_0^t (D_*^{\alpha_2} y^k(\tau) + x^k(\tau) - y^k(\tau))\, d\tau. \tag{6.9}$$

By the above variational iteration formulas, begin with $x^0(t) = 0$ and $y^0(t) = 1$, we can obtain the following approximations

$$x^1 = t,$$
$$y^1 = 1 + t,$$
$$x^2 = 2t + t^2 - \frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)},$$
$$y^2 = 1 + 2t - \frac{t^{2-\alpha_2}}{\Gamma(3-\alpha_2)},$$
$$x^3 = 3t + 3t^2 + \frac{t^3}{3} - 3\frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)} - 3\frac{t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + \frac{t^{3-2\alpha_1}}{\Gamma(4-2\alpha_1)} - \frac{t^{3-\alpha_2}}{\Gamma(4-\alpha_2)},$$
$$y^3 = 1 + 3t - \frac{t^3}{3} - 3\frac{t^{2-\alpha_2}}{\Gamma(3-\alpha_2)} - \frac{t^{3-\alpha_2}}{\Gamma(4-\alpha_2)} + \frac{t^{3-2\alpha_2}}{\Gamma(4-2\alpha_2)} + \frac{t^{3-\alpha_1}}{\Gamma(4-\alpha_1)},$$
$$x^4 = 4t + 6t^2 + \frac{4t^3}{3} - 6\frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)} - 12\frac{t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} - 4\frac{t^{4-\alpha_1}}{\Gamma(5-\alpha_1)} + 4\frac{t^{3-2\alpha_1}}{\Gamma(4-2\alpha_1)} + 4\frac{t^{4-2\alpha_1}}{\Gamma(5-2\alpha_1)}$$
$$- \frac{t^{4-3\alpha_1}}{\Gamma(5-3\alpha_1)} - 4\frac{t^{3-\alpha_2}}{\Gamma(4-\alpha_2)} - 2\frac{t^{4-\alpha_2}}{\Gamma(5-\alpha_2)} + \frac{t^{4-2\alpha_2}}{\Gamma(5-2\alpha_2)} + \frac{t^{4-\alpha_1-\alpha_2}}{\Gamma(5-\alpha_1-\alpha_2)},$$
$$y^4 = 1 + 4t - \frac{4t^3}{3} - \frac{t^4}{6} - 6\frac{t^{2-\alpha_2}}{\Gamma(3-\alpha_2)} - 4\frac{t^{3-\alpha_2}}{\Gamma(4-\alpha_2)} + 2\frac{t^{4-\alpha_2}}{\Gamma(5-\alpha_2)} + 4\frac{t^{3-2\alpha_2}}{\Gamma(4-2\alpha_2)} + 2\frac{t^{4-2\alpha_2}}{\Gamma(5-2\alpha_2)}$$
$$- \frac{t^{4-3\alpha_2}}{\Gamma(5-3\alpha_2)} + 4\frac{t^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + 4\frac{t^{4-\alpha_1}}{\Gamma(5-\alpha_1)} - \frac{t^{4-2\alpha_1}}{\Gamma(5-2\alpha_1)} - \frac{t^{4-\alpha_1-\alpha_2}}{\Gamma(5-\alpha_1-\alpha_2)},$$
$$\vdots$$

To solve the problem using the decomposition method, we simply substitute (6.7) and the initial conditions (6.8) into (3.6), to obtain the following recurrence relations

$$x^0(t) = 0, \quad x^{k+1}(t) = J^{\alpha_1}(x^k + y^k), \quad k \geqslant 0,$$
$$y^0(t) = 1, \quad y^{k+1}(t) = J^{\alpha_2}(-x^k + y^k), \quad k \geqslant 0. \tag{6.10}$$

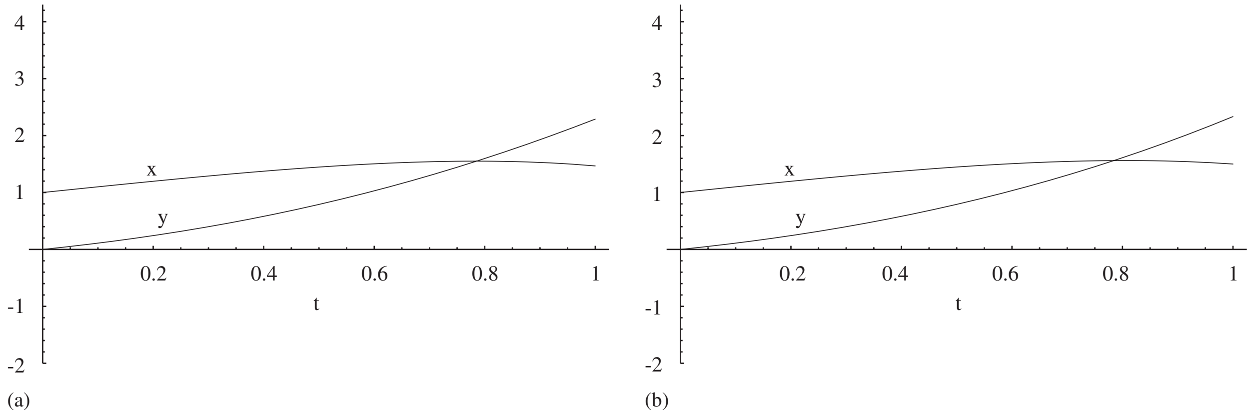

Fig. 1. Plots of system (6.7) when $\alpha_1 = 1.0$ and $\alpha_2 = 1.0$: (a) ADM; (b) VIM.

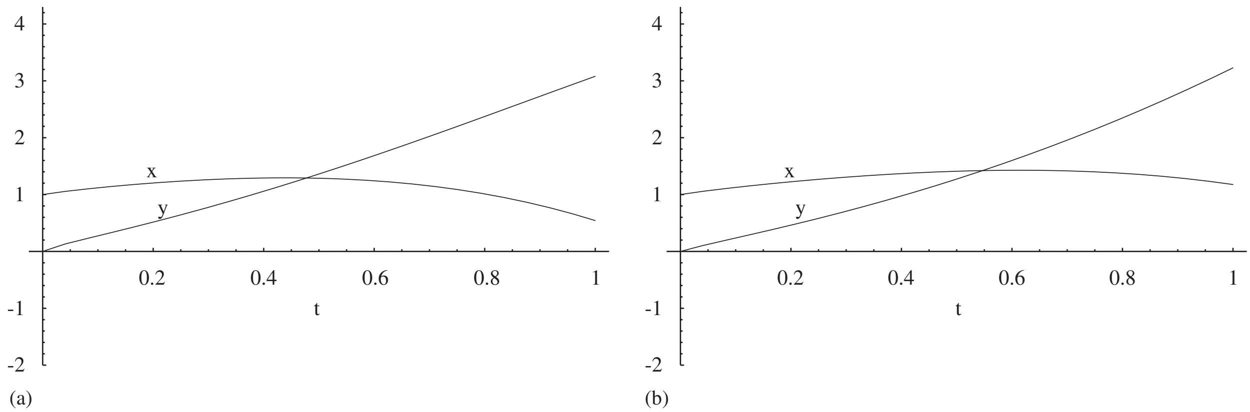

Fig. 2. Plots of system (6.7) when $\alpha_1 = 0.7$ and $\alpha_2 = 0.9$: (a) ADM; (b) VIM.

In view of (6.10), the first few terms of the decomposition series are given by

$$\begin{aligned}
x(t) &= \frac{t^{\alpha_1}}{\alpha_1 \Gamma(\alpha)} + \frac{t^{2\alpha_1}}{\Gamma(1+2\alpha_1)} + \frac{t^{3\alpha_1}}{\Gamma(1+3\alpha_1)} + \frac{t^{4\alpha_1}}{\Gamma(1+4\alpha_1)} + \frac{t^{5\alpha_1}}{\Gamma(1+5\alpha_1)} \\
&\quad + \frac{t^{6\alpha_1}}{\Gamma(1+6\alpha_1)} + \frac{t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} - \frac{t^{3\alpha_1+\alpha_2}}{\Gamma(1+3\alpha_1+\alpha_2)} + \cdots, \\
y(t) &= 1 + \frac{t^{\alpha_2}}{\Gamma(1+\alpha_2)} - \frac{t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} - \frac{t^{2\,\alpha_1+\alpha_2}}{\Gamma(1+2\,\alpha_1+\alpha_2)} - \frac{t^{3\,\alpha_1+\alpha_2}}{\Gamma(1+3\,\alpha_1+\alpha_2)} \\
&\quad - \frac{t^{4\,\alpha_1+\alpha_2}}{\Gamma(1+4\,\alpha_1+\alpha_2)} - \frac{t^{5\,\alpha_1+\alpha_2}}{\Gamma(1+5\,\alpha_1+\alpha_2)} + \frac{t^{2\,\alpha_2}}{\Gamma(1+2\,\alpha_2)} + \cdots\ .
\end{aligned}$$

Figs. 1 and 2 show the approximate solutions for system (6.7) obtained for different values of α_1 and α_2 using the ADM and the VIM. The values of $\alpha_1 = \alpha_2 = 1$ is the only case for which we know the exact solution ($x(t) = \mathrm{e}^t \sin t,\ \ y(t) = \mathrm{e}^t \cos t$) and our approximate solutions using the two methods are in good agreement with the exact solution. It is to be noted that only the fifth-order term of the variational iteration solution and only five terms of the decomposition series were used in evaluating the approximate solutions for Figs. 1 and 2. It is evident that the efficiency of these approaches can be dramatically enhanced by computing further terms or further components of $x(t)$, $y(t)$ when the VIM or the decomposition method are used. From the numerical results in Figs. 1 and 2, it is easy to conclude that the solution continuously depends on the time-fractional derivatives.

Example 6.3. Consider the nonlinear fractional predator–prey system

$$\begin{aligned} &D_*^{\alpha_1}x(t) = x(t) - x(t)y(t), \\ &D_*^{\alpha_2}y(t) = -y(t) + x(t)y(t), \end{aligned} \tag{6.11}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 0.5. \tag{6.12}$$

According to the formulas (4.5), the iteration formulas for system (6.11) are given by

$$\begin{aligned} &x^{k+1}(t) = x^k(t) - \int_0^t (D_*^{\alpha_1}x^k(\tau) - x^k(\tau) + x^k(\tau)y^k(\tau))\,\mathrm{d}\tau, \\ &y^{k+1}(t) = y^k(t) - \int_0^t (D_*^{\alpha_2}y^k(\tau) + y^k(\tau) - x^k(\tau)y^k(\tau))\,\mathrm{d}\tau. \end{aligned} \tag{6.13}$$

By the above variational iteration formulas, begin with $x^0(t) = 1$ and $y^0(t) = 0.5$, we can obtain the following approximations

$$\begin{aligned} &x^1 = 1 + \frac{t}{2}, \\ &y^1 = \tfrac{1}{2}, \\ &x^2 = 1 + t + \frac{t^2}{8} - \frac{t^{2-\alpha_1}}{2\Gamma(3-\alpha_1)}, \\ &y^2 = \frac{1}{2} + \frac{t^2}{8}, \\ &x^3 = 1 + \frac{3t}{2} + \frac{3t^2}{8} - \frac{t^3}{48} - \frac{t^4}{32} - \frac{t^5}{320} - \frac{3t^{2-\alpha_1}}{2\Gamma(3-\alpha_1)} - \frac{t^{3-\alpha_1}}{2\Gamma(4-\alpha_1)} + \frac{t^{3-2\alpha_1}}{2\Gamma(4-2\alpha_1)} + \frac{\Gamma(5-\alpha_1)t^{5-\alpha_1}}{16\Gamma(3-\alpha_1)\Gamma(6-\alpha_1)}, \\ &y^3 = \frac{1}{2} + \frac{3t^2}{8} + \frac{t^3}{48} + \frac{t^4}{32} + \frac{t^5}{320} - \frac{t^{3-\alpha_2}}{4\Gamma(4-\alpha_2)} - \frac{t^{3-\alpha_1}}{4\Gamma(4-\alpha_1)} - \frac{\Gamma(5-\alpha_1)t^{5-\alpha_1}}{16\Gamma(3-\alpha_1)\Gamma(6-\alpha_1)}, \\ &\vdots \end{aligned}$$

To solve the problem using the decomposition method, we substitute (6.11) and the initial conditions (6.12) into (3.6), we obtain the recurrence relations

$$\begin{aligned} &x^0(t) = 1, \\ &x^{k+1}(t) = J^{\alpha_1}(x^k - A^k), \quad k \geqslant 0, \end{aligned} \tag{6.14}$$

$$\begin{aligned} &y^0(t) = \tfrac{1}{2}, \\ &y^{k+1}(t) = J^{\alpha_2}(-y^k + A^k), \quad k \geqslant 0, \end{aligned} \tag{6.15}$$

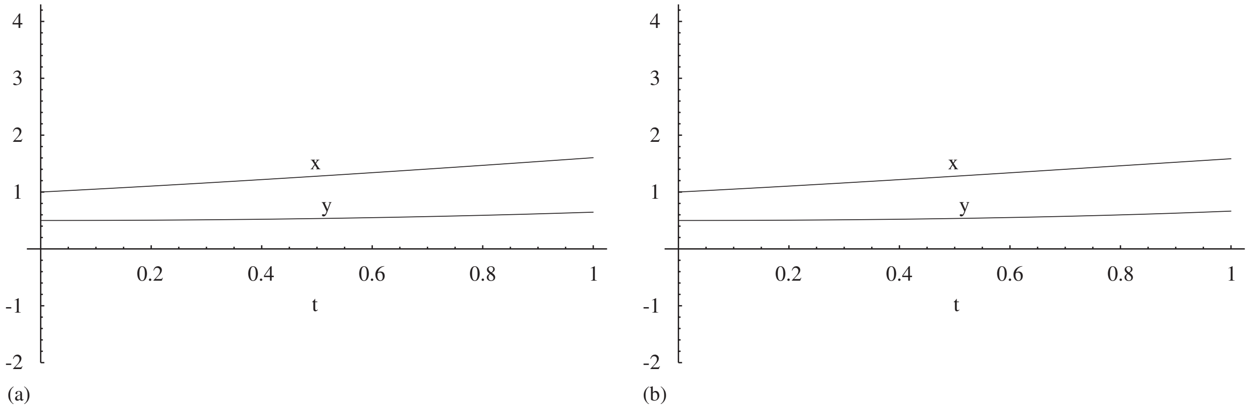

Fig. 3. Plots of system (6.11) when $\alpha_1 = 1.0$ and $\alpha_2 = 1.0$: (a) ADM; (b) VIM.

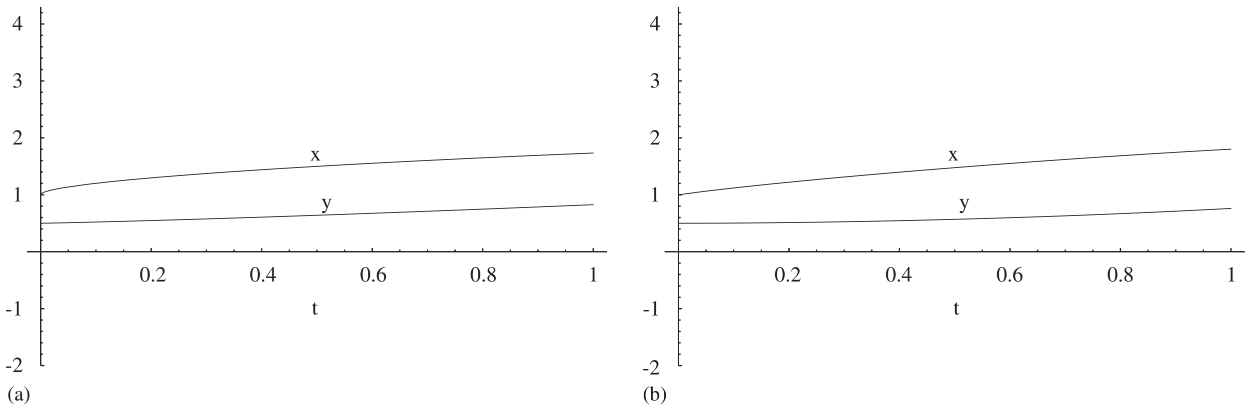

Fig. 4. Plots of system (6.11) when $\alpha_1 = 0.5$ and $\alpha_2 = 0.6$: (a) ADM; (b) VIM.

where the Adomian polynomials for the nonlinearity $g(x, y) = xy$ are

$$\begin{aligned}
&A^0 = x^0 y^0,\\
&A^1 = x^0 y^1 + x^1 y^0,\\
&A^2 = x^0 y^2 + x^1 y^1 + x^2 y^0,\\
&A^2 = x^0 y^3 + x^1 y^2 + x^2 y^1 + x^3 y^0.
\end{aligned} \tag{6.16}$$

Using the above recursive relationship and Mathematica, the first four terms of the decomposition series are given by

$$\begin{aligned}
x(t) &= 1.0 + \frac{0.5}{\Gamma(1+\alpha_1)} t^{\alpha_1} + \frac{4^{-\alpha_1}(0.443113)}{\Gamma(1+\alpha_1)\Gamma(0.5+\alpha_1)} t^{2\alpha_1} + \cdots,\\
y(t) &= 0.5 + \frac{4^{-\alpha_2}}{\Gamma(1+\alpha_2)\Gamma(0.5+\alpha_2)} t^{2\alpha_2} + \frac{0.25}{\Gamma(1+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2} + \cdots.
\end{aligned} \tag{6.17}$$

Setting $\alpha_1 = \alpha_2 = \alpha$ into (6.17), we obtain the solution obtained by Momani and Qaralleh [42] which corresponds to a system of fractional differential equations of single order. Figs. 3 and 4 show the approximate solutions for system (6.11) obtained for different values of α_1 and α_2 using the decomposition method and the VIM. As per the previous example, the approximate solution obtained using the VIM is in good agreement with the approximate solution obtained using the decomposition method for all values of α_1 and α_2.

Example 6.4. Consider the following system of nonlinear fractional differential equations

$$\begin{aligned} &D_*^{\alpha_1} x = 2y^2, \quad 0 < \alpha_1 \leqslant 1, \\ &D_*^{\alpha_2} y = tx, \quad 0 < \alpha_2 \leqslant 1, \\ &D_*^{\alpha_3} z = yz, \quad 0 < \alpha_3 \leqslant 1, \end{aligned} \tag{6.18}$$

subject to the initial conditions

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = 1. \tag{6.19}$$

According to formulas (4.5), the iteration formulas for system (6.18) are given by

$$\begin{aligned} x^{k+1}(t) &= x^k(t) - \int_0^t (D_*^{\alpha_1} x^k(\tau) - 2(y^k)^2(\tau))\,\mathrm{d}\tau, \\ y^{k+1}(t) &= y^k(t) - \int_0^t (D_*^{\alpha_2} y^k(\tau) - \tau x^k(\tau))\,\mathrm{d}\tau, \\ z^{k+1}(t) &= z^k(t) - \int_0^t (D_*^{\alpha_3} z^k(\tau) - y^k(\tau) z^k(\tau))\,\mathrm{d}\tau. \end{aligned} \tag{6.20}$$

By the above variational iteration formulas, begin with $x^0(t)=0$, $y^0(t)=1$ and $z^0(t)=1$, we can obtain the following approximations

$$\begin{aligned} x^1 &= 2t, \\ y^1 &= 1, \\ z^1 &= 1 + t, \\ x^2 &= 4t - 2\,\frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)}, \\ y^2 &= 1 + \frac{2t^3}{3}, \\ z^2 &= 1 + 2t + \frac{t^2}{2} - \frac{t^{2-\alpha_3}}{\Gamma(3-\alpha_3)}, \\ x^3 &= 6t + \frac{2t^4}{3} + \frac{8t^7}{63} - 6\,\frac{t^{2-\alpha_1}}{\Gamma(3-\alpha_1)} + 2\,\frac{t^{3-2\alpha_1}}{\Gamma(4-2\alpha_1)}, \\ y^3 &= 1 + 2t^3 - 4\,\frac{t^{4-\alpha_2}}{\Gamma(5-\alpha_2)} - 2\,\frac{t^{4-\alpha_1}}{(4-\alpha_1)\Gamma(3-\alpha_1)}, \\ z^3 &= 1 + 3t + \frac{3t^2}{2} + \frac{t^3}{6} + \frac{t^4}{6} + \frac{4t^5}{15} + \frac{t^6}{18} - 3\,\frac{t^{2-\alpha_3}}{\Gamma(3-\alpha_3)} - 2\,\frac{t^{3-\alpha_3}}{\Gamma(4-\alpha_3)} + \frac{t^{3-2\alpha_3}}{\Gamma(4-2\alpha_3)} \\ &\quad - 2\,\frac{\Gamma(6-\alpha_3)t^{6-\alpha_3}}{3\Gamma(3-\alpha_3)\Gamma(7-\alpha_3)}. \end{aligned}$$

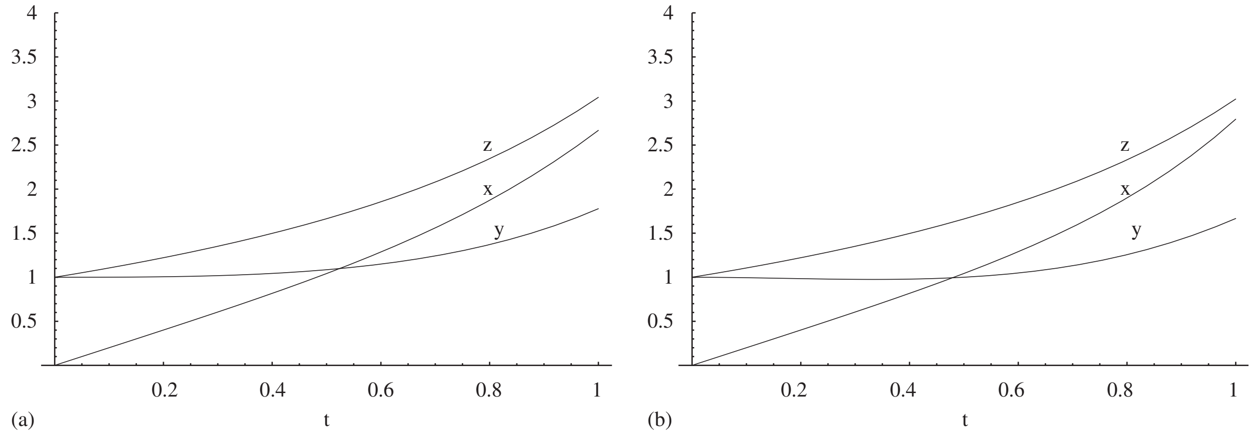

Fig. 5. Plots of system (6.18) when $\alpha_1 = 1.0$, $\alpha_2 = 1.0$ and $\alpha_3 = 1$: (a) ADM; (b) VIM.

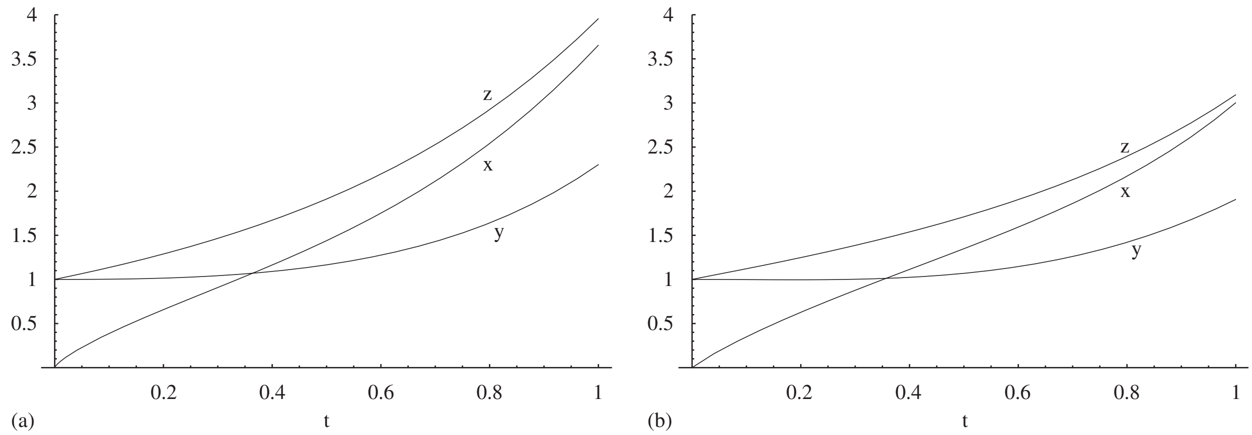

Fig. 6. Plots of system (6.18) when $\alpha_1 = 0.75$, $\alpha_2 = 0.85$ and $\alpha_3 = 0.95$: (a) ADM; (b) VIM.

To solve the problem using the decomposition method, we substitute (6.18) and the initial conditions (6.19) into (3.6), we obtain the recurrence relations

$$x^0 = 0, \quad x^{k+1} = 2J^{\alpha_1}(B^k), \quad k \geqslant 0,$$
$$y^0 = 1, \quad y^{k+1} = J^{\alpha_2}(tx^k), \quad k \geqslant 0,$$
$$z^0 = 1, \quad z^{k+1} = J^{\alpha_3}(C^k), \quad k \geqslant 0,$$

where $y^2 = \sum_{k=0}^{\infty} B^k$, $yz = \sum_{k=0}^{\infty} C^k$ and the B^k and C^k are the appropriate Adomian polynomials generated for the specific nonlinearities in this system.

The solution in a series form is given by

$$x(t) = \frac{2t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \cdots,$$
$$y(t) = 1 + \frac{2(1+\alpha+1)t^{1+\alpha_1+\alpha_2}}{\Gamma(2+\alpha_1+\alpha_2)} + \cdots,$$
$$z(t) = 1 + \frac{t^{\alpha_3}}{\Gamma(1+\alpha_3)} + \frac{4^{-\alpha_3}\sqrt{\pi}t^{2\alpha_3}}{\Gamma(1+\alpha_3)\Gamma(\frac{1}{2}+\alpha_3)} + \cdots. \tag{6.21}$$

In particular, when $(\alpha, \beta, \gamma)^t = (0.5, 0.4, 0.3)^t$, the solution (6.21) reduces to the solution obtained in [10] by using an iterative method and by Momani and Qarralleh [42] using the decomposition method. Figs. 5 and 6 show the approximate solutions for Eq. (6.21) obtained for different values of α_1, α_2 and α_3 using the decomposition method and the VIM.

7. Concluding remarks

The fundamental goal of this work has been to construct an approximate solution of linear and nonlinear systems of differential equations of fractional order. The goal has been achieved by using the variational iteration method (VIM) and the Adomian decomposition method (ADM). The methods were used in a direct way without using linearization, perturbation or restrictive assumptions.

There are six important points to make here. First, the VIM and the decomposition method provide the solutions in terms of convergent series with easily computable components. Second, it is clear and remarkable that the approximate solutions in all examples using the two methods are in good agreement. Third, the approximate solutions obtained using the VIM are exactly the same as those obtained by using the decomposition method for linear systems of ordinary differential equations. Fourth, the VIM is more effective and overcomes the difficulty arising in calculating Adomian polynomials. Fifth, the two techniques require less computational work than existing approaches while supplying quantitatively reliable results. It is also shown that the solutions of the fractional equations reduces to the solutions of the corresponding integer order equations.

Finally, the recent appearance of fractional differential equations as models in some fields of applied mathematics makes it necessary to investigate methods of solution for such equations (analytical and numerical) and we hope that this work is a step in this direction.

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