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Convergence behaviors of multisplitting methods with $K + 1$ relaxed parameters

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ABSTRACT

In this paper, two multisplitting methods with $K + 1$ relaxed parameters are established for solving a linear system whose coefficient matrix is a large sparse M-matrix or H-matrix and the corresponding convergence behaviors are studied. Then the implementation of these two methods with ILU factorizations as inner splittings is investigated. Finally, some numerical experiments are presented to illustrate the effectiveness of the preconditioners obtained from our methods when combined with BiCGSTAB.

1. Introduction

Consider the numerical solution of the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a known nonsingular matrix, $b \in \mathbb{R}^n$ is known and $x \in \mathbb{R}^n$ is unknown. O'leary and White have introduced the parallel multisplitting iterative method for obtaining the solution to (1.1) in [1], where several basic convergence results may be found. Neumann and Plemmons [2] developed some more refined convergence results for one of the cases considered in [1]. It has already been observed in [1] that the introduction of a relaxed parameter may considerably improve the multisplitting methods, but convergence results were not given for these modifications of multisplitting methods by the authors. So, Frommer and Mayer studied two relaxed variants of multisplitting methods and established the convergence of these methods under certain restrictions on a relaxed parameter and on the underlying multisplittings in [3]. The multisplitting method was also further studied by many authors [4–7].

In this paper, we will establish two multisplitting methods with $K + 1$ relaxed parameters for solving large nonsingular systems of Eq. (1.1), in which the coefficient matrix A is an M-matrix or an H-matrix, and we will study the convergence of these methods under certain restrictions on the relaxed parameters and on the underlying multisplittings. In this manner we obtain convergence results including and extending the convergence results which have been considered in the literature before.

This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results which we refer to later and establish two multisplitting methods with $K+1$ relaxed parameters. In Section 3, we present convergence results of the $K + 1$ parameter multisplitting methods on M-matrices and H-matrices. Some numerical examples are given in Section 4. Finally, conclusions are drawn in Section 5.

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2. Notation and preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} |a_{ij}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases} \quad i, j = 1, \ldots, n.$$

A matrix A is called an H-matrix if $\langle A\rangle$ is an M-matrix. A splitting $A = M - N$ of A is said to be regular if $M^{-1} \geq 0$ and $N \geq 0$; weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$; H-splitting if the matrix $\langle M\rangle - |N|$ is monotone; H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$; convergent if $\rho(M^{-1}N) < 1$, where $\rho(\cdot)$ denotes the spectral radius of a square matrix $\cdot$. It was shown in [8] that if A is an H-matrix and $A = M - N$ is an H-compatible splitting, then M is also an H-matrix. By $\mathrm{diag}(A)$ we denote the $n \times n$ diagonal matrix coinciding in its diagonal with $n \times n$ matrix A.

Definition 2.1 (*[1]*). Let A be a nonsingular real $n \times n$ matrix, and suppose that for some $K \in \mathbb{N}$ we are given matrices $M_k, N_k, E_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, K$, satisfying (i) $A = M_k - N_k$, (ii) M_k is nonsingular, (iii) E_k is a diagonal matrix with nonnegative entries and $\sum_{k=1}^{k=K} E_k = I$, where I is $n \times n$ identity matrix. Then a set $\{M_k, N_k, E_k\}_{k=1}^{k=K}$ is called a multisplitting of A. The corresponding multisplitting method to solve $Ax = b$ is defined by the iteration

$$x^{(i+1)} = \sum_{k=1}^{k=K} E_k M_k^{-1} N_k x^{(i)} + \sum_{k=1}^{k=K} E_k M_k^{-1} b, \quad i = 0, 1, 2, \ldots. \tag{2.1}$$

Putting

$$T = \sum_{k=1}^{k=K} E_k M_k^{-1} N_k, \qquad G = \sum_{k=1}^{k=K} E_k M_k^{-1},$$

T is called the iteration matrix.

In the following, we will give the relaxed nonstationary multisplitting method associated with this multisplitting and a positive relaxed parameter ω for solving a linear system $Ax = b$.

Algorithm 2.1 (*[10]*). Relaxed nonstationary multisplitting method

Given an initial vector $x^{(0)}$
For $i = 1, 2, \ldots$, until convergence
  For $k = 1$ to K
    $y^{(k,0)} = x^{(i-1)}$
    For $j = 1$ to $s(k, i)$
      $M_k y^{(k,j)} = N_k y^{(k,j-1)} + b$
  $x^{(i)} = \omega \sum_{k=1}^{k=K} E_k y^{(k,s(k,i))} + (1 - \omega) x^{(i-1)}$.

Remark 2.1. Notice that Algorithm 2.1 with $\omega = 1$ is called the nonstationary multisplitting method. Mas et al. [10] showed the convergence of Algorithm 2.1 under the condition when A is an H-matrix. When $\{M_k, N_k, E_k\}_{k=1}^{k=K}$ is a multisplitting of A and $M_k = B_k - C_k$ is a splitting of M_k for each k, the relaxed nonstationary two-stage multisplitting method with a positive relaxed parameter ω for solving a linear system $Ax = b$ is as follows.

Algorithm 2.2 (*[11]*). Relaxed nonstationary two-stage multisplitting method

Given an initial vector $x^{(0)}$
For $i = 1, 2, \ldots$, until convergence
  For $k = 1$ to K
    $y^{(k,0)} = x^{(i-1)}$
    For $j = 1$ to $s(k, i)$
      $y^{(k,j)} = \omega B_k^{-1}(C_k y^{(k,j-1)} + N_k x^{(i-1)} + b) + (1 - \omega) y^{(k,j-1)}$
  $x^{(i)} = \sum_{k=1}^{k=K} E_k y^{(k,s(k,i))}$.

Remark 2.2. In Algorithm 2.2, the splittings $A = M_k - N_k$ are called outer splittings and the splittings $M_k = B_k - C_k$ are called inner splittings. Bru et al. [11] showed the convergence of Algorithm 2.2 when A is a monotone matrix (i.e., $A^{-1} \geq 0$) or A is an H-matrix. If $\omega = 1$ in Algorithm 2.2, then Algorithm 2.2 reduces to the nonstationary two-stage multisplitting method. Observe that the loop k of Algorithms 2.1 and 2.2 can be executed completely in parallel by different processors. Also notice that the number of inner iterations $s(k, i)$ in Algorithms 2.1 and 2.2 depends on the iteration i and the splitting $A = M_k - N_k$. If $s(k, i) = 1$ for all k and i in Algorithm 2.1, Algorithm 2.1 is called the relaxed multisplitting method.

From Algorithms 2.1 and 2.2, the different splittings have the same parameter ω. In this case, the parameter ω is difficult to choose. For solving this problem, we establish the multisplitting methods with $K + 1$ relaxed parameters as follows.

Algorithm 2.3. Nonstationary multisplitting method with $K + 1$ relaxed parameters

Given an initial vector $x^{(0)}$
For $i = 1, 2, \ldots$, until convergence
  For $k = 1$ to K
    $y^{(k,0)} = x^{(i-1)}$
    For $j = 1$ to $s(k, i)$
      $y^{(k,j)} = \omega_k M_k^{-1} N_k y^{(k,j-1)} + (1-\omega_k) y^{(k,j-1)} + \omega_k M_k^{-1} b$
  $x^{(i)} = \omega \sum_{k=1}^{k=K} E_k y^{(k,s(k,i))} + (1-\omega) x^{(i-1)}$.

Remark 2.3. Notice that Algorithm 2.3, when $\omega_k = 1, k = 1, \ldots, K$, reduces to Algorithm 2.1.

Algorithm 2.4. Nonstationary two-stage multisplitting method with $K + 1$ relaxed parameters

Given an initial vector $x^{(0)}$
For $i = 1, 2, \ldots$, until convergence
  For $k = 1$ to K
    $y^{(k,0)} = x^{(i-1)}$
    For $j = 1$ to $s(k, i)$
      $y^{(k,j)} = \omega_k B_k^{-1} (C_k y^{(k,j-1)} + N_k x^{(i-1)} + b) + (1-\omega_k) y^{(k,j-1)}$
  $x^{(i)} = \omega \sum_{k=1}^{k=K} E_k y^{(k,s(k,i))} + (1-\omega) x^{(i-1)}$.

Remark 2.4. Notice that Algorithm 2.4, when $\omega_1 = \cdots = \omega_K$ and $\omega = 1$, reduces to Algorithm 2.2.
Now we give some well-known results which will be used later.

Lemma 2.1 ([3]). *Let $A = D - B$ be an H-matrix with $D = \mathrm{diag}(A)$. Then*

(1) *A and D are nonsingular and $\rho(|D|^{-1}B) < 1$;*
(2) *$|A^{-1}| \le \langle A \rangle^{-1}$.*

Lemma 2.2 ([9]). *Let $A, B \in \mathbb{R}^{n\times n}$ such that $|A| \le B$. Then $\rho(A) \le \rho(B)$.*

Lemma 2.3 ([12]). *Let A be a nonnegative matrix. Then*

(1) *If $\alpha x \le Ax$ for some nonnegative vector x, $x \ne 0$, then $\alpha \le \rho(A)$;*
(2) *If $Ax \le \beta x$ for some positive vector x, then $\rho(A) \le \beta$. Moreover, if A is irreducible and if $0 \ne \alpha x \le Ax \le \beta x$, $\alpha \ne Ax$ and $Ax \ne \beta x$ for some nonnegative vector x, then*

$$\alpha < \rho(A) < \beta$$

and x is a positive vector.

Lemma 2.4 ([8]). *Let $A = M - N$ be a splitting of A.*

(i) *If the splitting is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \le \rho(\langle M \rangle^{-1}|N|) < 1$.*
(ii) *If the splitting is an H-compatible splitting and A is an H-matrix, then it is an H-splitting and hence, a convergent splitting.*

Lemma 2.5 ([12]). *Let $A \ge 0$ be an irreducible $n \times n$ matrix. Then*

(1) *A has a positive real eigenvalue equal to its spectral radius;*
(2) *to $\rho(A)$ there corresponds an eigenvector $x > 0$;*
(3) *$\rho(A)$ is a simple eigenvalue of A.*

A general algorithm for building ILU factorization can be derived by performing Gaussian elimination and dropping some of the elements in predetermined off-diagonal positions. Let S_n denote the set of all pairs of indices of off-diagonal matrix entries, i.e.

$$S_n = \{(i, j) | i \ne j, 1 \le i, j \le n\}.$$

Lemma 2.6 ([13]). *Let A be an $n \times n$ H-matrix. Then, for every zero pattern set $Q \subset S_n$, there exist a unit lower triangular matrix $L = (l_{ij})$, an upper triangular matrix $U = (u_{ij})$, and a matrix $N = (n_{ij})$, with $l_{ij} = u_{ij} = 0$ if $(i, j) \notin Q$, such that $A = LU - N$. Moreover, the factors L and U are also H-matrices.*

Lemma 2.7 ([13,14]). *Let A be an $n \times nH$-matrix. Let $A = LU - N$ and $\langle A\rangle = \widetilde{L}\widetilde{U} - \widetilde{N}$ be the ILU factorizations of A and $\langle A\rangle$ corresponding to a zero pattern set $Q \subset S_n$, respectively. Then each of the following holds:*

$$(a)\ |L^{-1}| \le |\widetilde{L}|^{-1}, \qquad (b)\ |U^{-1}| \le \widetilde{U}^{-1}, \qquad (c)\ |N| \le \widetilde{N}, \qquad (d)\ |(LU)^{-1}N| \le (\widetilde{L}\widetilde{U})^{-1}\widetilde{N}.$$

3. Convergence behaviors of nonstationary multisplitting methods with $K + 1$ relaxed parameters

In this section, we present convergence results of Algorithms 2.3 and 2.4 when the coefficient matrices are M-matrices or H-matrices. First, we consider Algorithm 2.3. Let

$$R_{\omega_k} = \omega_k M_k^{-1} N_k + (1 - \omega_k)I,$$

then Algorithm 2.3 can be written as

$$x^{(i)} = H_{\omega,\omega_k,i}x^{(i-1)} + P_{\omega,\omega_k,i}b, \quad i = 1, 2, \ldots, \tag{3.1}$$

where

$$H_{\omega,\omega_k,i} = \omega \sum_{k=1}^{K} E_k R_{\omega_k}^{s(k,i)} + (1 - \omega)I, \quad i = 1, 2, \ldots \tag{3.2}$$

and

$$P_{\omega,\omega_k,i} = \omega \sum_{k=1}^{K} \omega_k E_k \left(\sum_{j=0}^{s(k,i)-1} R_{\omega_k}^{j} \right) M_k^{-1}, \quad i = 1, 2, \ldots. \tag{3.3}$$

The $H_{\omega,\omega_k,i}$'s are called iteration matrices for Algorithm 2.3. Then, it is easy to show that $P_{\omega,\omega_k,i}A = I - H_{\omega,\omega_k,i}$ for each i. Hence, the exact solution ξ of $Ax = b$ satisfies

$$\xi = H_{\omega,\omega_k,i}\xi + P_{\omega,\omega_k,i}b, \quad i = 1, 2, \ldots. \tag{3.4}$$

Theorem 3.1. *Let $A = D - B$ be an $n \times n$ H-matrix with $D = \operatorname{diag}(A)$ and $J = |D|^{-1}|B|$. Assume that the multisplitting $\{M_k, N_k, E_k\}_{k=1}^{k=K}$ is H-compatible and $\operatorname{diag}(|M_k|) \le |D|$. If*

$$0 < \omega < 2/(1 + \bar{\rho}) \quad \text{and} \quad 0 < \omega_k < 2/(1 + \rho), \quad k = 1, 2, \ldots, K,$$

where $\bar{\rho} = \max\{\omega_k\rho_\varepsilon + |1 - \omega_k||\rho_\varepsilon = \rho(J + \varepsilon ee^{\mathrm{T}}), k = 1, \ldots, K\}$, $\varepsilon \longrightarrow 0^+$ and $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$, $\rho = \rho(J)$, and $s(k, i) \ge 1$ for all nonnegative integers k and $1 \le i \le K$, then Algorithm 2.3 converges for all $x^{(0)} \in \mathbb{R}^n$ to ξ, the solution of the linear system $Ax = b$.

Proof. By Lemma 2.4, the matrices M_k are H-matrices. Then using Lemma 2.1, some manipulation yields

$$|M_k^{-1}N_k| \le I - \langle M_k\rangle|D|(I - J), \quad k = 1, 2, \ldots, K. \tag{3.5}$$

Consider the vector $e = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. Since $J = |D|^{-1}|B|$ is nonnegative, the matrix $J + \varepsilon ee^{\mathrm{T}}$ is irreducible for all $\varepsilon > 0$, then from Lemma 2.5, there exists a positive vector x_ε corresponding to the spectral radius such that

$$(J + \varepsilon ee^{\mathrm{T}})x_\varepsilon = \rho_\varepsilon x_\varepsilon, \tag{3.6}$$

where $\rho_\varepsilon = \rho(J + \varepsilon ee^{\mathrm{T}})$. The continuity of the spectral radius and Lemma 2.2 ensures that there exists ε_0 such that $\rho_\varepsilon < 1$, for all $0 < \varepsilon \le \varepsilon_0$. By $\langle M_k\rangle \le |D|$, $k = 1, 2, \ldots, K$, from (3.5) and (3.6) we obtain that

$$\begin{aligned}
|R_{\omega_k}|x_\varepsilon &= \left|\omega_k M_k^{-1}N_k + (1 - \omega_k)I\right| x_\varepsilon \\
&\le \omega_k|M_k^{-1}N_k|x_\varepsilon + |1 - \omega_k|x_\varepsilon \\
&\le \omega_k(x_\varepsilon - \langle M_k\rangle^{-1}|D|(1 - \rho_\varepsilon)x_\varepsilon) + |1 - \omega_k|x_\varepsilon \\
&\le (\omega_k\rho_\varepsilon + |1 - \omega_k|)x_\varepsilon \\
&\le \bar{\rho}x_\varepsilon,
\end{aligned} \tag{3.7}$$

since $0 < \omega_k < 2/(1+\rho)$, $k = 1, 2, \ldots, K$, and $\varepsilon_0 \longrightarrow 0$, then $\bar{\rho} < 1$. Hence, from (3.2) and (3.7), one obtains

$$\begin{aligned}
|H_{\omega,\omega_k,i}|x_\varepsilon &= \left|\omega \sum_{k=1}^{K} E_k R_{\omega_k}^{s(k,i)} + (1-\omega)I\right| x_\varepsilon \\
&\le \omega \sum_{k=1}^{K} E_k \left|R_{\omega_k}^{s(k,i)}\right| x_\varepsilon + |1-\omega| x_\varepsilon \\
&\le \omega \sum_{k=1}^{K} E_k \bar{\rho}^{s(k,i)} x_\varepsilon + |1-\omega| x_\varepsilon \\
&\le \omega \left|\sum_{k=1}^{K} E_k \bar{\rho}\right| x_\varepsilon + |1-\omega| x_\varepsilon \\
&= (\omega\bar{\rho} + |1-\omega|) x_\varepsilon \\
&< x_\varepsilon. \qquad (3.8)
\end{aligned}$$

Since $0 < \omega < 2/(1+\bar{\rho})$, according to Lemmas 2.2 and 2.3, we have $\rho(H_{\omega,\omega_k,i}) \le \rho(|H_{\omega,\omega_k,i}|) < 1$ for $i = 1, 2, \ldots$. The proof is complete. □

In what follows, we consider Algorithm 2.4. Let $R_{\omega_k}^{*} = \omega_k B_k^{-1} C_k + (1-\omega_k)I$, then Algorithm 2.4 can be written as

$$x^{(i)} = H_{\omega,\omega_k,i}^{*} x^{(i-1)} + P_{\omega,\omega_k,i}^{*} b, \quad i = 1, 2, \ldots, \qquad (3.9)$$

where

$$H_{\omega,\omega_k,i}^{*} = \omega \sum_{k=1}^{K} E_k {R_{\omega_k}^{*}}^{s(k,i)} + \omega \sum_{k=1}^{K} \omega_k E_k \left(\sum_{j=0}^{s(k,i)-1} {R_{\omega_k}^{*}}^{j}\right) B_k^{-1} N_k + (1-\omega)I, \quad i = 1, 2, \ldots \qquad (3.10)$$

and

$$P_{\omega,\omega_k,i}^{*} = \omega \sum_{k=1}^{K} \omega_k E_k \left(\sum_{j=0}^{s(k,i)-1} {R_{\omega_k}^{*}}^{j}\right) B_k^{-1}, \quad i = 1, 2, \ldots. \qquad (3.11)$$

The $H_{\omega,\omega_k,i}^{*}$'s are called iteration matrices for Algorithm 2.4. Then, it is easy to show that $P_{\omega,\omega_k,i}^{*} A = I - H_{\omega,\omega_k,i}^{*}$ for each i.

Theorem 3.2. *Let A be an $n \times n$ H-matrix. For each $1 \le k \le K$, let $A = M_k - N_k$ and $M_k = B_k - C_k$ be H-compatible splittings. Then the nonstationary two-stage multisplitting method with $K + 1$ relaxed parameters using $A = M_k - N_k$ as outer splittings and $M_k = B_k - C_k$ as inner splittings converges to the exact solution of the linear system $Ax = b$ for any initial vector $x^{(0)}$ if $0 < \omega \le 1$ and $0 < \omega_k \le 1$.*

Proof. The proof is similar to the proof of Theorem 3.4 in [11]. □

Theorem 3.3. *Let $A = D - B$ be an $n \times n$ H-matrix with $D = \mathrm{diag}(A)$. Let $J = |D|^{-1}|B|$ and let $Q_1, Q_2, \ldots, Q_K$ be zero pattern sets which are subsets of S_n. For each $1 \le k \le K$, let $A = L_k U_k - N_k$ be the ILU factorization of A corresponding to Q_k. Then, the nonstationary multisplitting method with $K + 1$ relaxed parameters associated with the multisplittings $\{L_k U_k, N_k, E_k\}_{k=1}^{K}$ converges to the exact solution of $Ax = b$ for any initial vector x^0 if $0 < \omega < 2/(1+\bar{\rho})$ and $0 < \omega_k < 2/(1+\rho)$, where $\rho = \rho(J)$, $\bar{\rho}$ is defined by Theorem 3.1.*

Proof. Since $A = D - B$ and $D = \mathrm{diag}(A)$,

$$\langle A \rangle = |D| - |B| = |D|(I - J). \qquad (3.12)$$

For each $1 \le k \le K$, let $\langle A \rangle = \widetilde{L}_k \widetilde{U}_k - \widetilde{N}_k$ be the ILU factorization of $\langle A \rangle$ corresponding to Q_k. By some manipulation, it can be shown that $\left|D^{-1}\right| \le \left(\widetilde{L}_k \widetilde{U}_k\right)^{-1}$ for all $k = 1, 2, \ldots, K$. It follows that for all $k = 1, 2, \ldots, K$

$$I \le \left(\widetilde{L}_k \widetilde{U}_k\right)^{-1} |D|. \qquad (3.13)$$

By Lemma 2.7, we have

$$\left|\omega_k (L_k U_k)^{-1} N_k + (1-\omega_k)I\right| \le \omega_k (\widetilde{L}_k \widetilde{U}_k)^{-1} \widetilde{N}_k + |1-\omega_k| I. \qquad (3.14)$$

Let $e = (1, 1, \ldots, 1)^{\mathrm{T}}$. Since $J \ge 0$, $J + \varepsilon e e^{\mathrm{T}} > 0$ for any $\varepsilon > 0$ and from Lemma 2.5 there exists a vector $x_\varepsilon > 0$ corresponding to the spectral radius such that

$$(J + \varepsilon e e^{\mathrm{T}}) x_\varepsilon = \rho_\varepsilon x_\varepsilon, \qquad (3.15)$$

where $\rho_\varepsilon = \rho(J + \varepsilon ee^{\mathrm{T}})$. From Lemma 2.1, $\rho < 1$. By continuity of the spectral radius, there exists an ε_0 such that $\rho_\varepsilon < 1$ for all $0 < \varepsilon \le \varepsilon_0$. Now, we choose an ε such that $0 < \varepsilon \le \varepsilon_0$. Then, from (3.13)–(3.15), we have

$$\begin{aligned}\left|\omega_k(L_kU_k)^{-1}N_k + (1-\omega_k)I\right| x_\varepsilon &\le \omega_k\left(\widetilde{L}_k\widetilde{U}_k\right)^{-1}\widetilde{N}_k x_\varepsilon + |1-\omega_k|x_\varepsilon\\ &\le \omega_k\rho_\varepsilon x_\varepsilon + |1-\omega_k|x_\varepsilon\\ &= (\omega_k\rho_\varepsilon + |1-\omega_k|)x_\varepsilon\\ &= \bar{\rho}x_\varepsilon,\end{aligned}\tag{3.16}$$

since $0 < \omega_k < 2/(1+\rho)$ for all $k = 1, 2, \ldots, K$, and $\varepsilon_0 \longrightarrow 0$, then $\bar{\rho} < 1$. Hence, we obtain

$$\begin{aligned}\left|H_{\omega,\omega_k,i}\right| x_\varepsilon &= \left|\omega\sum_{k=1}^{K}E_k\left(\omega_k(L_kU_k)^{-1}N_k + (1-\omega_k)I\right)^{s(k,i)} + (1-\omega)I\right| x_\varepsilon\\ &\le \omega\sum_{k=1}^{K}E_k\left(\omega_k(\widetilde{L}_k\widetilde{U}_k)^{-1}\widetilde{N}_k + |1-\omega_k|I\right)^{s(k,i)} x_\varepsilon + |1-\omega|x_\varepsilon\\ &\le \omega\sum_{k=1}^{K}E_k\bar{\rho}^{s(k,i)}x_\varepsilon + |1-\omega|x_\varepsilon\\ &\le \omega\sum_{k=1}^{K}E_k\bar{\rho}x_\varepsilon + |1-\omega|x_\varepsilon\\ &= (\omega\bar{\rho} + |1-\omega|)x_\varepsilon.\end{aligned}\tag{3.17}$$

Since $0 < \omega < 2/(1+\bar{\rho})$, from Lemmas 2.2 and 2.3, we obtain $\rho(H_{\omega,\omega_k,i}) \le \rho(|H_{\omega,\omega_k,i}|) < 1$. So the proof is complete. □

Theorem 3.4 (*[15]*). *Let A be an $n \times n$ H-matrix. Let $Q_1, Q_2, \ldots, Q_K$ be zero pattern sets which are subsets of S_n. For each $1 \le k \le K$, let $A = M_k - N_k$ be an H-compatible splitting and $M_k = L_kU_k - C_k$ be the ILU factorization of M_k corresponding to Q_k. Then, the relaxed nonstationary two-stage multisplitting method with $A = M_k - N_k$ as outer splittings and $M_k = L_kU_k - C_k$ as inner splittings converges to the exact solution of $Ax = b$ for any initial vector x^0 if $0 < \omega \le 1$.*

Theorem 3.5. *Let A be an $n \times n$ H-matrix. Let $Q_1, Q_2, \ldots, Q_K$ be zero pattern sets which are subsets of S_n. For each $1 \le k \le K$, let $A = M_k - N_k$ be an H-compatible splitting and $M_k = L_kU_k - C_k$ be the ILU factorization of M_k corresponding to Q_k. Then, the nonstationary two-stage multisplitting method with $K + 1$ relaxed parameters using $A = M_k - N_k$ as outer splittings and $M_k = L_kU_k - C_k$ as inner splittings converges to the exact solution of $Ax = b$ for any initial vector x^0 if $0 < \omega_k \le 1$ and $0 < \omega < 2/(1+\rho^*)$, where $\rho^* = \max_{k=1,2,\ldots,K}\{\rho(|H^*_{\omega_k,i}|)\}$, and $H^*_{\omega_k,i}$'s are iteration matrices of the relaxed nonstationary two-stage multisplitting method with $A = M_k - N_k$ as outer splittings and $M_k = L_kU_k - C_k$ as inner splittings for solving a linear system whose coefficient matrix is A.*

Proof. From (3.10), we have

$$\begin{aligned}\left|H^*_{\omega,\omega_k,i}\right| &= \left|\omega\sum_{k=1}^{K}E_k{R^*_{\omega_k}}^{s(k,i)} + \omega\sum_{k=1}^{K}\omega_kE_k\left(\sum_{j=0}^{s(k,i)-1}{R^*_{\omega_k}}^{j}\right)(L_kU_k)^{-1}C_k + (1-\omega)I\right|\\ &\le \omega\left|\sum_{k=1}^{K}E_k{R^*_{\omega_k}}^{s(k,i)} + \sum_{k=1}^{K}\omega_kE_k\left(\sum_{j=0}^{s(k,i)-1}{R^*_{\omega_k}}^{j}\right)(L_kU_k)^{-1}C_k\right| + |1-\omega|I\\ &= \omega|H^*_{\omega_k,i}| + |1-\omega|I,\end{aligned}\tag{3.18}$$

where $H^*_{\omega_k,i} = \sum_{k=1}^{K}E_k{R^*_{\omega_k}}^{s(k,i)} + \sum_{k=1}^{K}\omega_kE_k\left(\sum_{j=0}^{s(k,i)-1}{R^*_{\omega_k}}^{j}\right)(L_kU_k)^{-1}C_k$. From Theorem 3.4, we have $\rho(H^*_{\omega_k,i}) < 1$ if $0 < \omega_k \le 1$ for all $k = 1, 2, \ldots, K$. According to Lemma 2.5 there exists a nonnegative vector x corresponding to the spectral radius such that $|H^*_{\omega_k,i}|x = \rho\left(|H^*_{\omega_k,i}|\right)x$. Hence, we obtain

$$\begin{aligned}|H^*_{\omega,\omega_k,i}|x &\le \omega|H^*_{\omega_k,i}|x + |1-\omega|x\\ &= (\omega\rho(|H^*_{\omega_k,i}|) + |1-\omega|)x.\end{aligned}\tag{3.19}$$

Since $0 < \omega < 2/(1+\rho(|H^*_{\omega_k,i}|))$ for all $k = 1, 2, \ldots, K$, from Lemmas 2.2 and 2.3, we have $\rho(H^*_{\omega,\omega_k,i}) \le \rho(|H^*_{\omega,\omega_k,i}|) < 1$. So the proof is complete. □

Corollary 3.6. *Suppose that the matrix A satisfies one of the following conditions.*

(i) *A is an M-matrix.*

(ii) *A is strictly or irreducibly diagonally dominant.*
(iii) *$\langle A\rangle$ is symmetric and positive definite.*

Then A is an H-matrix and therefore Theorems 3.1–3.4 hold.

Proof. From [3] and Theorems 3.1–3.3 and 3.5, it is easy to obtain them. $\square$

Remark 3.1. If $\omega_k = 1$ for all $k = 1, 2, \ldots, K$ in Theorem 3.1, it reduces to Theorem 3.1 in [10]; If $\omega_1 = \omega_2 = \cdots = \omega_K$ and $\omega = 1$ in Theorem 3.2, it reduces to theorem 3.4 in [11]; If $\omega_k = 1$ for all $k = 1, 2, \ldots, K$ in Theorem 3.3, it reduces to Theorem 3.1 in [15]; If $\omega_1 = \omega_2 = \cdots = \omega_K$ and $\omega = 1$ in Theorem 3.5, it reduces to Theorem 3.6 in [15]. Hence, our results include and extend some previous results.

4. Numerical experiments

In this section, we consider an application of Algorithm 2.3 with $s(k, i) = s(k)$ to preconditioned Krylov subspace method. From (3.2) and (3.3), we have $H_{\omega,\omega_k,i} = H_{\omega,\omega_k}$ and $P_{\omega,\omega_k,i} = P_{\omega,\omega_k}$ for all $i = 1, 2, \ldots$, where

$$H_{\omega,\omega_k} = \omega \sum_{k=1}^{K} E_k R_{\omega_k}^{s(k)} + (1-\omega)I, \quad i = 1, 2, \ldots \tag{4.1}$$

and

$$P_{\omega,\omega_k} = \omega \sum_{k=1}^{K} \omega_k E_k \left(\sum_{j=0}^{s(k)-1} R_{\omega_k}^{j} \right) M_k^{-1}, \quad i = 1, 2, \ldots. \tag{4.2}$$

If Algorithm 2.3 with $s(k, i) = s(k)$ converges to the exact solution of $Ax = b$ for any initial vector x_0, then $\rho(H_{\omega,\omega_k}) < 1$. It follows that the matrix P_{ω,ω_k} such that $P_{\omega,\omega_k}A = I - H_{\omega,\omega_k}$ is nonsingular. Hence, ${P_{\omega,\omega_k}}^{-1}$ can be used as a preconditioner of Krylov subspace methods.

For simplicity of exposition, suppose that $K = 3$ and the zero pattern set Q_k is the zero pattern set of A, $k = 1, 2, 3$. Then, the H-matrix $A = L_kU_k - N_k$ is an ILU factorization of A corresponding to the zero pattern set $Q_k \subset S_n$, and we construct a multisplitting $\{L_kU_k, N_k, E_k\}$, $k = 1, 2, 3$. Clearly, $A = L_kU_k - N_k$ is an H-compatible splitting for each k. L_i's are lower triangular matrices and U_i's are upper triangular matrices. Let $B_k = LU$ and $C_k = C$ for $k = 1, 2, 3$. The symbol MPre means that P_{ω,ω_k} in (4.2) is used as a preconditioner.

In the following, some numerical experiments will be given. The goals of these experiments are to examine the effectiveness of the preconditioners generated by the multisplitting methods with K+1 relaxed parameters when combined with BiCGSTAB Krylov subspace method [5].

All the numerical experiments were performed with MATLAB 6.5. The machine we have used is a PC-Pentium(R)4, CPU 3.06 GHz, 512 M of RAM. In all of our runs we used a zero initial guess. Unless otherwise stated, BiCGSTAB is used with left preconditioning. The stopping criterion is $\|r^{(k)}\|_2/\|r^{(0)}\|_2 \leqslant 10^{-6}$, where $r^{(k)}$ is the residual vector after the kth iteration.

The test matrix A used in this paper is obtained by five-point discretization of the following elliptic second-order PDE:

$$-(au_{xx} + bu_{yy}) + cu_x + du_y + fu = g \tag{4.3}$$

with $a(x, y) > 0$, $b(x, y) > 0$, $c(x, y)$, $d(x, y)$, and $f(x, y)$ defined on the unit square region $\Omega = (0, 1) \times (0, 1)$, and with Dirichlet boundary condition $u(x, y) = 0$ on the boundary of Ω. Only the discretized matrix A is of importance, so the right-hand side b is created from Ae, where $e = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. Therefore, the right-hand side function $g(x, y)$ in (4.3) is not relevant.

Example 4.1. This example considers Eq. (4.3) with $a(x, y) = b(x, y) = 1$, $c(x, y) = -10(x + y)$, $d(x, y) = -10(x - y)$, and $f(x, y) = 0$. We have used a uniform mesh of $\Delta x = \Delta y = 1/(m + 1)$, which leads to a matrix of order $n = m \times m$, where Δx and Δy refer to the mesh sizes in the x- and y-direction, respectively. We use two uniform meshes of $\Delta x = \Delta y = 31$ and $\Delta x = \Delta y = 61$, which lead to two matrices of order $n = 30 \times 30$ and $n = 60 \times 60$. The corresponding matrices are called PDE1 and PDE2 which are given in Table 4.1.

Example 4.2. Similarly, this example considers Eq. (4.3) with $a(x, y) = b(x, y) = 1$, $c(x, y) = 10e^{xy}$, $d(x, y) = 10e^{-xy}$ and $f(x, y) = 0$. The corresponding matrices are called PDE3 and PDE4 respectively, which are given in Table 4.2.

5. Conclusion

In this paper, we established the convergence results of $K + 1$ relaxed multisplitting methods using ILU factorizations, and we provided performance results of BiCGSTAB with the preconditioners P_{ω,ω_k}^{-1}, which are derived from Algorithm 2.3. Numerical experiments showed that Algorithm 2.3 with Krylov subspace methods such as BiCGSTAB works very well

Table 4.1
Comparisons of the iteration number for different parameters $\omega, \omega_1, \omega_2, \omega_3$ in (4.2). The symbol "NPre" means that no preconditioner is used, the symbol "ILU(0)" means that the incomplete LU(0) factorization preconditioner is used.

$(\omega, \omega_1, \omega_2, \omega_3)$	PDE1	PDE2
(0.5, 0.8, 0.5, 1.0)	14.5	27.5
(1.0, 1.1, 0.8, 0.8)	12.5	26
(1.0, 1.0, 1.0, 1.0)	12	22
(1.0, 1.1, 1.0, 1.1)	11	20.5
(1.5, 1.1, 1.2, 1.5)	10	19
(1.5, 1.4, 1.4, 1.5)	9	20.5
(1.5, 1.5, 1.5, 1.5)	9	17
(1.6, 1.5, 1.5, 1.6)	9.5	17.5
ILU(0)	17.5	36
NPre	69	123

Table 4.2
Comparisons of the iteration number for different parameters $\omega, \omega_1, \omega_2, \omega_3$ in (4.2). The symbol "NPre" means that no preconditioner is used, the symbol "ILU(0)" means that the incomplete LU(0) factorization preconditioner is used.

$(\omega, \omega_1, \omega_2, \omega_3)$	PDE3	PDE4
(0.5, 0.8, 0.5, 1.0)	12.5	26
(1.0, 1.1, 0.8, 0.8)	12	22.5
(1.0, 1.0, 1.0, 1.0)	11.5	19
(1.0, 1.1, 1.0, 1.1)	11	19
(1.5, 1.1, 1.2, 1.5)	9	18
(1.5, 1.4, 1.4, 1.5)	8	17
(1.5, 1.5, 1.5, 1.5)	8	15.5
(1.6, 1.5, 1.5, 1.6)	11.5	17
ILU(0)	17	30.5
NPre	49.5	102.5

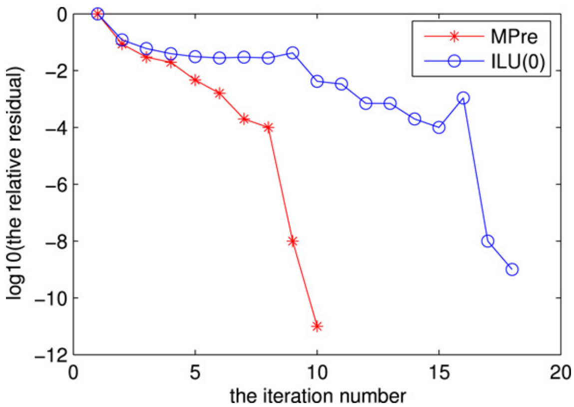

Fig. 1. The relative residuals obtained with BiCGSTAB using (4.2) as a preconditioner and ILU(0) for matrix PDE1, the parameters $\omega = \omega_1 = \omega_2 = \omega_3 = 1.5$.

(i.e., BiCGSTAB with the preconditioner P^{-1}_{ω,ω_k} performs very well as compared with Algorithm 2.3), which is faster than BiCGSTAB with the preconditioner ILU(0). For test problems used in this paper, BiCGSTAB with the preconditioner P^{-1}_{ω,ω_k} performs best if the range of all parameters ω and ω_k is near 1.5, i.e., the optimal parameters are approximately 1.5 (see Figs. 1 and 2).

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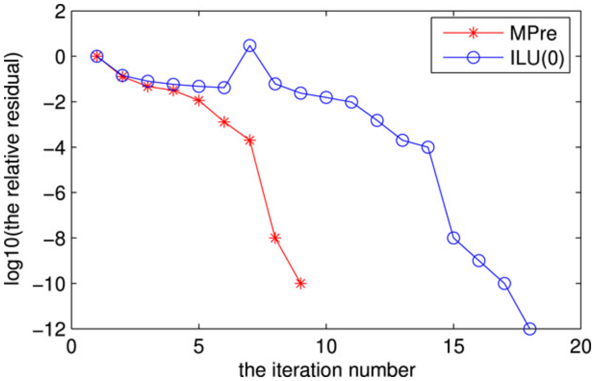

Fig. 2. The relative residuals obtained with BiCGSTAB using (4.2) as preconditioner and ILU(0) for matrix PDE3, the parameters $\omega = \omega_1 = \omega_2 = \omega_3 = 1.5$.

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