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A regularization method for a Cauchy problem of the Helmholtz equation

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ABSTRACT

We investigate a Cauchy problem for the Helmholtz equation. A modified boundary method is used for solving this ill-posed problem. Some Hölder-type error estimates are obtained. The numerical experiment shows that the modified boundary method works well.

1. Introduction

The Cauchy problem of an elliptic equation arises from many physical and engineering problems such as nondestructive testing techniques [1], geophysics [2], and cardiology [3].

It is well known that the Cauchy problem of an elliptic equation is ill-posed in the sense that arbitrarily "small" differences in the data can induce arbitrarily "large" errors in the solution. Under an additional condition, a continuous dependence of the solution on the Cauchy data can be obtained. This is called conditional stability [4]. Other results on conditional stability for the elliptic equation can be found in [5,6].

Due to the severe ill-posedness of the problem, it is impossible to solve the Cauchy problem of the elliptic equation by using classical numerical methods and it requires special techniques, e.g., regularization strategies. Theoretical concepts and computational implementation related to the Cauchy problem of the elliptic equation have been discussed by many authors, and a lot of methods have been provided. For computational aspects, the readers can consult D.N. Hào [7], H.J. Reinhardt et al. [8], J. Cheng [5] and Y.C. Hong [9]. For theoretical aspects, the readers can refer to X.T. Xiong [10] and Zhi Qian [11].

The Helmholtz equation arises in many areas, especially in practical physical applications, such as acoustic, wave propagation and scattering, vibration of the structure, electromagnetic scattering and so on. Several numerical methods have been proposed to solve this problem, such as alternating iterative algorithm based on the boundary element method (BEM) [12], the conjugate gradient method [13] and the method of fundamental solutions (MFS) [14,15,13,16–18]. Although there exists a vast literature on the Cauchy problem for the Helmholtz equation, to the authors' knowledge, there are much fewer papers devoted to the error estimates. Although in [19], the authors gave a quasi-reversibility method for solving a Cauchy problem of the Helmholtz equation in a rectangle domain where they considered a homogeneous Neumann boundary condition, the results were less encouraging. The main aim of this paper is to present a simple and effective regularization method, and investigate the error estimate between the regularization solution and the exact one.

This paper is organized as follows. In Section 2, the regularization method-modified boundary method is introduced; in Sections 3 and 4, a stability estimate is proved under an a priori condition; in Section 5, some numerical results are reported.

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2. Regularization for a Cauchy problem of the Helmholtz equation

Consider two Cauchy problems for the Helmholtz equation:

$$\begin{aligned} &\Delta u(x,y)+k^2u(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\ &u(0,y)=\phi(y), \quad y\in\mathbb{R},\\ &u_x(0,y)=0, \quad y\in\mathbb{R}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\Delta v(x,y)+k^2v(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\ &v(0,y)=0, \quad y\in\mathbb{R},\\ &v_x(0,y)=h(y), \quad y\in\mathbb{R}, \end{aligned} \tag{2.2}$$

where $k>0$ is the wave number (constant). We need to seek the solutions $u(x,y)$ and $v(x,y)$ from the given data ϕ, h, respectively. Physically, ϕ, h can only be measured, there will be measurement errors, and we would actually have some data functions $\phi^\delta(\cdot), h^\delta(\cdot)\in L^2(\mathbb{R})$, for which

$$\|\phi^\delta-\phi\|+\|h^\delta-h\|\le\delta, \tag{2.3}$$

where the constant $\delta>0$ represents a bound on the measurement error, $\|\cdot\|$ denotes the L^2-norm, and there exists a constant $E>0$, such that the following a priori bounds exist (e.g., says, the energy of the solution $u(x,y)$ and $v(x,y)$ at the right boundary $x=a$ are finite.)

$$\|u(a,\cdot)\|\le E, \quad \text{and} \quad \|v(a,\cdot)\|\le E. \tag{2.4}$$

Thus, the following problem can be solved because $w(x,y)=u(x,y)+v(x,y)$.

Problem 1. Determine the solution $w(x,y)$ for $0<x<a$ from the input data $\phi(\cdot):=w(0,\cdot), h(\cdot):=w_x(0,\cdot)$,when $w(x,y)$ satisfies

$$\begin{aligned} &\Delta w(x,y)+k^2w(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\ &w(0,y)=\phi(y), \quad y\in\mathbb{R},\\ &w_x(0,y)=h(y), \quad y\in\mathbb{R}. \end{aligned} \tag{2.5}$$

Let

$$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)\mathrm{e}^{-\mathrm{i}\xi y}\mathrm{d}y \tag{2.6}$$

be the Fourier transform of the function $f(y)\in L^2(\mathbb{R})$. The solution of problem (2.1) can be formulated in the frequency domain:

$$\hat{u}(x,\xi)=\cosh(x\sqrt{\xi^2-k^2})\hat{\phi}(\xi). \tag{2.7}$$

We notice that if $\xi^2>k^2$, for fixed $0<x\le a$, $\cosh(x\sqrt{\xi^2-k^2})$ is unbounded as $\xi^2\to\infty$. There we want to seek a solution $u(x,\cdot)\in L^2(\mathbb{R})$, the exact data $\hat{\phi}(\xi)$ must decay rapidly as $|\xi|\to\infty$. But in practice, we can only get the noisy data $\hat{\phi}^\delta(\cdot)\in L^2(\mathbb{R})$. Hence for the noisy data $\hat{\phi}^\delta(\cdot)$, we cannot obtain a meaningful solution.

Similarly, for problem (2.2), the solution can be found in the frequency domain:

$$\hat{v}(x,\xi)=\frac{\sinh(x\sqrt{\xi^2-k^2})}{\sqrt{\xi^2-k^2}}\hat{h}(\xi). \tag{2.8}$$

Therefore the solution of (2.5) in the frequency domain is

$$\hat{w}(x,\xi)=\cosh(x\sqrt{\xi^2-k^2})\hat{\phi}(\xi)+\frac{\sinh(x\sqrt{\xi^2-k^2})}{\sqrt{\xi^2-k^2}}\hat{h}(\xi). \tag{2.9}$$

Obviously, Problem 1 is an ill-posed problem and requires special regularization method to be employed. We follow the idea from Ames et al. [20] where they used a quasi-boundary method (or so-called modified boundary method) for solving backward heat equation. Consider the following problems with noisy data:

$$\begin{aligned} &\Delta u_\alpha^\delta(x,y)+k^2u_\alpha^\delta(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\ &u_\alpha^\delta(0,y)+\alpha u_\alpha^\delta(a,y)=\phi^\delta(y), \quad y\in\mathbb{R},\\ &(u_\alpha^\delta)_x(0,y)=0, \quad y\in\mathbb{R}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\Delta v_\alpha^\delta(x,y)+k^2 v_\alpha^\delta(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\
&v_\alpha^\delta(0,y)+\alpha v_\alpha^\delta(a,y)=0, \quad y\in\mathbb{R},\\
&(v_\alpha^\delta)_x(0,y)=h^\delta(y), \quad y\in\mathbb{R},
\end{aligned} \tag{2.11}$$

where $\alpha>0$ is a small parameter.

Therefore, for (2.5), we have

$$\begin{aligned}
&\Delta w_\alpha^\delta(x,y)+k^2 w_\alpha^\delta(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\
&w_\alpha^\delta(0,y)+\alpha w_\alpha^\delta(a,y)=\phi^\delta(y), \quad y\in\mathbb{R},\\
&(w_\alpha^\delta)_x(0,y)=h^\delta(y), \quad y\in\mathbb{R}.
\end{aligned} \tag{2.12}$$

We should first answer two questions:

- Do the problems (2.10) and (2.11) approximate the following problems:

$$\begin{aligned}
&\Delta u^\delta(x,y)+k^2 u^\delta(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\
&u^\delta(0,y)=\phi^\delta(y), \quad y\in\mathbb{R},\\
&(u^\delta)_x(0,y)=0, \quad y\in\mathbb{R},
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&\Delta v^\delta(x,y)+k^2 v^\delta(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\
&v^\delta(0,y)=0, \quad y\in\mathbb{R},\\
&(v^\delta)_x(0,y)=h^\delta(y), \quad y\in\mathbb{R},
\end{aligned} \tag{2.14}$$

respectively? If the answer is yes, thus the problem (2.12) approximates the problem:

$$\begin{aligned}
&\Delta w^\delta(x,y)+k^2 w^\delta(x,y)=0, \quad 0<x<a, y\in\mathbb{R},\\
&w^\delta(0,y)=\phi^\delta(y), \quad y\in\mathbb{R},\\
&(w^\delta)_x(0,y)=h^\delta(y), \quad y\in\mathbb{R}.
\end{aligned} \tag{2.15}$$

- Are the problems (2.10) and (2.11) well-posed? If the answer is yes, problem (2.12) is well-posed.

In order to answer these two questions, we should pay attention to the assumption $\phi^\delta(\cdot), h^\delta(\cdot)\in L^2(\mathbb{R})$. By the technique of Fourier transform, we can get the solutions of the problems (2.10) and (2.11) in the frequency domain:

$$\hat{u_\alpha^\delta}=\frac{\cosh(x\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})}\hat{\phi^\delta}, \tag{2.16}$$

and

$$\hat{v_\alpha^\delta}=\frac{\sinh(x\sqrt{\xi^2-k^2})-\alpha\sinh((1-x)\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})}\frac{\hat{h^\delta}(\xi)}{\sqrt{\xi^2-k^2}}. \tag{2.17}$$

Hence the solution of problem (2.12) in frequency domain is

$$\hat{w_\alpha^\delta}=\frac{\cosh(x\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})}\hat{\phi_\delta}+\frac{\sinh(x\sqrt{\xi^2-k^2})-\alpha\sinh((1-x)\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})}\frac{\hat{h^\delta}(\xi)}{\sqrt{\xi^2-k^2}}. \tag{2.18}$$

Now for the first question, from (2.16)–(2.18) we can easily see that when $\alpha\to 0$, $\hat{u_\alpha^\delta}\to\hat{u^\delta}$, $\hat{v_\alpha^\delta}\to\hat{v^\delta}$ and $\hat{w_\alpha^\delta}\to\hat{w^\delta}$, uniformly in x.

As to the second question, we only need to prove the stability of the problems. For problem (2.10), we have the conclusion: if any two functions $\phi^{\delta,1}$ and $\phi^{\delta,2}$ satisfy $\|\phi^{\delta,1}-\phi^{\delta,2}\|\le\epsilon$, let $u_{\alpha,1}^\delta$ and $u_{\alpha,2}^\delta$ be the corresponding solutions, respectively, set $\alpha=O(\epsilon)$, then $\|u_{\alpha,1}^\delta-u_{\alpha,2}^\delta\|\to 0$, as $\epsilon\to 0$. In fact, by Parseval identity, we have

$$\begin{aligned}
\|u_{\alpha,1}^\delta-u_{\alpha,2}^\delta\| &= \|\widehat{u_{\alpha,1}^\delta}-\widehat{u_{\alpha,2}^\delta}\|=\left\|\frac{\cosh(x\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})}(\widehat{\phi^{\delta,1}}-\widehat{\phi^{\delta,2}})\right\|\\
&\le\sup_{\xi\in\mathbb{R}}\left|\frac{\cosh(x\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})}\right|\epsilon.
\end{aligned} \tag{2.19}$$

Case I. If $\xi^2 - k^2 \le 0$, then $\cosh(x\sqrt{\xi^2-k^2}) = \cos(x\sqrt{k^2-\xi^2})$, $\cosh(a\sqrt{\xi^2-k^2}) = \cos(a\sqrt{k^2-\xi^2})$, therefore $\|u^\delta_{\alpha,1} - u^\delta_{\alpha,2}\| \le \epsilon$.

Case II. If $\xi^2 - k^2 > 0$, then by the inequality (3.1) in Lemma 3.1,

$$\sup_{\xi\in\mathbb{R}} \left| \frac{\cosh(x\sqrt{\xi^2-k^2})}{1+\alpha\cosh(a\sqrt{\xi^2-k^2})} \right| \le \sup_{\xi\in\mathbb{R}} \frac{e^{x\sqrt{\xi^2-k^2}}}{1+\frac{\alpha}{2}e^{a\sqrt{\xi^2-k^2}}} \le \left(\frac{\alpha}{2}\right)^{-\frac{x}{a}}.$$

Hence if $\alpha = O(\epsilon)$, then

$$\|u^\delta_{\alpha,1} - u^\delta_{\alpha,2}\| \to 0, \quad \text{for } \epsilon \to 0.$$

Similarly, for problem (2.11), if any two data functions satisfy $\|h^{\delta,1} - h^{\delta,2}\| \le \epsilon$, and $\alpha = O(\epsilon)$, then $\|v^\delta_{\alpha,1} - v^\delta_{\alpha,2}\| \to 0$, as $\epsilon \to 0$. In fact, in Section 4, (4.10), (4.13) and (4.14) imply the result.

Now we prove the main conclusions.

3. Error estimates

First we need two inequalities.

Lemma 3.1. *Let $0 < x < a$, then*

$$\sup_{\eta>0} \frac{e^{\eta x}}{1+\alpha e^{a\eta}} \le \alpha^{-\frac{x}{a}}; \tag{3.1}$$

$$\sup_{\eta>0} \frac{\sinh(\eta x)}{\eta e^{x\eta}} \le x. \tag{3.2}$$

Proof. Let $f(\eta) = \frac{\sinh(x\eta)}{\eta e^{x\eta}}$, we have the first order derivative

$$f'(\eta) = \frac{x\eta - \frac{e^{2x\eta-1}}{2}}{(\eta e^{x\eta})^2}. \tag{3.3}$$

Since $e^r - 1 \ge r$ for $r \ge 0$, we get $x\eta - \frac{e^{2x\eta-1}}{2} \le 0$. It is obvious that $f(\eta)$ is a decreasing function. Therefore $f(\eta) \le \lim_{\eta\to 0+} f(\eta) = x$. □

For system (2.1), we can obtain the error estimate between the regularized solution u^δ_α and the exact one u.

Lemma 3.2. *Suppose u be the solution of problem (2.1) with the exact data ϕ and u^δ be the regularization solution defined by with the noisy data ϕ^δ, let ϕ^δ satisfies (2.3) and let the exact solution u at $x = a$ satisfy (2.4). If we select $\alpha = (\frac{\delta}{E})$, then for fixed $0 < x < a$ we get the error estimate*

$$\|u^\delta_\alpha(x,\cdot) - u(x,\cdot)\| \le 2\delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}, \quad \text{for } \delta \to 0. \tag{3.4}$$

For system (2.2), similarly we have:

Lemma 3.3. *Suppose v be the solution of problem (2.2) with the exact data h and v^δ be the regularization solution with the noise data h^δ, let h^δ satisfies (2.3) and let the exact solution v at $x = a$ satisfy (2.4). If we select $\alpha = (\frac{\delta}{E})$, then for fixed $0 < x < a$ we get the error estimate*

$$\|v^\delta_\alpha(\cdot,y) - v(\cdot,y)\| \le (2a+1)\delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}(1+o(1)), \quad \text{for } \delta \to 0. \tag{3.5}$$

Theorem 3.4. *Suppose that $w = u + v$ is the solution with exact data $[\phi, h]$ and that $w^\delta_\alpha = u^\delta_\alpha + v^\delta_\alpha$ is the solution with measured data $[\phi_\delta, h_\delta]$. If (2.4) holds, and the measured functions satisfy (2.3) and if we choose $\alpha = (\frac{\delta}{E})$, then for fixed $0 < x < a$, we get the error estimate*

$$\|w^\delta_\alpha(\cdot,y) - w(\cdot,y)\| \le (2a+3)\delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}(1+o(1)), \quad \text{for } \delta \to 0. \tag{3.6}$$

Proof. According to the Parseval identity, we have $\|\hat{w} - \hat{w_\alpha^\delta}\| = \|w - w_\alpha^\delta\|$, and $\|w - w^\delta\| = \|(u + v) - (u^\delta + v^\delta)\| \le \|u - u^\delta\| + \|v - v^\delta\|$; then the theorem is straightforward by using triangle inequality and Lemma 3.2, Lemma 3.3.

From Theorem 3.4, we find that w^δ is an approximation of exact solution w. The approximation error depends continuously on the measurement error for fixed $0 < x < a$. However, as $x \to a$, the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution.

To retain the continuous dependence of the solution at $x = a$, instead of (2.3), we introduce a stronger a priori assumption,

$$\|w(x, \cdot)|_{x=a}\|_p \le E, \tag{3.7}$$

where $\|\cdot\|_p$ denotes the norm of Sobolev space $H^p(\mathbb{R})$ and $p > 0$. □

Remark 3.5. We separately consider the case $0 \le x \le a$ and the case $x = a$ in order to emphasize the following facts. For the case $0 \le x < a$, the a priori bound $\|w(x, \cdot)\|$ is sufficient. However, for the case $x = a$, the stronger a priori bound for $\|w(x, \cdot)|_{x=a}\|_p$ where $p > 0$ must be imposed. By this assumption, if $\alpha = \frac{\delta}{E}(\ln\frac{E}{\delta})^p$ with $p > 0$, we can get the error estimate between $w(a, \cdot)$ and $w_\alpha^\delta(a, \cdot)$:

$$\|w_\alpha^\delta(a, \cdot) - w(a, \cdot)\| \le O\left(E\left(\ln\frac{E}{\delta}\right)^{-p}\right), \quad \text{for } \delta \to 0. \tag{3.8}$$

This is a logarithmic stability estimate. This often occurs in the boundary error estimate for ill-posed problems.

Remark 3.6. The modified method (2.12) can also be replaced by

$$\begin{aligned} &\Delta w_\alpha^\delta(x, y) + k^2 w_\alpha^\delta(x, y) = 0, \quad 0 < x < a, y \in \mathbb{R}, \\ &w_\alpha^\delta(0, y) = \phi^\delta(y), \quad y \in \mathbb{R}, \\ &(w_\alpha^\delta)_x(0, y) + \alpha(w_\alpha^\delta)_x(a, y) = h^\delta(y), \quad y \in \mathbb{R}. \end{aligned} \tag{3.9}$$

Remark 3.7. The error estimates (3.6) and (3.8) are order optimal according to the general regularization theory [21,22].

4. Proofs of Lemmas 3.1–3.3

In this section, we denote $\eta = \sqrt{\xi^2 - k^2}$.

Proof of Lemma 3.1. The proof is very easy by using the method in Carasso [23].

Proof of Lemma 3.2. *Case I: for $\xi^2 > k^2$ (ill-posed part), in this case, $\eta > 0$.*

By Parseval identity and triangle inequality, we have

$$\|u_\alpha^\delta(x, \cdot) - u(x, \cdot)\| = \|\hat{u_\alpha^\delta}(x, \cdot) - \hat{u}(x, \cdot)\| \le \|\hat{u_\alpha}(x, \cdot) - \hat{u}(x, \cdot)\| + \|\hat{u_\alpha^\delta}(x, \cdot) - \hat{u_\alpha}(x, \cdot)\|. \tag{4.1}$$

For the second term on the right-hand side of above inequality, we have

$$\|\hat{u_\alpha^\delta}(x, \cdot) - \hat{u_\alpha}(x, \cdot)\| = \left\|\frac{\cosh(x\eta)}{1 + \alpha\cosh(a\eta)}\hat{\phi}^\delta(\xi) - \frac{\cosh(x\eta)}{1 + \alpha\cosh(a\eta)}\hat{\phi}(\xi)\right\|, \tag{4.2}$$

i.e.,

$$\|\hat{u_\alpha^\delta}(x, \cdot) - \hat{u_\alpha}(x, \cdot)\| \le \sup_{\eta > 0}\frac{\cosh(x\eta)}{1 + \alpha\cosh(a\eta)}\delta \le \sup_{\eta > 0}\frac{e^{x\eta}}{1 + \frac{\alpha}{2}e^{a\eta}}\delta. \tag{4.3}$$

By Lemma 3.1, we get

$$\|\hat{u_\alpha^\delta}(x, \cdot) - \hat{u_\alpha}(x, \cdot)\| \le \sup_{\eta > 0}\frac{e^{x\eta}}{1 + \frac{\alpha}{2}e^{a\eta}}\delta \le \left(\frac{\alpha}{2}\right)^{-\frac{x}{a}}\delta = \left(\frac{\delta}{2E}\right)^{-\frac{x}{a}}\delta \le \delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}. \tag{4.4}$$

For the first term on the right-hand side of (4.1), we have

$$\|\hat{u_\alpha}(x, \cdot) - \hat{u}(x, \cdot)\| = \left\|\frac{\cosh(x\eta)}{1 + \alpha\cosh(a\eta)}\hat{\phi}(\xi) - \cosh(x\eta)\hat{\phi}(\xi)\right\| = \left\|\frac{\alpha\cosh(x\eta)\cosh(a\eta)}{1 + \alpha\cosh(a\eta)}\hat{\phi}(\xi)\right\|. \tag{4.5}$$

Via (2.7), we have

$$\hat{\phi}(\xi) = \frac{\hat{u}(a,\xi)}{\cosh(a\eta)},$$

$$\begin{aligned}\|\hat{u_\alpha}(x,\cdot) - \hat{u}(x,\cdot)\| &= \alpha \left\| \frac{\cosh(x\eta)}{1+\alpha\cosh(a\eta)} \hat{u}(a,\xi) \right\| \\ &\le \alpha \sup_{\eta>0} \frac{\cosh(x\eta)}{1+\alpha\cosh(a\eta)} \|\hat{u}(a,\cdot)\|.\end{aligned}$$

By the a priori assumption on $\|u(a,\cdot)\| \le E$, it yields

$$\|\hat{u_\alpha}(x,\cdot) - \hat{u}(x,\cdot)\| \le \alpha E \sup_{\eta>0} \frac{\cosh(x\eta)}{1+\alpha\cosh(a\eta)}. \tag{4.6}$$

Using Lemma 3.1, we have

$$\|\hat{u_\alpha}(x,\cdot) - \hat{u}(x,\cdot)\| \le \alpha E \sup_{\eta>0} \frac{\mathrm{e}^{x\eta}}{1+\frac{\alpha}{2}\mathrm{e}^{a\eta}} \le 2E\left(\frac{\alpha}{2}\right)^{1-\frac{x}{a}} = 2E\left(\frac{\delta}{2E}\right)^{1-\frac{x}{a}} \le \delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}. \tag{4.7}$$

Combining (4.4) and (4.7), we have

$$\|u_\alpha^\delta(x,\cdot) - u(x,\cdot)\| \le 2\delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}.$$

Case II: for $\xi^2 \le k^2$ (well-posed part), in this case, $\eta \le 0$, and $\cosh(\eta) = \cos(k^2 - \xi^2)$.
We can easily get

$$\|u_\alpha^\delta(x,\cdot) - u(x,\cdot)\| \le \delta. \tag{4.8}$$

Thus, we get Lemma 3.2. □

Proof of Lemma 3.3. *Case I: for $\xi^2 > k^2$ (ill-posed part), in this case, $\eta > 0$.*
By Parseval identity and triangle inequality, we have

$$\|v_\alpha^\delta(x,\cdot) - v(x,\cdot)\| = \|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v}(x,\cdot)\| \le \|\hat{v_\alpha}(x,\cdot) - v(x,\cdot)\| + \|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v_\alpha}(x,\cdot)\|; \tag{4.9}$$

for the second term on the right-hand side of above inequality, we have

$$\|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v_\alpha}(x,\cdot)\| = \left\| \frac{\sinh(x\eta) - \alpha\sinh((a-x)\eta)}{1+\alpha\cosh(a\eta)} \frac{\hat{h^\delta}(\xi)}{\eta} - \frac{\sinh(x\eta) - \alpha\sinh((a-x)\eta)}{1+\alpha\cosh(a\eta)} \frac{\hat{h}(\xi)}{\eta} \right\|, \tag{4.10}$$

i.e.,

$$\|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v_\alpha}(x,\cdot)\| \le \sup_{\eta>0} \left| \frac{\sinh(x\eta) - \alpha\sinh((a-x)\eta)}{\eta(1+\alpha\cosh(a\eta))} \right| \delta. \tag{4.11}$$

Then

$$\|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v_\alpha}(x,\cdot)\| \le \sup_{\eta>0} \left| \frac{\sinh(x\eta)}{\eta(1+\alpha\cosh(a\eta))} \right| \delta + \sup_{\eta>0} \left| \frac{\sinh((a-x)\eta)}{\eta(1+\alpha\cosh(a\eta))} \right| \alpha\delta, \tag{4.12}$$

i.e.,

$$\|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v_\alpha}(x,\cdot)\| \le \sup_{\eta>0} \frac{\sinh(x\eta)}{\eta\mathrm{e}^{x\eta}} \left| \frac{\mathrm{e}^{x\eta}}{(1+\frac{\alpha}{2}\mathrm{e}^{a\eta})} \right| \delta + \sup_{\eta>0} \frac{\sinh((a-x)\eta)}{\eta\mathrm{e}^{(a-x)\eta}} \left| \frac{\mathrm{e}^{(a-x)\eta}}{(1+\frac{\alpha}{2}\mathrm{e}^{a\eta})} \right| \alpha\delta. \tag{4.13}$$

By Lemma 3.1, we get

$$\begin{aligned}\|\hat{v_\alpha^\delta}(x,\cdot) - \hat{v_\alpha}(x,\cdot)\| &\le x\left(\frac{\alpha}{2}\right)^{-\frac{x}{a}} \delta + \alpha\delta(a-x)\left(\frac{\alpha}{2}\right)^{-\frac{a-x}{a}} \\ &\le (2a-x)\delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}} \le 2a\delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}.\end{aligned} \tag{4.14}$$

For the first term on the right-hand side of (4.9), we have

$$\|\hat{v_\alpha}(x,\cdot) - \hat{v}(x,\cdot)\| = \left\| \frac{\sinh(x\eta) - \alpha\sinh((a-x)\eta)}{1+\alpha\cosh(a\eta)} \frac{\hat{h}(\xi)}{\eta} - \sinh(x\eta)\frac{\hat{h}(\xi)}{\eta} \right\|, \tag{4.15}$$

i.e.,

$$\|\hat{v_\alpha}(x,\cdot)-\hat{v}(x,\cdot)\| = \left\| \frac{\alpha\sinh((a-x)\eta)+\alpha\sinh(x\eta)\cosh(a\eta)}{1+\alpha\cosh(a\eta)}\frac{\hat{h}(\xi)}{\eta}\right\| = \left\| \frac{\frac{\alpha}{2}(\sinh((a+x)\eta)+\sinh((a-x)\eta))}{1+\alpha\cosh(a\eta)}\frac{\hat{h}(\xi)}{\eta}\right\|.$$

By the a priori assumption on $\|v(a,\cdot)\|\le E$, it yields

$$\|\hat{v_\alpha}(x,\cdot)-\hat{v}(x,\cdot)\| \le \frac{\alpha}{2}E\sup_{\eta>0}\frac{\sinh((a+x)\eta)+\sinh((a-x)\eta)}{(\sinh(a\eta)(1+\alpha\cosh(a\eta)))} \le \frac{\alpha}{2}E\sup_{\eta>0}\left[\frac{\sinh((a+x)\eta)}{(\sinh(a\eta)(1+\alpha\cosh(a\eta)))}+\frac{\sinh((a-x)\eta)}{\sinh(a\eta)}\right].$$

Now we investigate the functions $f_1(\eta) := \frac{\sinh((a+x)\eta)}{\sinh(a\eta)}$, and $f_2(\eta) := \frac{\sinh((a-x)\eta)}{\sinh(a\eta)}$. Obviously, $f_2(\eta)\le 1$, but since $\tau(\eta) := \frac{1-e^{-2(a+x)\eta}}{1-e^{-2a\eta}}$ is a decreasing function for $\eta>0$,

$$f_1(\eta) = e^{x\eta}\frac{1-e^{-2(a+x)\eta}}{1-e^{-2a\eta}} \le e^{x\eta}\left(1+\frac{x}{a}\right)\le 2e^{x\eta}. \tag{4.16}$$

Therefore, we have

$$\|\hat{v_\alpha}(x,\cdot)-\hat{v}(x,\cdot)\| \le \alpha E\sup_{\eta>0}\frac{e^{x\eta}}{1+\frac{\alpha}{2}e^{a\eta}}+\frac{\alpha}{2}E \le 2E\left(\frac{\alpha}{2}\right)^{1-\frac{x}{a}}+\frac{\alpha}{2}E = \delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}(1+o(1)). \tag{4.17}$$

Combining (4.14) and (4.17), we get

$$\|v_\alpha^\delta(x,\cdot)-v(x,\cdot)\| \le \delta^{1-\frac{x}{a}}(2E)^{\frac{x}{a}}(1+o(1)). \tag{4.18}$$

Case II: for $\xi^2 \le k^2$ (well-posed part), in this case, $\eta\le 0$, *and* $\cosh(\eta)=\cos(k^2-\xi^2)$, $\sinh(\eta)=i\sin(k^2-\xi^2)$. We can easily get

$$\|v_\alpha^\delta(x,\cdot)-v(x,\cdot)\| \le \delta. \tag{4.19}$$

Thus, the proof of Lemma 3.3 completed. □

5. A numerical example

In this section, a simple example is devised for verifying the validity of the proposed method.

To test the accuracy of the approximate solution, we use the root mean square error (RSE) and the relative root mean square error (RRSE), which are defined for two vectors W and W^* (where W and W^* denote the exact and computed solutions at the test points, respectively):

$$RSE(W) = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(W_j-W_j^*)^2}; \tag{5.1}$$

$$RRSE(W) = \frac{\sqrt{\sum_{j=1}^{n}(W_j-W_j^*)^2}}{\sqrt{\sum_{j=1}^{n}(W_j)^2}}. \tag{5.2}$$

The forthcoming numerical example is devised in the following way: first we sample the Cauchy data pairs $(\phi(y),h(y))$ at the y-grid to get vectors Φ and H, then we add random distribution perturbation and obtain Φ_δ and H_δ,i.e.,

$$\Phi_\delta = \Phi+\sigma_1\,\mathrm{rand}(\mathrm{size}(\Phi)), \qquad H_\delta = H+\sigma_2\,\mathrm{rand}(\mathrm{size}(H)), \tag{5.3}$$

where

$$\Phi = (\phi(y_1),\ldots,\phi(y_n))^T, \tag{5.4}$$

$$H = (h(y_1),\ldots,h(y_n))^T, \tag{5.5}$$

$$\delta = RSE(\Phi_\delta-\Phi)+RSE(H_\delta-H), \tag{5.6}$$

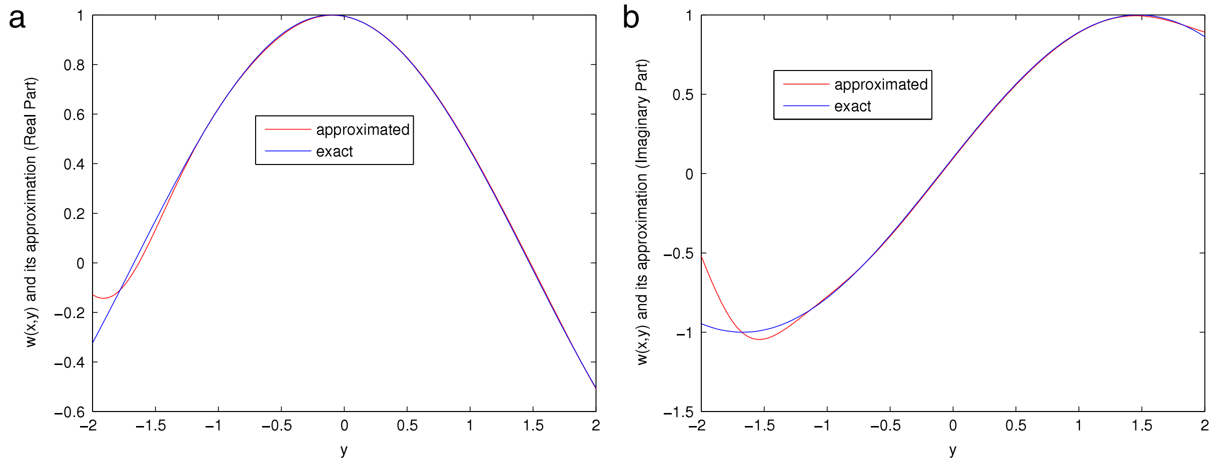

Fig. 1. $x = 0.1, k = 1, \alpha = 0.0006, \sigma_1 = 10^{-4}, \sigma_2 = 10^{-3}$ (1a): recovery of real part with $RSE = 0.0148$, $RRSE = 0.0465$; (1b): recovery of imaginary part with $RSE = 0.0360$, $RRSE = 0.0935$.

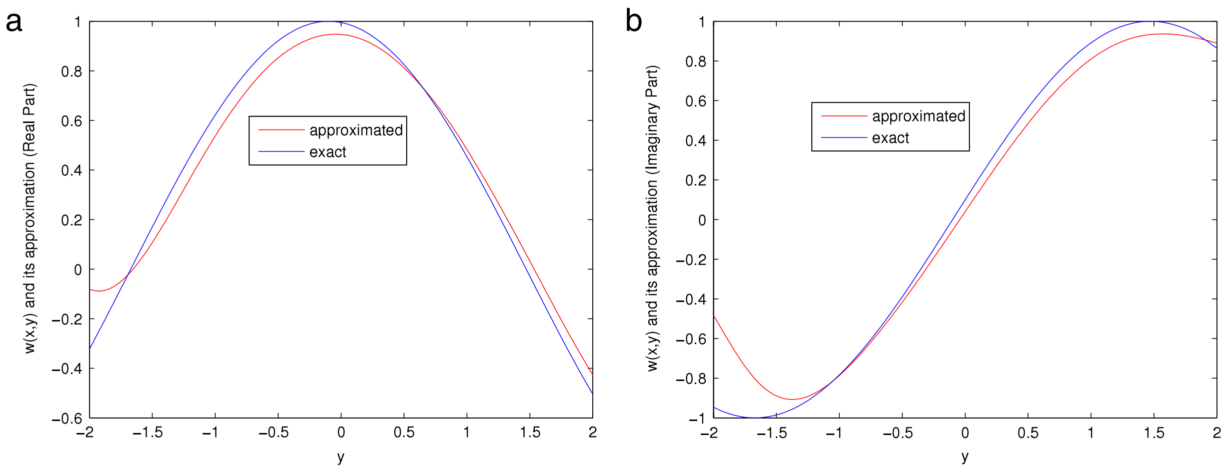

Fig. 2. $x = 0.1, k = 1, \alpha = 0.06, \sigma_1 = 10^{-2}, \sigma_2 = 10^{-1}$ (2a): recovery of real part with $RSE = 0.0342$, $RRSE = 0.1076$; (2b): recovery of imaginary part with $RSE = 0.0596$, $RRSE = 0.1544$.

σ_1 indicates the error level of Φ, and σ_2 denotes the error level of H. Considering that the flux error is usually much larger, we take σ_2 10 times as large as σ_1. The symbol $rand(size(\cdot))$ is a random number between $[-1, 1]$.

We consider the following Cauchy problem:

$$\begin{aligned} &\Delta u(x, y) + k^2 u(x, y) = 0, \quad 0 < x < 1, y \in \mathbb{R}, \\ &u(0, y) = e^{kiy}, \quad y \in \mathbb{R}, \\ &u_x(0, y) = kie^{kiy}, \quad y \in \mathbb{R}, \end{aligned} \tag{5.7}$$

The exact solution $u(x, y)$ is given by $u(x, y) = e^{ki(x+y)}$, where $i = \sqrt{-1}$.

We solve the discretized version of the Cauchy problem by using Matlab in IEEE double precision with unit round-off 1.1×10^{-16}. Since the solution is periodic, so in computation we take $-2 \le y \le 2$ and fix x. The regularized solution was computed by the Fast Fourier Transform (FFT) and inverse Fast Fourier Transform technique according to formula (2.18). The regularization parameter α is chosen by Theorem 3.4 where we take $E = 1$.

Figs. 1–3 show that inverse solutions are extremely sensitive to measurement errors, measurement locations. A small noise in the measurements (ϕ, h) tends to produce large oscillations, which becomes even more significant as the sensors are placed farther away from the surface at $x = 0$. The large oscillation appears in Figs. 1–4 at the ends of the test interval $[-2, 2]$ is due to Gibbs phenomenon. Fig. 4 shows that for larger wave number k, the regularization method also works well.

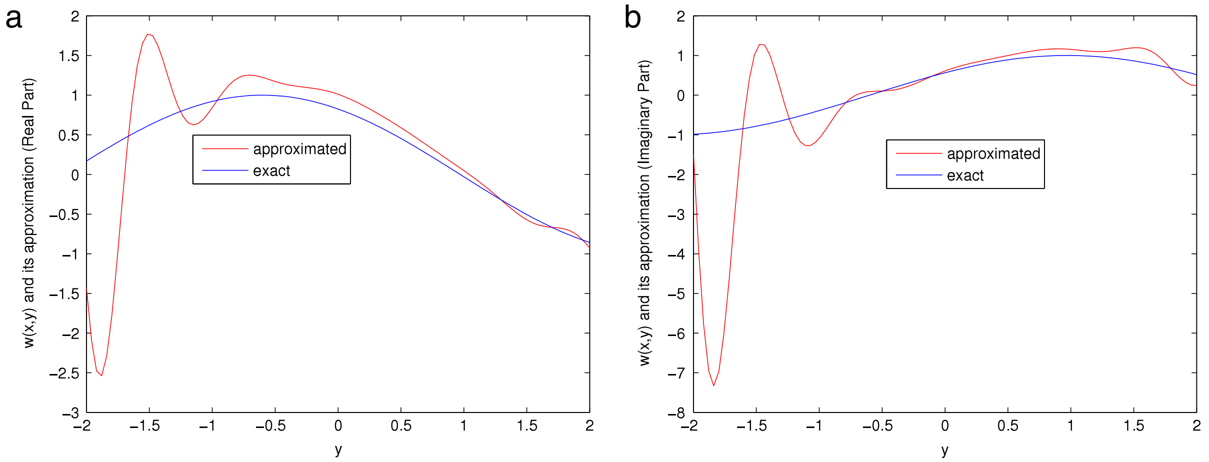

Fig. 3. $x = 0.6, k = 1, \alpha = 0.0006, \sigma_1 = 10^{-4}, \sigma_2 = 10^{-3}$ (3a): the recovery of real part with $RSE = 0.3378$, $RRSE = 0.9968$; (3b): recovery of imaginary part with $RSE = 0.7333$, $RRSE = 2.0048$.

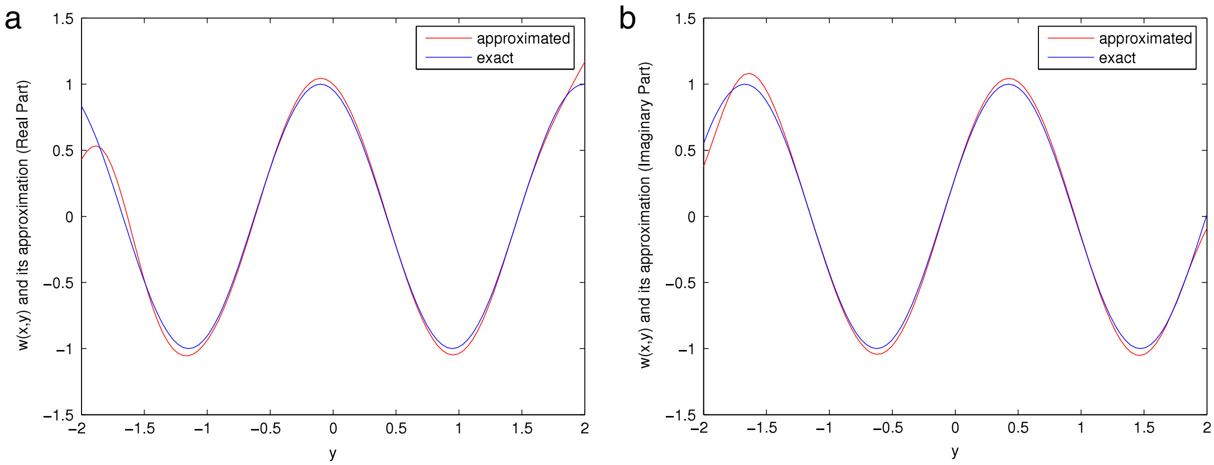

Fig. 4. $x = 0.1, k = 3, \alpha = 0.0006, \sigma_1 = 10^{-4}, \sigma_2 = 10^{-3}$ (4a): recovery of real part with $RSE = 0.0348$, $RRSE = 0.1000$; (4b): recovery of imaginary part with $RSE = 0.0244$, $RRSE = 0.0680$.

6. Concluding remarks

In this paper, we consider the non-characteristic Cauchy problem for the Helmholtz equation. Some Hölder-type stability estimates are proved. This is an improvement for the results obtained in [19]. As for the case of finite domain, the regularization solution can be solved by the boundary element method [15]. The modified method can be easily generalized to the three-dimensional case [24] where we have used the cut-off method and the Tikhonov method.

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