Journal of Computational and Applied Mathematics 236 (2011) 1423–1434

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The MMOC and MMOCAA schemes for the finite volume element method of convection–diffusion problems☆

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ARTICLE INFO

Article history:
Received 19 January 2010
Received in revised form 13 August 2011

Keywords:
MMOC
MMOCAA
Finite volume element method
Error estimates

ABSTRACT

We present and analyze the *modified method of characteristics* (MMOC) and the *modified method of characteristics with adjusted advection* (MMOCAA) for the *finite volume element* (FVE) method of convection–diffusion problems. These two schemes maintain the advantages of both the MMOC and the FVE method. And the MMOCAA scheme discussed herein conserves the conservation law globally at a minor additional computational cost. Optimal-order error estimates in the H^1-norm are proved for these schemes. A numerical example is presented to confirm the estimates.

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1. Introduction

We are concerned with the numerical approximation of the solutions of convection–diffusion problems in which the convection or transport dominates the diffusion. Convection–diffusion transport *partial differential equations* (PDEs) arise in petroleum reservoir simulation, subsurface contaminant transport, and many other important applications. Standard finite difference, finite element, finite volume methods or upwind schemes may respectively tend to generate numerical solutions with severe nonphysical oscillation or excessive numerical diffusion. The *modified method of characteristics* (MMOC) was first formulated for a convection–diffusion equation by Douglas and Russell in [1], and then extended in [2] to nonlinear coupled systems in two and three spatial dimensions. In the MMOC scheme the time derivative and the convection term are combined as a directional derivative along the characteristics, leading to a characteristic time-stepping procedure. Consequently, the MMOC symmetrizes and stabilizes the governing PDEs, allowing for large time steps in a simulation without loss of accuracy, and eliminates the excessive numerical dispersion and grid orientation effects presented in many upwind methods. But the MMOC has one fundamental flaw, the failure to preserve as an algebraic identity a desired conservation law associated with underlying physical problem. So Douglas et al. [3] formulated a variant of MMOC called MMOCAA which does preserve the desired conservation law and the advantage of MMOC. But there is no formal analysis of the MMOCAA in [3]. In [4] Douglas, Huang and Pereira gave an analysis for the MMOCAA. They obtained a suboptimal-order error estimate in the L^2-norm of $O(h+\Delta t)$ for piecewise linear MMOCAA scheme for a convection–diffusion equation with nondegenerate diffusion. Recently, [5] gave an analysis for the piecewise linear MMOC and MMOCAA schemes for multidimensional advection–reaction equations uniformly and obtained an optimal-order error estimate in the L^2-norm for these schemes under superconvergence estimate.

☆ This work is supported in part by Shandong Province Natural Science Foundation (No. ZR2010AQ010, ZR2011AQ021) and Projects of Shandong Province Higher Educational Science and Technology Program (No. J11LA09, J10LA01).

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doi:10.1016/j.cam.2011.09.006

However, these MMOC and MMOCAA schemes are based on the finite difference or finite element methods. As far as we know, there is no such schemes that can be applied to the finite volume element method. In this paper, we discuss the MMOC and MMOCAA schemes based on the finite volume element method which partly fills this gap.

The *finite volume element* (FVE) method, as a type of important numerical tool for solving differential equations, has a long history, and the method has been widely used in several engineering fields, such as fluid mechanics, heat and mass transfer and petroleum engineering. The theoretical framework and the basic tools for the analysis of the FVE method have been developed in the past two decades (see, e.g. [6–13]).

In this paper, we combine the MMOC and MMOCAA with the FVE method to give two new schemes called MMOCFVE and MMOCAAFVE, respectively. These two schemes maintain the advantages of both the MMOC and the FVE method. By introducing a higher-order perturbation, we construct a new global mass conservation scheme—MMOCAAFVE. It is proved to be optimal-order error estimates in the H^1-norm for the MMOCFVE and MMOCAAFVE schemes under general condition. The rest of this paper is given as follows. In Section 2, we construct the MMOCFVE and MMOCAAFVE schemes. Some useful lemmas are introduced in Section 3, and optimal-order error estimates in the H^1-norm are proved for the two schemes. In Section 4, a numerical example is presented to confirm the estimates.

Throughout this paper, C denotes a generic positive constant which does not depend on the spatial and time discretization parameters and may be different at its different occurrences.

2. Construction of the MMOCFVE and MMOCAAFVE schemes

We consider the following two dimensional problem

$$\frac{\partial u}{\partial t} + \mathbf{b}(x)\cdot\nabla u - \nabla\cdot(a(x)\nabla u) = f(x,t), \quad (x,t)\in\Omega\times(0,T], \tag{1}$$

$$u(x,0) = u_0(x), \quad x\in\Omega, \tag{2}$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain, $x=(x_1,x_2)$. $\mathbf{b}(x)=(b_1(x),b_2(x))^{\mathrm{T}}$ and $a(x)$ satisfy

$$0 < a_* \le a(x) \le a^*, \qquad |\mathbf{b}(x)| + |\nabla\mathbf{b}(x)| \le b^*,$$

where, a_*, a^* and b^* are all positive constants.

For convenience, we assume that problem (1) is Ω-periodic, that is, we assume that all functions in (1) are spatially Ω-periodic [3,4]. We also assume that the solution of (1) satisfies

$$u\in L^\infty(0,T;W^{q,2}(\Omega)), \qquad \frac{\partial u}{\partial t}\in L^\infty(0,T;W^{q,2}(\Omega)), \qquad \frac{\partial^2 u}{\partial t^2}\in L^2(0,T;L^2(\Omega)), \quad (q=1,2).$$

2.1. Preliminaries and notation

We will use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with $1\le p\le\infty$ consisting of functions that have generalized derivatives of order s in the space $L^p(\Omega)$. The norm of $W^{s,p}(\Omega)$ is defined by

$$\|u\|_{s,p,\Omega} = \|u\|_{s,p} = \left(\int_\Omega \sum_{|\alpha|\le s} |D^\alpha u|^p dx\right)^{\frac{1}{p}}, \quad 1\le p\le\infty,$$

with the standard modification for $p=\infty$. In order to simplify the notation, we denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p=2$ and Ω whenever possible, i.e., $\|u\|_{s,2,\Omega} = \|u\|_{s,2} = \|u\|_s$.

For the polygonal domain Ω, we consider a quasi-uniform regular triangulation T_h consisting of closed triangle elements K such that $\bar\Omega = \cup_{K\in T_h} K$. We will use $\mathcal{N}_h$ to denote the set of all nodes or vertices of T_h,

$$\mathcal{N}_h = \{p : p \text{ is a vertex of element } K\in T_h \text{ and } p\in\bar\Omega\},$$

and $\mathcal{N}_h^0 = \mathcal{N}_h\cap\Omega$.

Then we introduce a dual mesh T_h^* based on T_h. There are various ways to introduce the dual mesh. Almost all approaches can be described by the following general scheme: in each element $K\in T_h$ consisting of vertices x_i, x_j, x_k, select a point Q in K, and select a point x_{ij} on each of the three edges $\overline{x_ix_j}$ of K. Then connect Q to the points x_{ij} by straight lines r_{ij}. Then for a vertex x_i, we let V_i be the polygon whose edges are r_{ij} in which x_i is a vertex of the element K. We call V_i a control volume centered at x_i. Obviously we have $\cup_{x_i\in\mathcal{N}_h} V_i = \bar\Omega$, and the dual mesh T_h^* is then defined as the set of these control volumes (see Fig. 1). We call the control volume mesh T_h^* quasi-uniform regular if there exists a positive constant $C>0$, such that

$$C^{-1}h^2 \le \mathrm{meas}(V_i) \le Ch^2, \quad \forall V_i\in T_h^*,$$

where h is the maximum diameter of all elements $K\in T_h$.

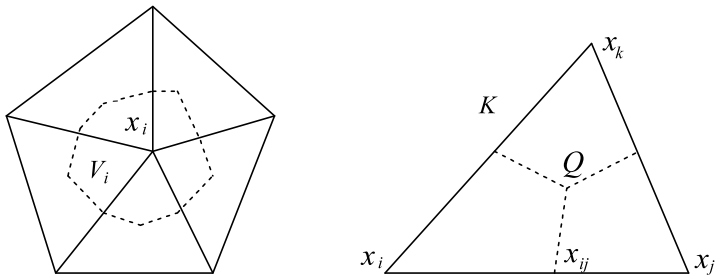


Fig. 1. *Left*: The dotted line shows the boundary of the corresponding control volume V_i with x_i a common vertex. *Right*: a triangle K partitioned into three subregions.

There are various ways to introduce a regular dual mesh T_h^* depending on the choice of the point Q in an element $K \in T_h$ and the points x_{ij} on its edges. In this paper, we use a popular configuration in which Q is chosen to be the barycenter of an element $K \in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K. This type of control volume can be introduced for any triangulation T_h and leads to relatively simple calculations for both two- and three-dimensional problems. In addition, if T_h is locally regular, then the corresponding dual mesh T_h^* is also locally regular. Let S_h be the standard piecewise linear finite element space defined on the triangulation T_h,

$$S_h = \{v \in C(\bar{\Omega}) : v|_K \text{ is linear for all } K \in T_h, \Omega\text{-periodic}\},$$

and its dual volume element space S_h^* on T_h^*,

$$S_h^* = \{v \in L^2(\Omega) : v|_V \text{ is constant for all } V \in T_h^*, \Omega\text{-periodic}\}.$$

Then we obtain $S_h = \text{span}\{\phi_i(x) : x_i \in \mathcal{N}_h^0\}$ and $S_h^* = \text{span}\{\phi_i^*(x) : x_i \in \mathcal{N}_h^0\}$, where $\phi_i(x)$ is the standard nodal basis functions associated with the node x_i, and $\phi_i^*(x)$ is the characteristic functions of the volume V_i.

We define an interpolation operator as $I_h : C(\bar{\Omega}) \to S_h$, such that

$$I_h v = \sum_{x_i \in \mathcal{N}_h} v(x_i)\phi_i(x).$$

For any $v_h \in S_h$, we define another interpolation operator $I_h^* : S_h \to S_h^*$, such that

$$I_h^* v_h = \sum_{x_i \in \mathcal{N}_h} v_h(x_i)\phi_i^*(x).$$

By the interpolation theorem in Sobolev space, we have

$$\|v_h - I_h^* v_h\|_0 \le Ch|v_h|_1. \tag{3}$$

2.2. The MMOCFVE scheme

In the modified method of characteristics, the time derivative and the convection term in (1) is written as a directional derivative

$$\frac{\partial u}{\partial t} + \mathbf{b}(x) \cdot \nabla u = \sqrt{1 + |\mathbf{b}(x)|^2}\frac{\partial u}{\partial \tau},$$

along the characteristic direction $\tau = \tau(x)$ associated with the operator $u_t + \mathbf{b} \cdot \nabla u$ given by

$$\frac{\partial}{\partial \tau} = \frac{1}{\psi(x)}\left(\frac{\partial}{\partial t} + \mathbf{b}(x) \cdot \nabla\right), \quad \psi(x) = \sqrt{1 + |\mathbf{b}(x)|^2}.$$

Then, (1) can be put in the form

$$\psi(x)\frac{\partial u}{\partial \tau} - \nabla \cdot (a(x)\nabla u) = f(x, t), \quad (x, t) \in \Omega \times (0, T]. \tag{4}$$

We define a partition of the time interval $[0, T]$ by $t^n = n\Delta t$, $n = 0, 1, 2, \ldots, N$, with $\Delta t = T/N$. In the standard MMOC [1], the characteristic derivative is approximated by

$$\psi(x)\frac{\partial u^n}{\partial \tau} \approx \psi(x)\frac{u(x, t^n) - u(\bar{x}, t^{n-1})}{\sqrt{(x - \bar{x})^2 + \Delta t^2}} = \frac{u(x, t^n) - u(\bar{x}, t^{n-1})}{\Delta t}, \tag{5}$$

where $\bar{x} = x - \mathbf{b}(x)\Delta t$. Denote $u_h^n = u(x, t^n)$, $\bar{u}_h^{n-1} = u(\bar{x}, t^{n-1})$, $f^n = f(x, t^n)$. Then, the continuous-in-space MMOC procedure is given by

$$\frac{u_h^n - \bar{u}_h^{n-1}}{\Delta t} - \nabla \cdot (a(x)\nabla u_h^n) = f^n, \quad x \in \Omega, \tag{6}$$

with $u_h^0 = u_0(x)$. Combining with FVE, we integrate (6) on each associate control volume V_i and use Green's formula, and sum over all $x_i \in \mathcal{N}_h$ and obtain

$$\sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u_h^n dx - \Delta t \int_{\partial V_i} (a\nabla u_h^n) \cdot \mathbf{n} ds \right) = \sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} \bar{u}_h^{n-1} dx + \Delta t \int_{V_i} f^n dx \right), \tag{7}$$

where $\mathbf{n}$ denotes the unit outward normal on ∂V_i.

Now we rewrite (7) into a variational form. For any $v_h \in S_h$, we multiply the integral form by $I_h^* v_h$ to obtain the MMOCFVE scheme

$$\left(\frac{u_h^n - \bar{u}_h^{n-1}}{\Delta t}, I_h^* v_h \right) + a_h(u_h^n, I_h^* v_h) = (f^n, I_h^* v_h), \tag{8}$$

with $u_h^0 = u(x, 0)$, where $a_h(\cdot, I_h^* \cdot)$ is defined by, for any $u_h, v_h \in S_h$,

$$a_h(u_h, I_h^* v_h) = -\sum_{x_i \in \mathcal{N}_h} \int_{\partial V_i} (a\nabla u_h) \cdot \mathbf{n} I_h^* v_h ds = -\sum_{x_i \in \mathcal{N}_h} v_h(x_i) \int_{\partial V_i} (a\nabla u_h) \cdot \mathbf{n} ds.$$

2.3. The MMOCAAFVE scheme

We note that scheme (8), which combined the MMOC scheme with FVE, maintains the advantages of both MMOC and FVE. However, when we apply it, the conservation law is not automatically satisfied. Indeed, (7) is the equation of the global mass conservation, which the MMOCFVE solution u_h^n satisfies for the given initial condition u_h^{n-1} at time t^n. On the other hand, for each control volume V_i, integrating (1) on $V_i \times (t^{n-1}, t^n)$, and using the divergence theorem, we have

$$\begin{aligned} &\sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u(x, t^n) dx - \int_{t^{n-1}}^{t^n} \int_{V_i} \nabla \mathbf{b}(x) \cdot u(x, t) dx dt - \int_{t^{n-1}}^{t^n} \int_{\partial V_i} a(x) \nabla u(x, t) \cdot \mathbf{n} dx dt \right) \\ &= \sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u(x, t^{n-1}) dx + \int_{t^{n-1}}^{t^n} \int_{V_i} f(x, t) dx dt \right). \end{aligned} \tag{9}$$

Applying Euler quadrature at time t^n to evaluate the space–time integrals in (9), we obtain the following equation of global mass conservation in the integral form (up to the order of the truncation error)

$$\begin{aligned} &\sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u(x, t^n) dx - \Delta t \int_{V_i} \nabla \mathbf{b}(x) \cdot u(x, t^n) dx - \Delta t \int_{\partial V_i} a(x) \nabla u(x, t^n) \cdot \mathbf{n} dx \right) \\ &= \sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u_h(x, t^{n-1}) dx + \Delta t \int_{V_i} f^n dx \right). \end{aligned} \tag{10}$$

Let

$$Q^{n-1} = \sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u_h(x, t^{n-1}) dx + \Delta t \int_{V_i} \nabla \mathbf{b}(x) \cdot u(x, t^n) dx \right), \tag{11}$$

$$\bar{Q}^{n-1} = \sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} u_h(\bar{x}, t^{n-1}) dx \right). \tag{12}$$

In (11), $u(x, t^n)$ is unknown in the evaluation of Q^{n-1}, it is approximated by an extrapolation Eu_h^n instead

$$Eu_h^n = \begin{cases} 2u_h^{n-1} - u_h^{n-2}, & n \geq 2, \\ u_h^0, & n = 1. \end{cases}$$

From the above analysis, we see that in order to maintain mass balance globally, there must be $Q^{n-1} = \bar{Q}^{n-1}$. But in general $Q^{n-1} \neq \bar{Q}^{n-1}$. So we adjust the convection part of the time step to obtain the conservation. Define two higher-order perturbations $\bar{x}^+$ and $\bar{x}^-$ of $\bar{x}$,

$$\bar{x}^+ = x - \mathbf{b}(x)\Delta t + \gamma \mathbf{b}(x)(\Delta t)^2 = \bar{x} + \gamma \mathbf{b}(x)(\Delta t)^2,$$
$$\bar{x}^- = x - \mathbf{b}(x)\Delta t - \gamma \mathbf{b}(x)(\Delta t)^2 = \bar{x} - \gamma \mathbf{b}(x)(\Delta t)^2,$$

where $\gamma > 0$ is a fixed constant, normally chosen to be less than one. Then we let

$$\tilde{u}_h^{n-1} = \tilde{u}_h(\bar{x}, t^{n-1}) = \begin{cases} \max\{u_h(\bar{x}^+, t^{n-1}), u_h(\bar{x}^-, t^{n-1})\}, & \text{if } Q^{n-1} \geq \bar{Q}^{n-1}, \\ \min\{u_h(\bar{x}^+, t^{n-1}), u_h(\bar{x}^-, t^{n-1})\}, & \text{if } Q^{n-1} < \bar{Q}^{n-1}. \end{cases} \tag{13}$$

And set

$$\tilde{Q}^{n-1} = \sum_{x_i \in \mathcal{N}_h} \left(\int_{V_i} \tilde{u}_h^{n-1} dx \right). \tag{14}$$

If $\tilde{Q}^{n-1} = \bar{Q}^{n-1}$, let $\hat{u}_h^{n-1} = u_h^{n-1}$; otherwise, one needs to find θ^{n-1} such that

$$Q^{n-1} = \theta^{n-1}\bar{Q}^{n-1} + (1 - \theta^{n-1})\tilde{Q}^{n-1},$$

and set

$$\hat{u}_h^{n-1} = \theta^{n-1}\bar{u}_h^{n-1} + (1 - \theta^{n-1})\tilde{u}_h^{n-1}.$$

Then we obtain the MMOCAAFVE scheme as follows

$$\left(\frac{u_h^n - \hat{u}_h^{n-1}}{\Delta t}, I_h^* v_h \right) + a_h(u_h^n, I_h^* v_h) = (f^n, I_h^* v_h), \quad \forall v_h \in S_h, \tag{15}$$

with $u_h^0 = u(x, 0)$.

From the above analysis we know that the MMOCAAFVE scheme (15) preserves mass conservation globally.

3. Error estimates for the MMOCFVE and MMOCAAFVE schemes

Before we estimate the errors of the MMOCFVE and MMOCAAFVE schemes, we give some preliminaries and useful lemmas.

3.1. Some preliminaries and lemmas

To describe convergence features, we give some useful lemmas. The first two lemmas have been proved in [9,13], where Lemma 2 shows that $\|\!| \cdot |\!\|_0$ and $\| \cdot \|$ are equivalent. While Lemma 2 indicates that the bilinear form $a(\cdot, I_h^* \cdot)$ is continuous and coercive on S_h.

Lemma 1. *Define $\|\!|u_h|\!\|_0^2 = (u_h, I_h^* u_h)$. Further $(u_h, I_h^* v_h)$, $\forall u_h, v_h \in S_h$ is symmetric and positive definite and the corresponding discrete norm is equivalent to the L^2-norm, i.e., that there exist two positive constants $C_*, C^* > 0$, independent of h such that*

$$C_* \|u_h\| \leq \|\!|u_h|\!\|_0 \leq C^* \|u_h\|, \quad \forall u_h \in S_h. \tag{16}$$

Lemma 2. *There exist two positive constants $\alpha, M > 0$ such that, for all $u_h, v_h \in S_h$, the coercive property*

$$a_h(u_h, I_h^* u_h) \geq \alpha \|u_h\|_1^2$$

and the boundedness property

$$|a_h(u_h, I_h^* v_h)| \leq M \|u_h\|_1 \|v_h\|_1$$

hold true.

The next lemma shows that $a(\cdot, I_h^* \cdot)$ is generally not symmetric and how far it is from being symmetric.

Lemma 3 ([14])**.** *For any $u_h, v_h \in S_h$, there is*

$$|a_h(u_h, I_h^* v_h) - a_h(v_h, I_h^* u_h)| \leq Ch \|u_h\|_1 \|v_h\|_1.$$

Lemma 4. *Assume that $u \in L^\infty(0, T; H^1(\Omega))$. Let $\bar{u}(x) = u(\bar{x}, t^{n-1})$, $\bar{x}$ is defined by (5), and let $\tilde{u}^{n-1}(x) = u(x^*, t^{n-1})$, x^* is equal to $\bar{x}^+$ or $\bar{x}^-$, as required in (13) in the evaluation of $\tilde{u}^{n-1}(x)$. Then we have*

$$\|\tilde{u}^{n-1} - \bar{u}^{n-1}\| \leq C \|u(\cdot, t^{n-1})\|_1 (\Delta t)^2.$$

Proof. Let $\theta(x) = x^* - \bar{x}$ and represent $\tilde{u}^{n-1} - \bar{u}^{n-1}$ in the form

$$\begin{aligned}\tilde{u}^{n-1} - \bar{u}^{n-1} &= u(x^*, t^{n-1}) - u(\bar{x}, t^{n-1}) \\ &= \int_{\bar{x}}^{x^*} \frac{\partial u}{\partial \alpha}(\alpha, t^{n-1}) d\alpha \\ &= \int_0^1 \frac{\partial u}{\partial z}(\bar{x} + \theta(x)z, t^{n-1})\theta(x) dz \\ &= (x^* - \bar{x}) \int_0^1 \frac{\partial u}{\partial z}(\bar{x} + \theta(x)z, t^{n-1}) dz. \end{aligned} \tag{17}$$

Note that $|x^* - \bar{x}| \le C(\Delta t)^2$, thus we have

$$\begin{aligned}\|\tilde{u}^{n-1} - \bar{u}^{n-1}\|^2 &= \int_\Omega |\tilde{u}^{n-1} - \bar{u}^{n-1}|^2 dx \\ &= \int_\Omega |(x^* - \bar{x}) \int_0^1 \frac{\partial u}{\partial z}(\bar{x} + \theta(x)z, t^{n-1}) dz|^2 dx \\ &\le C(\Delta t)^4 \int_\Omega \int_0^1 \left| \frac{\partial u}{\partial z}(\bar{x} + \theta(x)z, t^{n-1}) \right|^2 dz dx \\ &\le C(\Delta t)^4 \|\nabla u(\cdot, t^{n-1})\|^2 \\ &\le C(\Delta t)^4 \|u(\cdot, t^{n-1})\|_1^2, \end{aligned} \tag{18}$$

from this we conclude that

$$\|\tilde{u}^{n-1} - \bar{u}^{n-1}\| \le C\|u(\cdot, t^{n-1})\|_1 (\Delta t)^2. \quad \square$$

Lemma 5. *Assume that $u \in L^\infty(0, T; H^1(\Omega))$. Let $\bar{u}(x) = u(\bar{x}, t^{n-1})$, $\bar{x}$ is defined by (5). Then we have*

$$\|u^{n-1} - \bar{u}^{n-1}\| \le C\|u^{n-1}\|_1 \Delta t. \tag{19}$$

Proof. Note that $|x - \bar{x}| \le C\Delta t$. So similar argument to that given to prove Lemma 4 leads to (19). $\square$

Lemma 6 ([13,14])**.** *Introduce an elliptic projection $P_h : C(\bar{\Omega}) \to S_h$, so that*

$$a_h(u - P_h u, I_h^* v_h) = 0, \quad \forall v_h \in S_h. \tag{20}$$

Then we have

$$\|u - P_h u\|_1 \le Ch\|u\|_2, \tag{21}$$

$$\left\| \frac{\partial u}{\partial t} - P_h \frac{\partial u}{\partial t} \right\|_1 \le Ch \left\| \frac{\partial u}{\partial t} \right\|_2. \tag{22}$$

3.2. Error estimate for the MMOCFVE scheme

Now we consider the error estimate for the MMOCFVE scheme.

Theorem 1. *Let u and u_h be the solutions of (1) and the MMOCFVE scheme (8), respectively. For Δt small enough, if $u_h^0 = P_h u_0$ with P_h defined by (20), we have the error estimate result*

$$\max_{1 \le n \le N} \|u^n - u_h^n\|_1 \le C(\Delta t + h). \tag{23}$$

Proof. For convenience, we set

$$\xi^n = u_h^n - P_h u^n, \qquad \eta^n = u^n - P_h u^n, \qquad u_h^n - u^n = \xi^n - \eta^n.$$

Then from (4), (8) and Lemma 6, we get the following error equation at $t = t^n$

$$\left(\frac{\xi^n - \bar{\xi}^{n-1}}{\Delta t}, I_h^* v_h \right) + a_h(\xi^n, I_h^* v_h) = \left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}, I_h^* v_h \right) + \left(\frac{\eta^n - \bar{\eta}^{n-1}}{\Delta t}, I_h^* v_h \right), \quad \forall v_h \in S_h. \tag{24}$$

Letting $\partial_t \xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$ and choosing $v_h = \partial_t \xi^n$, we get

$$\left(\frac{\xi^n - \bar{\xi}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + a_h(\xi^n, I_h^* \partial_t \xi^n) = \left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + \left(\frac{\eta^n - \bar{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right). \tag{25}$$

Rewrite (25) as

$$\begin{aligned}(\partial_t \xi^n, I_h^* \partial_t \xi^n) + a_h(\xi^n, I_h^* \partial_t \xi^n) = &\left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \\ &+ \left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + \left(\frac{\bar{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right).\end{aligned} \tag{26}$$

Now we estimate (26). First

$$\begin{aligned}a_h(\xi^n, I_h^* \partial_t \xi^n)\Delta t &= \frac{1}{2}\left[a_h(\xi^n + \xi^{n-1}, I_h^*(\xi^n - \xi^{n-1})) + a_h(\xi^n - \xi^{n-1}, I_h^*(\xi^n - \xi^{n-1}))\right] \\ &\geq \frac{1}{2} a_h(\xi^n + \xi^{n-1}, I_h^*(\xi^n - \xi^{n-1})) \\ &= \frac{1}{2}\left[a_h(\xi^n, I_h^* \xi^n) - a_h(\xi^{n-1}, I_h^* \xi^{n-1})\right] - \frac{1}{2}\left[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)\right]\Delta t.\end{aligned} \tag{27}$$

Combining with (27), multiplying (26) by Δt and summing over n from 1 to $l (1 \leq l \leq N)$ at both sides of (26), we have

$$\begin{aligned}&\sum_{n=1}^{l} (\partial_t \xi^n, I_h^* \partial_t \xi^n)\Delta t + \frac{1}{2}\left[a_h(\xi^l, I_h^* \xi^l) - a_h(\xi^0, I_h^* \xi^0)\right] \\ &\leq \sum_{n=1}^{l}\left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\Delta t + \sum_{n=1}^{l}\left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\Delta t \\ &\quad + \sum_{n=1}^{l}\left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\Delta t + \sum_{n=1}^{l}\left(\frac{\bar{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\Delta t \\ &\quad + \frac{1}{2}\sum_{n=1}^{l}\left[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)\right]\Delta t \equiv \sum_{i=1}^{5} T_i.\end{aligned} \tag{28}$$

By Lemmas 1 and 2 and noting that $\xi^0 = 0$, we obtain the error equation

$$\|\xi^l\|_1^2 + \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t \leq C \sum_{i=1}^{5} T_i. \tag{29}$$

Then we estimate the right hand terms of (29). For T_1, from the result in [2], we have

$$\left\|\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}\right\|^2 \leq C\Delta t \int_{t^{n-1}}^{t^n}\int_{\Omega}\left|\frac{\partial^2 u}{\partial \tau^2}\right|^2 dxdt \leq C\Delta t \left\|\frac{\partial^2 u}{\partial \tau^2}\right\|^2_{L^2(t^n-1, t^n; L^2(\Omega))}, \tag{30}$$

so there is

$$\begin{aligned}|T_1| &\leq C\sum_{n=1}^{l}\left\|\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}\right\|^2 \Delta t + \frac{\epsilon}{5}\sum_{n=1}^{l}\|I_h^* \partial_t \xi^n\|^2 \Delta t \\ &\leq C\sum_{n=1}^{l}\left\|\frac{\partial^2 u}{\partial \tau^2}\right\|^2_{L^2(t^n-1, t^n; L^2(\Omega))}(\Delta t)^2 + \frac{\epsilon}{5}\sum_{n=1}^{l}\|\partial_t \xi^n\|^2 \Delta t \\ &\leq C\left\|\frac{\partial^2 u}{\partial \tau^2}\right\|^2_{L^2(0,T; L^2(\Omega))}(\Delta t)^2 + \frac{\epsilon}{5}\sum_{n=1}^{l}\|\partial_t \xi^n\|^2 \Delta t,\end{aligned} \tag{31}$$

where ϵ denotes a small positive constant.

For T_2, from (22), we get

$$|T_2| \leq C\sum_{n=1}^{l}\left\|\frac{\partial \eta^n}{\partial t}\right\|^2 \Delta t + \frac{\epsilon}{5}\sum_{n=1}^{l}\|\partial_t \xi^n\|^2 \Delta t$$

$$\leq C\sum_{n=1}^{l} h^2 \left\| \frac{\partial u^n}{\partial t} \right\|_2^2 \Delta t + \frac{\epsilon}{5}\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t$$

$$\leq Ch^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^\infty(0,T;H^2(\Omega))} + \frac{\epsilon}{5}\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t. \tag{32}$$

By Lemma 5 and (21), we obtain

$$|T_3| = \left| \sum_{n=1}^{l} \left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n \right) \Delta t \right| \leq \sum_{n=1}^{l} \left\| \frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t} \right\| \|I_h^* \partial_t \xi^n\| \Delta t$$

$$\leq C\sum_{n=1}^{l} \|\eta^{n-1}\|_1 \|\partial_t \xi^n\| \Delta t \leq C\sum_{n=1}^{l} \|\eta^{n-1}\|_1^2 \Delta t + \frac{\epsilon}{5}\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t$$

$$\leq Ch^2 \|u\|^2_{L^\infty(0,T;H^2(\Omega))} + \frac{\epsilon}{5}\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t. \tag{33}$$

For T_4, we have the similar estimate,

$$|T_4| = \left| \sum_{n=1}^{l} \left(\frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n \right) \Delta t \right| \leq \sum_{n=1}^{l} \left\| \frac{\xi^{n-1} - \bar{\xi}^{n-1}}{\Delta t} \right\| \|I_h^* \partial_t \xi^n\| \Delta t$$

$$\leq C\sum_{n=1}^{l} \|\xi^{n-1}\|_1 \|\partial_t \xi^n\| \Delta t \leq C\sum_{n=1}^{l} \|\xi^{n-1}\|_1^2 \Delta t + \frac{\epsilon}{5}\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t. \tag{34}$$

By Lemma 3 and the inverse estimate, we have

$$|T_5| \leq C\sum_{n=1}^{l} h\|\xi^n\|_1 \|\partial_t \xi^n\|_1 \Delta t$$

$$\leq C\sum_{n=1}^{l} \|\xi^n\|_1 \|\partial_t \xi^n\| \Delta t$$

$$\leq C\sum_{n=1}^{l} \|\xi^n\|_1^2 \Delta t + \frac{\epsilon}{5}\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t. \tag{35}$$

Combining the error estimates of $T_i (1 \leq i \leq 5)$ with (29), we have

$$\|\xi^l\|_1^2 + \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t \leq C\left\{ \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(0,T;L^2(\Omega))} (\Delta t)^2 + Ch^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^\infty(0,T;H^2(\Omega))} + (h\Delta t)^2 \|u\|^2_{L^\infty(0,T;H^2(\Omega))} \right.$$
$$\left. + (\Delta t)^2 \sum_{n=1}^{l} \|\xi^{n-1}\|_1^2 \Delta t \right\} + C\sum_{n=1}^{l} \|\xi^n\|_1^2 \Delta t + C\epsilon \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t. \tag{36}$$

Choosing proper ϵ and kicking the last term into the left hand side of (36), and applying the Gronwall lemma, we get

$$\|\xi^l\|_1^2 + \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t \leq C\left\{ \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|^2_{L^2(0,T;L^2(\Omega))} (\Delta t)^2 + Ch^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^\infty(0,T;H^2(\Omega))} + (h\Delta t)^2 \|u\|^2_{L^\infty(0,T;H^2(\Omega))} \right\}. \tag{37}$$

Then, we have

$$\|\xi^l\|_1 + \left(\sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t \right)^{1/2} \leq C(h + \Delta t), \tag{38}$$

together with (21) and the triangular inequality yields (23). □

3.3. Error estimate for the MMOCAAFVE scheme

Theorem 2. *Let u and u_h be the solutions of (1) and the MMOCAAFVE scheme (15), respectively. Assume that θ^{n-1} are bounded by n, h and Δt. For Δt small enough, if $u_h^0 = P_h u_0$ with P_h defined by (20), we have the error estimate result*

$$\max_{1\le n\le N} \|u^n - u_h^n\|_1 \le C(\Delta t + h). \tag{39}$$

Proof. From (4), (15) and Lemma 6, we get the following error equation at $t = t^n$

$$\left(\frac{\xi^n - \hat{\xi}^{n-1}}{\Delta t}, I_h^* v_h\right) + a_h(\xi^n, I_h^* v_h) = \left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \hat{u}^{n-1}}{\Delta t}, I_h^* v_h\right) + \left(\frac{\eta^n - \hat{\eta}^{n-1}}{\Delta t}, I_h^* v_h\right), \quad \forall v_h \in S_h. \tag{40}$$

Letting $\partial_t \xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$ and choosing $v_h = \partial_t \xi^n$, we get

$$\left(\frac{\xi^n - \hat{\xi}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + a_h(\xi^n, I_h^* \partial_t \xi^n) = \left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \hat{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n \Delta t\right) + \left(\frac{\eta^n - \hat{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n \Delta t\right). \tag{41}$$

Rewrite (41) as

$$\begin{aligned}
(\partial_t \xi^n, I_h^* \partial_t \xi^n) + a_h(\xi^n, I_h^* \partial_t \xi^n) = {} & \left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \\
& + (1 - \theta^{n-1}) \left(\frac{\tilde{u}^{n-1} - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \\
& + \left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + (1 - \theta^{n-1}) \left(\frac{\bar{\eta}^{n-1} - \tilde{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \\
& + \left(\frac{\bar{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) + (1 - \theta^{n-1}) \left(\frac{\tilde{\xi}^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right).
\end{aligned} \tag{42}$$

Now we estimate (42). Combining with (27), multiplying (42) by Δt and summing over n from 1 to l ($1 \le l \le N$), we have

$$\begin{aligned}
& \sum_{n=1}^{l} (\partial_t \xi^n, I_h^* \partial_t \xi^n) \Delta t + \frac{1}{2} \left[a_h(\xi^l, I_h^* \xi^l) - a_h(\xi^0, I_h^* \xi^0)\right] \\
& \le \sum_{n=1}^{l} \left(\psi \frac{\partial u^n}{\partial \tau} - \frac{u^n - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \Delta t + \sum_{n=1}^{l} \left[(1 - \theta^{n-1}) \left(\frac{\tilde{u}^{n-1} - \bar{u}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\right] \Delta t \\
& \quad + \sum_{n=1}^{l} \left(\frac{\eta^n - \eta^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \Delta t + \sum_{n=1}^{l} \left(\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \Delta t \\
& \quad + \sum_{n=1}^{l} \left[(1 - \theta^{n-1}) \left(\frac{\bar{\eta}^{n-1} - \tilde{\eta}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\right] \Delta t + \sum_{n=1}^{l} \left(\frac{\bar{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right) \Delta t \\
& \quad + \sum_{n=1}^{l} \left[(1 - \theta^{n-1}) \left(\frac{\tilde{\xi}^{n-1} - \bar{\xi}^{n-1}}{\Delta t}, I_h^* \partial_t \xi^n\right)\right] \Delta t + \frac{1}{2} \sum_{n=1}^{l} \left[a_h(\partial_t \xi^n, I_h^* \xi^n) - a_h(\xi^n, I_h^* \partial_t \xi^n)\right] \Delta t \equiv \sum_{i=1}^{8} Q_i.
\end{aligned} \tag{43}$$

By Lemmas 1 and 2 and noting that $\xi^0 = 0$, we obtain the error equation

$$\|\xi^l\|_1^2 + \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t \le C \sum_{i=1}^{8} Q_i. \tag{44}$$

Then we estimate the right hand terms of (44). For Q_1, Q_3, Q_4, Q_6, Q_8, we have the similar estimates as in Theorem 1. For Q_2, Q_5, Q_7, by Lemmas 4 and 6 and the boundedness of θ^{n-1}, we obtain

$$|Q_2| + |Q_5| + |Q_7| \le C \sum_{n=1}^{l} \left(\left\|\frac{\tilde{u}^{n-1} - \bar{u}^{n-1}}{\Delta t}\right\|^2 + \left\|\frac{\bar{\eta}^{n-1} - \tilde{\eta}^{n-1}}{\Delta t}\right\|^2 + \left\|\frac{\tilde{\xi}^{n-1} - \bar{\xi}^{n-1}}{\Delta t}\right\|^2\right) \Delta t + \frac{3\epsilon}{8} \sum_{n=1}^{l} \|\partial_t \xi^n\|^2 \Delta t$$

Table 1
The H^1-norm error when $\epsilon = 1$.

h	H^1-norm error	Error reduction
1/8	0.003815	/
1/16	0.002002	1.90
1/32	0.001191	1.68

$$\leq C\sum_{n=1}^{l}\left(\|u^{n-1}\|_1^2+\|\eta^{n-1}\|_1^2+\|\xi^{n-1}\|_1^2\right)(\Delta t)^3+\frac{3\epsilon}{8}\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t$$
$$\leq C\left(\|u\|^2_{L^\infty(0,T;H^1(\Omega))}(\Delta t)^2+\|u\|^2_{L^\infty(0,T;H^2(\Omega))}h^2\right)(\Delta t)^2$$
$$+C\left((\Delta t)^2\sum_{n=1}^{l}\|\xi^{n-1}\|_1^2\right)\Delta t+\frac{3\epsilon}{8}\sum_{n=1}^{l}\|\partial_t\xi^n\|\Delta t. \tag{45}$$

So we have

$$\|\xi^l\|_1^2+\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t \leq C\left\{\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(0,T;L^2(\Omega))}(\Delta t)^2+Ch^2\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;H^2(\Omega))}\right.$$
$$+\|u\|^2_{L^\infty(0,T;H^1(\Omega))}(\Delta t)^2+(h\Delta t)^2\|u\|^2_{L^\infty(0,T;H^2(\Omega))}$$
$$\left.+(\Delta t)^2\sum_{n=1}^{l}\|\xi^{n-1}\|_1^2\Delta t\right\}+C\sum_{n=1}^{l}\|\xi^n\|_1^2\Delta t+C\epsilon\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t. \tag{46}$$

Choosing proper ϵ and kicking the last term into the left hand side of (44), and applying the Gronwall lemma, we get

$$\|\xi^l\|_1^2+\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t$$
$$\leq C\left\{\left\|\frac{\partial^2 u}{\partial\tau^2}\right\|^2_{L^2(0,T;L^2(\Omega))}(\Delta t)^2+Ch^2\left\|\frac{\partial u}{\partial t}\right\|^2_{L^\infty(0,T;H^2(\Omega))}+\|u\|^2_{L^\infty(0,T;H^1(\Omega))}(\Delta t)^2+(h\Delta t)^2\|u\|^2_{L^\infty(0,T;H^2(\Omega))}\right\}. \tag{47}$$

Then, we have

$$\|\xi^l\|_1+\left(\sum_{n=1}^{l}\|\partial_t\xi^n\|^2\Delta t\right)^{1/2}\leq C(h+\Delta t), \tag{48}$$

together with (21) and the triangular inequality yields (39). □

4. Numerical example

In this section, we consider the following parabolic problem

$$\frac{\partial u}{\partial t}+\nabla u-\nabla\cdot(\epsilon\nabla u(x))=f(x,t),\quad x\in\Omega,\ t>0, \tag{49}$$

where $\Omega=[0,1]^2$. Let the time step $\Delta t=h$ and the space step $h=1/N$ for some given integer N. Denote by U the numerical approximations to $u(x,t)$ and let $n=8, 16, 32$. We use the MMOCAAFVE scheme in this section to solve problem (49) and obtain the discrete solution U.

First, we let $\epsilon=1$, and the following function is used as the exact solution of (49)

$$u(x,t)=\exp(-t)x_1x_2(1-x_1)(1-x_2). \tag{50}$$

When $t=0.25$, this solution is shown in Fig. 2.

When $t=0.25$, $n=16$, the computational results of U and the results of the error $|u-U|$ are as in Figs. 3 and 4. Numerical results of the H^1-norm error when $t=0.25$ are as follows.

Next, let $\epsilon=0.01$, $\epsilon=0.001$, $\epsilon=0.0001$ and $\epsilon=0.00001$. Similarly, choose a certain $f(x,t)$ so that the exact solutions of (49) is (50). We use the same notations and parameters as the case $\epsilon=1$. The numerical results are shown in Tables 2 and 3.

From Tables 1–5, we can see that the method in this section is useful and effective even when ϵ is very small.

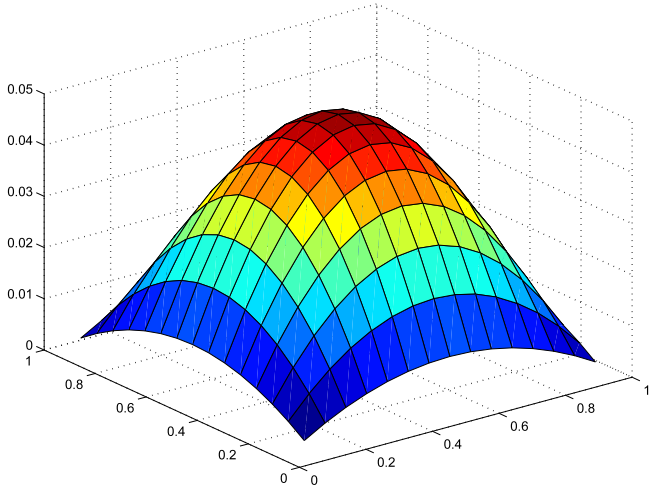


Fig. 2. The exact solution of (49).

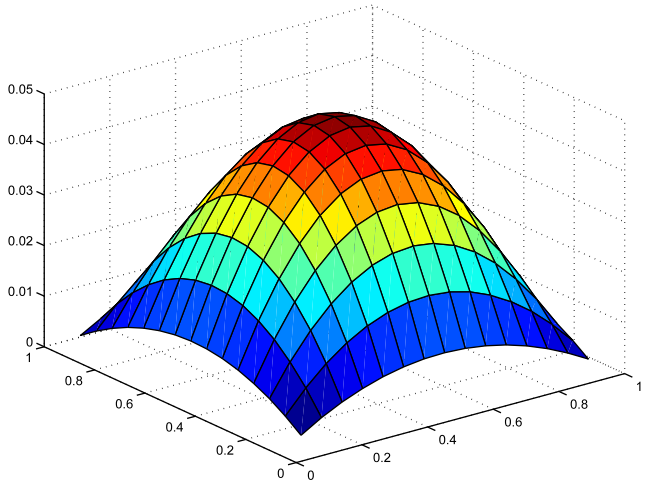


Fig. 3. The approximation U.

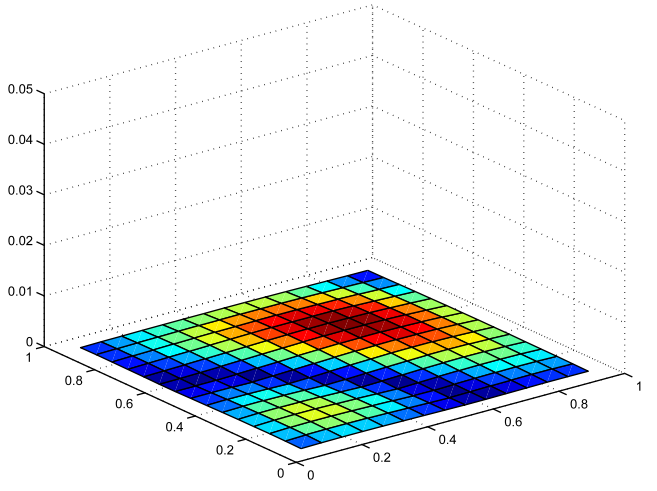


Fig. 4. The error $|u - U|$.

Table 2
The H^1-norm error when $\epsilon = 0.01$.

h	H^1-norm error	Error reduction
1/8	0.004030	/
1/16	0.002217	1.82
1/32	0.001372	1.62

Table 3
The H^1-norm error when $\epsilon = 0.001$.

h	H^1-norm error	Error reduction
1/8	0.004163	/
1/16	0.002341	1.78
1/32	0.001470	1.59

Table 4
The H^1-norm error when $\epsilon = 0.0001$.

h	H^1-norm error	Error reduction
1/8	0.004314	/
1/16	0.002475	1.74
1/32	0.001572	1.57

Table 5
The H^1-norm error when $\epsilon = 0.00001$.

h	H^1-norm error	Error reduction
1/8	0.004475	/
1/16	0.002603	1.72
1/32	0.001680	1.55

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