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A new approach to characterize the solution set of a pseudoconvex programming problem

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ABSTRACT

A new approach to characterize the solution set of a nonconvex optimization problem via its dual problem is proposed. Some properties of the Lagrange function associated to the problem are investigated. Then characterizations of the solution set of the problem are established.

1. Introduction

The study of characterizations of solution sets of mathematical programming problems is a topic which has attracted several authors for years. For programs that have multiple solutions, based on understanding characterizations of solution sets, solution methods can be developed. Mangasarian firstly presented characterizations of solution sets for convex problems by using subdifferentials of convex functions [1]. Since the appearance of the paper, there were several results on characterizations of solution sets of convex/nonconvex optimization problems published such as [2–12]. The method above was developed to give characterizations of solution sets of cone-constrained convex problems [7] and of convex infinite problems [10]. This method was also used to give characterizations of solution sets of nonconvex problems such as [13,14] with generalized convex functions involved.

Our aim is to present an approach to establish characterizations of solution sets of a class of nonconvex programs via dual problems. We also show that characterizations of solution sets of some class of convex/nonconvex optimization problems given in several papers before can be obtained by this approach.

Let us consider the following problem

$$(\mathrm{P})\quad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & f_t(x) \le 0, \quad t \in T, \\ & x \in C, \end{array}$$

where the functions $f, f_t : X \to \mathbb{R},\ t \in T$, are locally Lipschitz on a Banach space X, T is an arbitrary (possibly infinite) index set, and C is a closed convex subset of X. Problems with an infinite number of constraints have been considered recently in several papers with various requirements on the functions $f, f_t,\ t \in T$, and the space X [10,15–20].

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It is well known that generalized convexity plays crucial role in nonconvex optimization. In this paper we use a concept of generalized convexity, named pseudoconvexity, in case nondifferentiable functions proposed by Hiriart-Urruty [21] for the functions involved. Note that the concept of pseudoconvex function for differentiable functions was firstly introduced by Mangasarian [22]. The reader is referred to [23], where the properties of generalized convexity are surveyed. Theory of nonsmooth optimization used in this paper is based on the well known books of Clarke [24,25].

To give characterizations of solution sets of inequality constrained convex/nonconvex programming problems, it usually needs at least one given solution which satisfies some optimality condition. Then Lagrange functions, corresponding to some Lagrange multipliers given by optimality conditions, are proved to be constant on the solution sets of the problems [7–9,14, 13,10]. Based on these conditions, characterizations of solution sets of the problems are established. In our approach, to give characterizations of the solution set of (P), we formulate a dual problem of (P) in a mixed type of Wolfe type and Mond–Weir type. Then characterizations of the solution set will be established upon the assumptions that there exists a given solution of the dual problem of (P) and the two problems are of equal optimal value.

We now describe the results. The first part is devoted to present some new properties of Lagrange function. Note that the Lagrange function associated to (P) is also the objective function of its dual problem in Wolfe type. We show that the Lagrange function (corresponding to some Lagrange multiplier) can be constant on a subset of X which is wider than the solution set of (P). Moreover, with a given solution of (P), the Lagrange function is constant on the set of all corresponding Lagrange multipliers. In the second part, we establish several characterizations of the solution set via the dual problems of (P). We also show that, by using similar approaches, several results on characterizations of solution sets of optimization problems studied before can be found.

The paper is organized as follows. The next section is devoted to preliminaries. Our main results are in Section 3. They are included in two part. In the first one, some new properties of Lagrange function are presented. The last one is devoted to give the characterizations of solution set of (P). Some examples are given.

2. Preliminaries

Let us denote by $\mathbb{R}^{(T)}$ the linear space of generalized finite sequences $\lambda = (\lambda)_{t\in T}$ such that $\lambda_t \in \mathbb{R}$ for all $t \in T$ but only finitely many $\lambda_t \neq 0$ (see [26]),

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

For each $\lambda \in \mathbb{R}^{(T)}$, the supporting set corresponding to λ is $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$. It is a finite subset of T. We denote $\mathbb{R}^{(T)}_+ := \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0,\ t \in T\}$. It is a nonnegative cone of $\mathbb{R}^{(T)}$. For $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space, we understand that

$$\sum_{t\in T} \lambda_t z_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t & \text{if } T(\lambda) \neq \emptyset, \\ 0 & \text{if } T(\lambda) = \emptyset. \end{cases}$$

Throughout this paper X is a Banach space, C is a nonempty closed convex subset of X, T is a compact topological space, $f : X \to \mathbb{R}$ is a locally Lipschitz function, and $f_t : X \to \mathbb{R}$, $t \in T$, are locally Lipschitz with respect to x uniformly in t, i.e.,

$$\forall x \in X,\ \exists U(x),\ \exists K > 0, \quad |f_t(u) - f_t(v)| \leq K\|u - v\|,\ \forall u, v \in U(x),\ \forall t \in T.$$

The following concepts can be found in the Clarke's books [24,25]. Let D be a nonempty closed convex subset of X. The normal cone to D at a point $z \in D$ coincides with the normal cone in the sense of convex analysis and given by

$$N(D, z) := \left\{v \in X^* \mid v(x - z) \leq 0,\ \forall x \in D\right\}.$$

Let $g : X \to \mathbb{R}$ be a locally Lipschitz function. The directional derivative of g at $z \in X$ in direction $d \in X$, is

$$g'(z; d) = \lim_{t\to 0^+} \frac{g(z + td) - g(z)}{t}$$

if the limit exists. The Clarke generalized directional derivative of g at $z \in X$ in direction $d \in X$ is

$$g^c(z; d) := \limsup_{\substack{y\to z \\ t\to 0^+}} \frac{g(y + td) - g(y)}{t}.$$

The Clarke subdifferential of g at $z \in X$, denoted by $\partial^c g(z)$, is defined by

$$\partial^c g(z) := \left\{v \in X^* \mid v(d) \leq g^c(z; d),\ \forall d \in X\right\}.$$

A locally Lipschitz function g is said to be regular (in the sense of Clarke) at $z \in X$ if $g'(z; d)$ exists and

$$g^c(z; d) = g'(z; d), \quad \forall d \in X.$$

The following definition is due to Hiriart-Urruty [21].

Definition 2.1. A locally Lipschitz function $f : X \to \mathbb{R}$ is said to be pseudoconvex if for all $x, y \in X$,

$$f^c(x; y - x) \geq 0 \Longrightarrow f(y) \geq f(x).$$

Definition 2.2. A locally Lipschitz function $f : X \to \mathbb{R}$ is said to be quasiconvex if for all $x, y \in X$,

$$f(y) \le f(x) \Longrightarrow f^c(x; y - x) \le 0.$$

We need the following lemmas.

Lemma 2.1. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz pseudoconvex function. If there exists $u \in \partial^c f(x)$ such that $u(y-x) \ge 0$, $y, x \in X$, then $f(y) \ge f(x)$.*

Lemma 2.2. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. If f is pseudoconvex then f is quasiconvex.*

3. Main results

Let us denote by A the feasible set of (P) and denote by Sol(P) the solution set of (P),

$$\mathrm{Sol(P)} = \{z \in A \mid f(z) \le f(x),\ \forall x \in A\}.$$

We suppose that $\mathrm{Sol(P)} \neq \emptyset$. Suppose further that, under some constraint qualification condition, if $z \in \mathrm{Sol(P)}$ then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that the following condition holds (see [14,19]),

$$0 \in \partial^c f(z) + \sum_{t\in T} \lambda_t \partial^c f_t(z) + N(C, z), \lambda_t f_t(z) = 0, \quad \forall t \in T. \tag{1}$$

The Lagrange function associated to (P) is formulated by

$$L(x, \lambda) = \begin{cases} f(x) + \sum_{t\in T} \lambda_t f_t(x), & (x, \lambda) \in C \times \mathbb{R}_+^{(T)} \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $\lambda \in \mathbb{R}_+^{(T)}$, the function $L(\cdot, \lambda)$ is locally Lipschitz on X.

The dual problem of (P) in a mixed type of Wolfe type and in Mond–Weir type is formulated by

$$\begin{array}{lll} \text{(D)} & \text{Maximize} & L(y, \lambda) := f(y) + \sum_{t\in T} \lambda_t f_t(y) \\ & \text{s.t.} & 0 \in \partial^c f(y) + \sum_{t\in T} (\lambda_t + \mu_t) \partial^c f_t(y) + N(C, y), \\ & & \mu_t f_t(y) \ge 0, \quad t \in T, \\ & & (y, \lambda, \mu) \in C \times \mathbb{R}_+^{(T)} \times \mathbb{R}_+^{(T)}. \end{array}$$

Let us denote by G the feasible set of (D). The optimal values of the problems (P) and (D) are denoted by $V(\mathrm{P})$ and $V(\mathrm{D})$, respectively.

It is obvious that, for $\mu = 0$, the problem (D) is equivalent to the dual problem of (P) in Wolfe type, denoted by $(\mathrm{D_W})$,

$$\begin{array}{lll} (\mathrm{D_W}) & \text{Maximize} & L(y, \lambda) := f(y) + \sum_{t\in T} \lambda_t f_t(y) \\ & \text{s.t.} & 0 \in \partial^c f(y) + \sum_{t\in T} \lambda_t \partial^c f_t(y) + N(C, y), \\ & & (y, \lambda) \in C \times \mathbb{R}_+^{(T)}. \end{array}$$

For $\lambda = 0$, the problem (D) is equivalent to the dual problem of (P) in Mond–Weir type, denoted by $(\mathrm{D_M})$,

$$\begin{array}{lll} (\mathrm{D_M}) & \text{Maximize} & f(y) \\ & \text{s.t.} & 0 \in \partial^c f(y) + \sum_{t\in T} \mu_t \partial^c f_t(y) + N(C, y), \\ & & \mu_t f_t(y) \ge 0, \quad t \in T, \\ & & (y, \mu) \in C \times \mathbb{R}_+^{(T)}. \end{array}$$

From now on, we suppose that the function $L(\cdot, \lambda)$ is pseudoconvex on X for every $\lambda \in \mathbb{R}_+^{(T)}$, and $f, f_t,\ t \in T$, are regular on X.

Lemma 3.1. *If z is a solution of (P) and there exists $\bar\lambda \in \mathbb{R}_+^{(T)}$ such that the condition (1) holds for $(z, \bar\lambda)$, then $(z, \bar\lambda, 0)$ and $(z, 0, \bar\lambda)$ are solutions of (D) and $V(\mathrm{P}) = V(\mathrm{D})$.*

Proof. If z is a solution of (P) and there exists $\bar\lambda$ such that the condition (1) holds for $(z, \bar\lambda)$, then $(z, \bar\lambda, 0)$ and $(z, 0, \bar\lambda)$ are feasible solutions of (D). Firstly, we prove that $L(z, \bar\lambda) \ge L(x, \lambda)$ for all $(x, \lambda, \mu) \in G$. Indeed, since $(z, \bar\lambda)$ satisfies (1), we have that $\bar\lambda_t f_t(z) = 0$ for all $t \in T$. Then,

$$L(z, \bar\lambda) - L(x, \lambda) = f(z) - L(x, \lambda). \tag{2}$$

On the other hand, since $(x, \lambda, \mu) \in G$, there exist $u \in \partial^c f(x)$, $u_t \in \partial^c f_t(x)$, $t \in T$ and $w \in N(C, x)$ such that $\mu_t f_t(x) \geq 0$ for all $t \in T$ and

$$u + \sum_{t\in T}(\lambda_t + \mu_t)u_t + w = 0.$$

Hence,

$$\left[u + \sum_{t\in T}(\lambda_t + \mu_t)u_t\right](y - x) = -w(y - x) \geq 0, \quad \forall y \in C.$$

Since the function $L(\cdot, \lambda)$ is pseudoconvex on X for every $\lambda \in \mathbb{R}_+^{(T)}$, and $f, f_t, t \in T$, are regular on X, it is easy to deduce that

$$L(y, \lambda + \mu) - L(x, \lambda + \mu) \geq 0, \quad \forall y \in C.$$

Since $z \in A$ and $\mu_t f_t(x) \geq 0$ for all $t \in T$, we get

$$f(z) \geq L(z, \lambda + \mu) \geq L(x, \lambda + \mu) \geq L(x, \lambda).$$

This and (2) implies that $L(z, \bar{\lambda}) \geq L(x, \lambda)$ for all $(x, \lambda, \mu) \in G$. We conclude that $(z, \bar{\lambda}, 0)$ is a solution of (D).

For the feasible point $(z, 0, \bar{\lambda})$, we need to prove that $L(z, 0) \geq L(x, \lambda)$ for all $(x, \lambda, \mu) \in G$. Indeed, note that $L(z, 0) - L(x, \lambda) = f(z) - L(x, \lambda)$. Using an argument as above, we obtain $f(z) \geq L(x, \lambda)$ for all $(x, \lambda, \mu) \in G$. This shows that $(z, 0, \bar{\lambda})$ is a solution of (D). $\square$

The following corollary can be deduced directly from Lemma 3.1. The proof is omitted.

Lemma 3.2. *If z is a solution of* (P) *and there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that* (1) *holds for $(z, \bar{\lambda})$, then $(z, \bar{\lambda})$ is a solution of* $(\mathrm{D_W})$ *and* $(\mathrm{D_M})$*, and* $V(\mathrm{P}) = V(\mathrm{D_W}) = V(\mathrm{D_M})$.

3.1. Some properties of Lagrange function

It is well known that, in mathematical programming, Lagrange function plays a key role to find maxima and minima of problems which have constraint functions. In this part we will introduce some new properties of the Lagrange function associated to (P) which relate to its dual problems.

Proposition 3.1. *Suppose that* (y^*, λ^*, μ^*) *is a solution of* (D).

(i) *It holds* $L(y, \lambda^* + \mu^*) = V(\mathrm{D})$ *for all* $y \in G_1 := \{y \in C \mid (y, \lambda^*, \mu^*) \in G\}$ *and* $\mu_t^* f_t(y^*) = 0$ *for all* $t \in T$.
(ii) *Furthermore, if* $V(\mathrm{D}) = V(\mathrm{P})$ *then* $L(y, \lambda^* + \mu^*) = V(\mathrm{D})$ *for all* $y \in \mathrm{Sol}(\mathrm{P})$ *and* $(\lambda_t^* + \mu_t^*)f_t(y) = 0$ *for all* $t \in T$.

Proof. (i) Let (y^*, λ^*, μ^*) be a solution of (D). We obtain $(y^*, \lambda^*, \mu^*) \in C \times \mathbb{R}_+^{(T)} \times \mathbb{R}_+^{(T)}$ such that

$$0 \in \partial^c f(y^*) + \sum_{t\in T}(\lambda_t^* + \mu_t^*)\partial^c f_t(y^*) + N(C, y^*), \ \mu_t^* f_t(y^*) \geq 0, \quad t \in T.$$

Thus, there exist $u \in \partial^c f(y^*)$, $u_t \in \partial^c f_t(y^*)$, $t \in T$, $w \in N(C, y^*)$ such that

$$u + \sum_{t\in T}(\lambda_t^* + \mu_t^*)u_t + w = 0. \tag{3}$$

Since $L(\cdot, \lambda)$ is pseudoconvex on X for every $\lambda \in \mathbb{R}_+^{(T)}$, and $f, f_t, t \in T$, are regular on X, using an argument as in the proof of Lemma 3.1, from (3) we deduce

$$L(y^*, \lambda^* + \mu^*) \leq L(y, \lambda^* + \mu^*), \quad \forall y \in C. \tag{4}$$

Hence,

$$L(y^*, \lambda^*) \leq L(y^*, \lambda^* + \mu^*) \leq L(y, \lambda^* + \mu^*), \quad \forall y \in G_1.$$

So,

$$L(y^*, \lambda^*) \leq \inf_{G_1} L(y, \lambda^* + \mu^*). \tag{5}$$

On the other hand, for λ^* and μ^* above,

$$\inf_{G_1} L(y, \lambda^* + \mu^*) \leq \sup_{G_1} L(y, \lambda^* + \mu^*) \leq \sup_{(y,\lambda,\mu)\in G} L(y, \lambda) = L(y^*, \lambda^*).$$

Combining this and (5) we get

$$L(y, \lambda^* + \mu^*) = L(y^*, \lambda^*) = V(\mathrm{D}), \quad \forall y \in G_1.$$

In addition, by taking $y = y^*$ in the left hand side of the equality $L(y, \lambda^* + \mu^*) = L(y^*, \lambda^*)$ we get $\mu_t^* f_t(y^*) = 0$ for all $t \in T$.

(ii) If $V(\mathrm{D}) = V(\mathrm{P})$ then by (4), we get

$$V(\mathrm{P}) = L(y^*, \lambda^* + \mu^*) \le L(y, \lambda^* + \mu^*) \le f(y) = V(\mathrm{P}), \quad \forall y \in \mathrm{Sol}(\mathrm{P}),$$

i.e., $L(y, \lambda^* + \mu^*) = V(\mathrm{D})$ for all $y \in \mathrm{Sol}(\mathrm{P})$. This implies that $f(y) + \sum_{t\in T}(\lambda_t^* + \mu_t^*)f_t(y) = f(y)$ for all $y \in \mathrm{Sol}(\mathrm{P})$. Hence, $(\lambda_t^* + \mu_t^*)f_t(y) = 0$ for all $t \in T$. □

Corollary 3.1. *Suppose that z is a solution of (P) and there exists $\bar\lambda$ such that the condition (1) holds for $(z, \bar\lambda)$. Then*

$$L(y, \bar\lambda) = f(z),\ \forall y \in G_1 := \{y \in C \mid (y, \bar\lambda, 0) \in G\}, \tag{6}$$

and $\bar\lambda_t f_t(y) = 0$ for all $y \in \mathrm{Sol}(\mathrm{P})$.

Proof. Let $z \in \mathrm{Sol}(\mathrm{P})$ and $\bar\lambda \in \mathbb{R}_+^{(T)}$ be such that the condition (1) holds for $(z, \bar\lambda)$. By Lemma 3.1, $(z, \bar\lambda, 0)$ is a solution of (D) and strong duality holds. Hence, by (i) of Proposition 3.1, the conclusion (6) holds and by (ii) of Proposition 3.1, we get $\bar\lambda_t f_t(y) = 0$ for all $y \in \mathrm{Sol}(\mathrm{P})$. □

Remark 3.1. (1) From the corollary above, we can see that the Lagrange function of (P) can be constant on a subset of X wider than the solution set of (P) and Corollary 3.1 covers Lemma 1 in [18].
(2) If the involved functions of (P) are convex, Corollary 3.1 covers Lemma 3.1 in [10].
(3) Using the same method, we can establish the results which cover Theorem 2.1 in [7] and Theorem 3.2 in [13].

Let $\bar G$ and $\bar{\bar G}$ be the feasible sets of $(\mathrm{D_W})$ and $(\mathrm{D_M})$, respectively. We deduce some more corollaries as follows.

Corollary 3.2. *Suppose that (y^*, λ^*) is a solution of $(\mathrm{D_W})$.*
(i) *It holds $L(y, \lambda^*) = V(\mathrm{D_W})$ for all $y \in \bar G_1$, where $\bar G_1 = \{y \in C \mid (y, \lambda^*) \in \bar G\}$.*
(ii) *Furthermore, if $V(\mathrm{D_W}) = V(\mathrm{P})$ then $L(y, \lambda^*) = V(\mathrm{D_W})$ for all $y \in \mathrm{Sol}(\mathrm{P})$ and $\lambda_t^* f_t(y) = 0$ for all $t \in T$.*

Proof. The conclusions of the corollary follow if the problem (D) is considered in the special case $\mu = 0$. In this case the fact that (y^*, λ^*) is a solution of $(\mathrm{D_W})$ is equivalent to $(y^*, \lambda^*, 0)$ is a solution of (D). □

Corollary 3.3. *Suppose that (y^*, μ^*) is a solution of $(\mathrm{D_M})$.*
(i) *It holds $L(y, \mu^*) = V(\mathrm{D_M})$, $\forall y \in \bar{\bar G}_1$ where $\bar{\bar G}_1 = \{y \in C \mid (y, \mu^*) \in \bar{\bar G}\}$.*
(ii) *Furthermore, if $V(\mathrm{D_M}) = V(\mathrm{P})$ then $L(y, \mu^*) = V(\mathrm{D_M})$ for all $y \in \mathrm{Sol}(\mathrm{P})$ and $\mu_t^* f_t(y) = 0$ for all $t \in T$.*

Proof. The conclusions of the corollary follow if the problem (D) is considered in the special case $\lambda = 0$. □

We now present another property of Lagrange function.

Proposition 3.2. *Suppose that (y^*, λ^*, μ^*) is a solution of (D). If $f(y^*) \ge V(\mathrm{P})$ then*

$$L(y^*, \lambda) = V(\mathrm{P}), \quad \forall \lambda \in G_2 := \{\lambda \in \mathbb{R}_+^{(T)} \mid (y^*, \lambda, \mu) \in G, \lambda_t f_t(y^*) \ge 0, t \in T\}.$$

Proof. Since (y^*, λ^*, μ^*) is a solution of (D) and $f(y^*) \ge V(\mathrm{P})$,

$$L(y^*, \lambda^*) \ge L(y^*, \lambda) \ge f(y^*) \ge V(\mathrm{P}), \quad \forall \lambda \in G_2. \tag{7}$$

On the other hand, since $(y^*, \lambda^*, \mu^*) \in G$, using an argument as in the proof of Proposition 3.1, we get $L(y^*, \lambda^*) \le L(y^*, \lambda^* + \mu^*) \le L(y, \lambda^* + \mu^*)$ for all $y \in C$. Hence, $L(y^*, \lambda^*) \le L(y, \lambda^* + \mu^*) \le f(y)$ for all $y \in A$. This implies that $L(y^*, \lambda^*) \le V(\mathrm{P})$. Combining this and (7), we obtain the desired result. □

Corollary 3.4. *If y^* is a solution of (P) then the function $L(y^*, \cdot)$ is constant on G_2.*

Proof. Suppose that y^* is a solution of (P). There exists λ^* such that (1) holds. By Lemma 3.1, $(y^*, \lambda^*, 0)$ is a solution of (D). By Proposition 3.2, we have that $L(y^*, \cdot)$ is constant on G_2. □

Corresponding to $(\mathrm{D_W})$ we have the following corollary.

Corollary 3.5. *Let (y^*, λ^*) be a solution of $(\mathrm{D_W})$. If $f(y^*) \ge V(\mathrm{P})$ then*

$$L(y^*, \lambda) = V(\mathrm{P}), \quad \forall \lambda \in \bar G_2 = \{\lambda \in \mathbb{R}_+^{(T)} \mid (y^*, \lambda) \in \bar G, \lambda_t f_t(y) \ge 0,\ t \in T\}.$$

Proof. Let (y^*, λ^*) be a solution of $(\mathrm{D_W})$. It is equivalent to $(y^*, \lambda^*, 0)$ is a solution of the problem (D) with $\mu = 0$. Applying Proposition 3.2, we deduce the function $L(y^*, \cdot)$ is constant on $\bar G_2$. □

Example.

$$\begin{array}{lll} (\mathrm{Q}_1) & \text{Minimize} & x - y + (x-y)^3 \\ & \text{s.t.} & y - x \le 0, \\ & & x^2 - y \le 0. \end{array}$$

A simple computation gives that $\mathrm{Sol}(\mathrm{Q}_1) = \{(x, y) \mid 0 \le x \le 1, x = y\}$. The dual problem of (Q_1) in Wolfe type is formulated by

$$(\mathrm{Q}_1^*) \quad \begin{array}{ll} \text{Maximize} & L(z,\lambda) = x - y + (x-y)^3 + \lambda_1(y-x) + \lambda_2(x^2 - y) \\ \text{s.t.} & (0,0) = \big(1 + 3(x-y)^2 - \lambda_1 + 2\lambda_2 x, -1 - 3(x-y)^2 + \lambda_1 - \lambda_2\big), \\ & (x,y) \in \mathbb{R}^2, \quad \lambda_1, \lambda_2 \ge 0. \end{array}$$

We can check that $\bar{z} = (0, 0)$ is a solution of (Q_1) and $(\bar{z}, \bar{\lambda})$ is a solution of (Q_1^*) where $\bar{\lambda} = (1, 0)$. Note that $L(\cdot, \bar{\lambda})$ is a pseudoconvex function. We will show that $L(\cdot, \bar{\lambda})$ is constant on a subset of X wider than solution set of (Q_1). Indeed, we can check that the set $\bar{G}_1$ corresponding to $\bar{\lambda} = (1, 0)$ is $\{(x, y) \mid x - y = 0\}$. Obviously, the function $L(\cdot, \bar{\lambda})$ equals to 0 on $\bar{G}_1$ and the range of x is out of $[0, 1]$.

Let z be a solution of (Q_1). From the feasible set of (Q_1^*) we can check that $\bar{G}_2 = \{(\lambda_1, \lambda_2) \mid \lambda_1 > 0, \lambda_2 = 0\}$. The function $L(z, \cdot)$ is constant on $\bar{G}_2$.

3.2. Characterizations of solution set of (P)

In this part, based on the results on properties of the Lagrange function established above, we will give characterizations of solution set of (P) via its dual problems. Then characterizations of solution sets of some aforesaid problems can be covered.

Theorem 3.1. *Suppose that* $(z, \bar{\lambda}, \bar{\mu})$ *is a solution of* (D). *If* $V(\mathrm{P}) = V(\mathrm{D})$ *then* $\mathrm{Sol}(\mathrm{P}) = S = S_1$ *where*

$$S = \{y \in C \mid \exists u \in \partial^c L(y, \bar{\lambda} + \bar{\mu}), u(z-y) = 0, f_t(y) = 0, t \in T(\bar{\lambda} + \bar{\mu}); f_t(y) \le 0, t \in T \setminus T(\bar{\lambda} + \bar{\mu})\},$$

$$S_1 = \{y \in C \mid \exists u \in \partial^c L(y, \bar{\lambda} + \bar{\mu}) \cap \partial^c L(z, \bar{\lambda} + \bar{\mu}), u(z-y) = 0, f_t(y) = 0, \\ t \in T(\bar{\lambda} + \bar{\mu}); f_t(y) \le 0, t \in T \setminus T(\bar{\lambda} + \bar{\mu})\}.$$

Proof. Since $S_1 \subset S$, it needs to prove that $\mathrm{Sol}(\mathrm{P}) \subset S_1$ and $S \subset \mathrm{Sol}(\mathrm{P})$.

(i) $S \subset \mathrm{Sol}(\mathrm{P})$: Let $y \in S$. Then $y \in C$ and there exists $u \in \partial^c L(y, \bar{\lambda} + \bar{\mu})$ such that $u(z - y) = 0$ and $f_t(y) = 0, t \in T(\bar{\lambda} + \bar{\mu}); f_t(y) \le 0, t \in T \setminus T(\bar{\lambda} + \bar{\mu})$, i.e., $y \in A$. Since the function $L(\cdot, \bar{\lambda} + \bar{\mu})$ is pseudoconvex on X, by Lemma 2.1, it deduces that $L(z, \bar{\lambda} + \bar{\mu}) \ge L(y, \bar{\lambda} + \bar{\mu})$, i.e.,

$$f(z) + \sum_{t \in T} (\bar{\lambda}_t + \bar{\mu}_t) f_t(z) \ge f(y) + \sum_{t \in T} (\bar{\lambda}_t + \bar{\mu}_t) f_t(y).$$

By Proposition 3.1, we have $\bar{\mu}_t f_t(z) = 0$ for all $t \in T$. Combining this and $f_t(y) = 0, t \in T(\bar{\lambda} + \bar{\mu})$, from the inequality above we get

$$V(\mathrm{D}) = f(z) + \sum_{t \in T} \bar{\lambda}_t f_t(z) \ge f(y).$$

Since $V(\mathrm{P}) = V(\mathrm{D})$ and $y \in A$, $V(\mathrm{P}) \ge f(y)$. Thus, $y \in \mathrm{Sol}(\mathrm{P})$.

(ii) $\mathrm{Sol}(\mathrm{P}) \subset S_1$: Let $y \in \mathrm{Sol}(\mathrm{P})$. By Proposition 3.1, we have $L(z, \bar{\lambda} + \bar{\mu}) = L(y, \bar{\lambda} + \bar{\mu})$ and $f_t(y) = 0$ for all $t \in T(\bar{\lambda} + \bar{\mu})$. Since $L(z, \bar{\lambda} + \bar{\mu}) = L(y, \bar{\lambda} + \bar{\mu})$, by Lemma 2.2 and Definition 2.2, we can deduce that $u(y - z) \le 0$ for all $u \in \partial^c L(z, \bar{\lambda} + \bar{\mu})$. Since $(z, \bar{\lambda}, \bar{\mu}) \in G$, there exist $v \in \partial^c f(z)$, $u_t \in \partial^c f_t(z)$, $t \in T$, and $w \in N(C, z)$ such that

$$v + \sum_{t \in T} (\bar{\lambda}_t + \bar{\mu}_t) u_t + w = 0 \tag{8}$$

and $\bar{\mu}_t f_t(z) \ge 0$ for all $t \in T$. Set $u = v + \sum_{t \in T} (\bar{\lambda}_t + \bar{\mu}_t) u_t$. It is easy to see that $u \in \partial^c L(\cdot, \bar{\lambda} + \bar{\mu})(z)$ and $u(x - z) \ge 0$ for all $x \in C$. Hence, there exists $u \in \partial^c L(\cdot, \bar{\lambda} + \bar{\mu})(z)$ such that $u(y - z) = 0$. To complete the proof, it needs to prove that $u \in \partial^c L(\cdot, \bar{\lambda} + \bar{\mu})(y)$. We claim that if $u(d) \le L^\circ(\cdot, \bar{\lambda} + \bar{\mu})(z; d)$ then $u(d) \le L^\circ(\cdot, \bar{\lambda} + \bar{\mu})(y; d)$. Indeed, suppose to contrary that $u(d) > L^\circ(\cdot, \bar{\lambda} + \bar{\mu})(y; d)$. Then $L^\circ(\cdot, \bar{\lambda} + \bar{\mu})(z; d) - L^\circ(\cdot, \bar{\lambda} + \bar{\mu})(y; d) > 0$. Since f, f_t, $t \in T$ are regular on X, $L(\cdot, \bar{\lambda} + \bar{\mu})$ are regular on X. There exist $\alpha > 0$ and $t > 0$ small is enough such that

$$\lim_{t \to 0} \frac{L(z + td, \bar{\lambda} + \bar{\mu}) - L(z, \bar{\lambda} + \bar{\mu})}{t} - \frac{L(y + td, \bar{\lambda} + \bar{\mu}) - L(y, \bar{\lambda} + \bar{\mu})}{t} > \alpha > 0.$$

This implies that $L(z + td, \bar{\lambda} + \bar{\mu}) - L(y + td, \bar{\lambda} + \bar{\mu}) > 0$. We get $L(z, \bar{\lambda} + \bar{\mu}) > L(y, \bar{\lambda} + \bar{\mu})$, a contradiction. Hence, $u \in \partial^c L(z, \bar{\lambda} + \bar{\mu})$ implies $u \in \partial^c L(y, \bar{\lambda} + \bar{\mu})$. $\square$

Corollary 3.6. *Suppose that z is a solution of* (P) *and there exists $\bar{\lambda}$ such that the condition* (1) *is satisfied. Then* $\mathrm{Sol}(\mathrm{P}) = S' = S'_1$ *where*

$$S' = \{y \in C \mid \exists u \in \partial^c L(y, \bar{\lambda}), u(z - y) = 0, f_t(y) = 0, t \in T(\bar{\lambda}); f_t(y) \le 0, t \in T \setminus T(\bar{\lambda})\},$$

$$S'_1 = \{y \in C \mid \exists u \in \partial^c L(y, \bar{\lambda}) \cap \partial^c L(z, \bar{\lambda}), u(z - y) = 0, f_t(y) = 0, t \in T(\bar{\lambda}); f_t(y) \le 0, t \in T \setminus T(\bar{\lambda})\}.$$

Proof. If z is a solution of (P) and there exists $\bar{\lambda}$ such that the condition (1) is satisfied then, by Lemma 3.1, $(z, \bar{\lambda}, 0)$ is a solution of (D). Applying Theorem 3.1, we get the desired results. □

Note that as $\mu = 0$, the problem (D) reduces to (D_W) and as $\lambda = 0$ it reduces to (D_M). In the following corollary we show that characterizations of solution set of (P) can be established via (D_W). It is derived directly from Theorem 3.1.

Corollary 3.7. *Suppose that $(z, \bar{\lambda})$ is a solution of* (D_W). *If* $V(\mathrm{P}) = V(\mathrm{D_W})$ *then* $\mathrm{Sol}(\mathrm{P}) = S' = S_1'$.

Proof. Let $(z, \bar{\lambda})$ be a solution of (D_W). It is equivalent to $(z, \bar{\lambda}, 0)$ is a solution of (D) in the case $\mu = 0$. By applying Theorem 3.1 we can deduce the desired result. □

Remark 3.2. Note that if z is a solution of (P) and the condition (1) is satisfied then, by Lemma 3.1, there exists $\bar{\lambda}$ such that $(z, \bar{\lambda})$ is solution of (D_W) and $\mathrm{V(P)} = \mathrm{V(D_W)}$. This shows that Corollary 3.6 can be deduced from Corollary 3.7.

Next, we deal with characterizations of solution set (P) by using the dual problem of (P) in Mond–Weir type.

Corollary 3.8. *Suppose that $(z, \bar{\lambda})$ is a solution of* (D_M). *If* $V(\mathrm{P}) = V(\mathrm{D_M})$ *then* $\mathrm{Sol}(\mathrm{P}) = S' = S_1'$.

Proof. Let $(z, \bar{\lambda})$ be a solution of (D_M). It is equivalent to $(z, 0, \bar{\lambda})$ is a solution of (D). By applying Theorem 3.1 we can deduce the desired result. □

Remark 3.3. We also note that if z is a solution of (P) and the condition (1) is satisfied then, by Lemma 3.1, there exists $\bar{\lambda}$ such that $(z, \bar{\lambda})$ is solution of (D_M) and $\mathrm{V(P)} = \mathrm{V(D_M)}$. This shows that Corollary 3.6 can be deduced from Corollary 3.8.

We now give the other characterizations of solution set of (P).

Theorem 3.2. *Suppose that $(z, \bar{\mu})$, is a solution of* (D_M), *the function f is pseudoconvex on X, and the functions f_t, $t \in T(\bar{\mu})$ are quasiconvex on X. If* $V(\mathrm{P}) = V(\mathrm{D_M})$ *then* $\mathrm{Sol}(\mathrm{P}) = S_2 = S_3$ *where*

$$S_2 = \{y \in C \mid \exists u \in \partial^c f(y), u(z - y) = 0, f_t(y) = 0, t \in T(\bar{\mu}); f_t(y) \leq 0, t \in T \setminus T(\bar{\mu})\},$$
$$S_3 = \{y \in C \mid \exists u \in \partial^c f(y) \cap \partial^c f(z), u(z - y) = 0, f_t(y) = 0, t \in T(\bar{\mu}); f_t(y) \leq 0, t \in T \setminus T(\bar{\mu})\}.$$

Proof. Since $S_3 \subset S_2$, it needs to prove that $S_2 \subset \mathrm{Sol(P)}$ and $\mathrm{Sol(P)} \subset S_3$.

(i) Let $y \in S_2$. Then $y \in A$ and there exists $u \in \partial^c f(y)$ such that $u(z - y) = 0$ and $f_t(y) = 0,\ t \in T(\bar{\mu}); f_t(y) \leq 0,\ t \in T \setminus T(\bar{\mu})$. Since the function f is pseudoconvex on X, from $u(z-y) = 0$, we get $f(z) \geq f(y)$. Since $V(\mathrm{P}) = V(\mathrm{D_M})$, $V(\mathrm{P}) \geq f(y)$. Hence, $y \in \mathrm{Sol(P)}$.

(ii) Let $y \in \mathrm{Sol(P)}$. We will prove that $y \in S_3$. Since $V(\mathrm{P}) = V(\mathrm{D_M})$, $f(z) = f(y)$. Combining this and the property of pseudoconvex f, by Lemma 2.2, we get $u(y - z) \leq 0$ for all $u \in \partial^c f(z)$. In addition, since $(z, \bar{\mu}) \in \bar{\bar{G}}$, there exist $u \in \partial^c f(z)$, $u_t \in \partial^c f_t(z)$, $t \in T$, $w \in N(C, z)$, and $\bar{\mu}_t f_t(z) \geq 0$ for all $t \in T$ such that

$$u + \sum_{t \in T} \bar{\mu}_t u_t + w = 0. \tag{9}$$

Since f_t, $t \in T(\bar{\mu})$, are quasiconvex on X and $f_t(z) \geq 0$ for all $t \in T(\bar{\mu})$, $f_t^{\circ}(z; y - z) \leq 0$ for $y \in A$. Hence,

$$u_t(y - z) \leq 0, \quad u_t \in \partial^c f_t(z),\ t \in T(\bar{\mu}).$$

This and (9) imply $u(y-z) \geq 0$. Thus $u(y-z) = 0$. We now prove that $f_t(y) = 0$ for all $t \in T(\bar{\mu})$. Note that, by Corollary 3.3, we get $\bar{\mu}_t f_t(y) = 0$ for all $t \in T$, i.e., $f_t(y) = 0$ for all $t \in T(\bar{\mu})$. To complete the proof we need to show that $u \in \partial^c f(y)$, but this conclusion is satisfied by using an argument similar to the one in the last part of the proof of Theorem 3.1. □

Corollary 3.9. *Suppose that z is a solution of* (P) *and there exists $\bar{\mu} \in \mathbb{R}_+^{(T)}$ such that the condition* (1) *holds. Suppose further that f is pseudoconvex on X and the functions and f_t, $t \in T(\bar{\mu})$, are quasiconvex on X. Then* $\mathrm{Sol}(\mathrm{P}) = S_2 = S_3$.

Proof. From Theorem 3.2 and based on Lemma 3.1, we can deduce the desired results. □

Remark 3.4. The conclusion of the corollary above coincides with Theorem 3.2 in [14]. In case the involved functions are convex we can find back the results established in [10].

In the theorem above, if the feasible set of (P) is a convex subset of X then, by using an argument similarly to the one in [14, Theorem 3.2], we can prove that $u_t(y - z) \leq 0$ for $x \in A$, $u_t \in \partial^c f_t(z)$, $t \in T$, without assuming that f_t, $t \in T$, are quasiconvex functions. We obtain the following theorem and corollary with the proofs omitted.

Theorem 3.3. *Suppose that $(z, \bar{\mu})$ is a solution of* (D_M) *and the function f is pseudoconvex over X. If* $V(\mathrm{P}) = V(\mathrm{D_M})$ *and the feasible set of* (P) *is a convex subset of X then* $\mathrm{Sol}(\mathrm{P}) = S_2 = S_3$.

Corollary 3.10. *Suppose that z is a solution of* (P) *and there exists $\bar{\mu} \in \mathbb{R}_+^{(T)}$ such that the condition (1) holds. If the function f is pseudoconvex over X and the feasible set of* (P) *is a convex subset of X then* $\mathrm{Sol(P)} = S_2 = S_3$.

Example.

$$
\begin{array}{lll}
(\mathrm{Q}_2) & \text{Minimize} & f(x,y) = (x-y) + (x-y)^3 \\
 & \text{subject to} & f_t(x,y) \le 0, \quad t \in T, \\
 & & (x,y) \in C := \{(x,y) \mid x^2+y^2 \le 1, x \ge 0\}
\end{array}
$$

where $f_t(x,y) = \begin{cases} \sin(ty-x), t \in (0,1] \\ x^3 - y, t = 0 \end{cases}$ and $T = [0,1]$. For $(x,y) \in C$, we can check that $ty - x \in [-\sqrt{2}, \sqrt{2}] \subset [-\pi/2, \pi/2]$ for all $t \in (0,1]$. Hence,

$$\sin(ty-x) \le 0 \Leftrightarrow ty - x \le 0, \quad t \in (0,1].$$

The feasible set of (Q_2) is $\{(x,y) \mid 0 \le x \le 1, y \le x, y \ge x^3\}$. We can see that the objective function of the problem is pseudoconvex and the feasible set is a convex subset in $\mathbb{R}^2$. Let λ be such that $\lambda_1 = 1$ and $\lambda_t = 0$ for all $t \in T \setminus \{1\}$. We can check that $(0,0)$ is a solution of (Q_2) and it satisfies the optimality condition (1) corresponding to λ. In this case we obtain $T(\lambda) = \{1\}$. By using the formulation S_2, the solution set of (Q_2) can be determined by

$$\mathrm{Sol}(\mathrm{Q}_2) = \{(x,y) \in C \mid (1 + 3(x-y)^2)(y-x) = 0, f_1(x,y) = 0, f_t(x,y) \le 0, t \in T \setminus \{1\}\}.$$

A simple computation gives that $\mathrm{Sol}(\mathrm{Q}_2) = \{(x,y) \mid 0 \le x \le 1, y = x\}$.

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