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Monotone iterates for solving nonlinear integro-parabolic equations of Volterra type

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ABSTRACT

The paper deals with numerical solving of nonlinear integro-parabolic problems based on the method of upper and lower solutions. A monotone iterative method is constructed. Existence and uniqueness of a solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Construction of initial upper and lower solutions is discussed. Numerical experiments are presented.

1. Introduction

Integro-parabolic differential equations of Volterra type arise in the chemical, physical and engineering sciences (see [1] for details). In this paper we give a numerical treatment for nonlinear integro-parabolic differential equations of Volterra type. The parabolic problem under consideration is given in the form

$$u_t - Lu + f(x,t,u) + \int_0^t g_0(x,t,s,u(x,s))ds = 0, \quad (x,t) \in \omega \times (0,T], \tag{1}$$
$$u(x,t) = h(x,t), \quad (x,t) \in \partial\omega \times (0,T],$$
$$u(x,0) = \psi(x), \quad x \in \overline{\omega},$$

where ω is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1, 2, \ldots$) with boundary $\partial\omega$. The linear differential operator L is given by

$$Lu = \sum_{\alpha=1}^{\kappa} \frac{\partial}{\partial x_\alpha}\left(D(x,t)\frac{\partial u}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{\kappa} v_\alpha(x,t)\frac{\partial u}{\partial x_\alpha},$$

where the coefficients of the differential operators are smooth and D is positive in $\overline{\omega} \times [0,T]$. It is also assumed that the functions f, g_0, h and ψ are smooth in their respective domains.

In solving such nonlinear problems by the finite difference or finite element methods, the corresponding discrete problem on each discrete time level is usually formulated as a nonlinear system of algebraic equations. A basic mathematical concern of this problem is whether the nonlinear system possesses a solution. This nonlinear system requires some iterative method for the computation of numerical solutions. This leads to the question of convergence of the sequence of iterations. The aim of this paper is to investigate the above questions concerning the existence and uniqueness of a solution to the nonlinear system, methods of iterations for the computation of the solution.

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Our iterative scheme is based on the method of upper and lower solutions and associated monotone iterates. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem.

Monotone iterative schemes for solving nonlinear parabolic equations were used in [2–8]. In [9], a monotone iterative method for solving nonlinear integro-parabolic equations of Fredholm type is presented. Here, the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were not given. In this paper, we investigate the monotone iterative method in the case when on each time level nonlinear difference schemes are solved inexactly, and give an analysis of convergence rates of the monotone iterative method.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1). A monotone iterative method is presented in Section 3. Existence and uniqueness of the solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Convergence of the nonlinear difference scheme to the nonlinear integro-parabolic problem (1) is established. Section 4 deals with construction of initial upper and lower solutions. Section 5 presents results of numerical experiments.

2. The nonlinear difference scheme

On the domains $\overline{\omega}$ and $[0,T]$, we introduce meshes $\overline{\omega}^h$ and $\overline{\omega}^\tau$, respectively. For solving (1), consider the nonlinear two-level implicit difference scheme

$$\mathcal{L}U(p,t_k)+f(p,t_k,U)+g(p,t_k,U)-\tau_k^{-1}U(p,t_{k-1})=0,\qquad (2)$$
$$(p,t_k)\in\omega^h\times(\overline{\omega}^\tau\setminus\{0\}),$$

with the boundary and initial conditions

$$U(p,t_k)=h(p,t_k),\quad (p,t_k)\in\partial\omega^h\times(\overline{\omega}^\tau\setminus\{0\}),$$
$$U(p,0)=\psi(p),\quad p\in\overline{\omega}^h,$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$ and time steps $\tau_k=t_k-t_{k-1},\ k\geq 1, t_0=0$.

The difference operator $\mathcal{L}$ is defined by

$$\mathcal{L}U(p,t_k)=\mathcal{L}^hU(p,t_k)+\tau_k^{-1}U(p,t_k),$$
$$\mathcal{L}^hU(p,t_k)=d(p,t_k)U(p,t_k)-\sum_{p'\in\sigma'(p)}a(p',t_k)U(p',t_k),$$

where $\sigma'(p)=\sigma(p)\setminus\{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p\in\omega^h$. We make the following assumptions on the coefficients of the difference operator $\mathcal{L}^h$:

$$d(p,t_k)>0,\qquad a(p',t_k)\geq 0,\quad p'\in\sigma'(p),\qquad (3)$$
$$d(p,t_k)-\sum_{p'\in\sigma'(p)}a(p',t_k)\geq 0,\quad (p,t_k)\in\omega^h\times(\overline{\omega}^\tau\setminus\{0\}).$$

The integral g in (1) is approximated by the finite sum g based on the Riemann sum (the rectangular rule)

$$g(p,t_k,U)=\sum_{l=1}^{k}\tau_l g_0(p,t_k,t_l,U(p,t_l)).$$

We also assume that the mesh $\overline{\omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1,p_2,\ldots,p_s\}$ such that

$$p_1\in\sigma'(\tilde{p}),\ p_2\in\sigma'(p_1),\ldots,\ p_s\in\sigma'(p_{s-1}),\quad \hat{p}\in\sigma'(p_s).\qquad (4)$$

On each time level $t_k,\ k\geq 1$, introduce the linear problem

$$(\mathcal{L}+\overline{c})W(p,t_k)=\Psi(p,t_k),\quad p\in\omega^h,\qquad (5)$$
$$\overline{c}(p,t_k)\geq 0,\qquad W(p,t_k)=h(p,t_k),\quad p\in\partial\omega^h.$$

We now formulate the maximum principle for the difference operator $\mathcal{L}+\overline{c}$ and give an estimate to the solution to (5).

Lemma 1. *Let the coefficients of the difference operator $\mathcal{L}^h$ satisfy (3) and the mesh $\overline{\omega}^h$ be connected.*

(i) *If a mesh function $W(p,t_k)$ satisfies the conditions*

$$(\mathcal{L}+\overline{c})W(p,t_k)\geq 0\ (\leq 0),\quad p\in\omega^h,$$
$$W(p,t_k)\geq 0\ (\leq 0),\quad p\in\partial\omega^h,$$

then $W(p,t_k)\geq 0\ (\leq 0)$ in $\overline{\omega}^h$.

(ii) *The following estimate to the solution to* (5) *holds true*

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} \le \max \left\{ \|h(\cdot, t_k)\|_{\partial\omega^h}, \max_{p\in\omega^h} \frac{|\Psi(p, t_k)|}{c(p, t_k) + \tau_k^{-1}} \right\}, \tag{6}$$

where

$$\|W(\cdot, t_k)\|_{\overline{\omega}^h} = \max_{p\in\overline{\omega}^h} |W(p, t_k)|, \qquad \|h(\cdot, t_k)\|_{\partial\omega^h} = \max_{p\in\partial\omega^h} |h(p, t_k)|.$$

The proof of the lemma can be found in [10].

3. The monotone iterative method

Two mesh functions $\widetilde{U}(p, t_k)$ and $\widehat{U}(p, t_k)$ are called ordered upper and lower solutions of (2), if they satisfy the relation $\widetilde{U}(p, t_k) \ge \widehat{U}(p, t_k)$, $(p, t_k) \in \overline{\omega}^h \times \overline{\omega}^\tau$, and

$$\begin{aligned}
&\mathcal{L}\widetilde{U}(p, t_k) + f(p, t_k, \widetilde{U}) + g(p, t_k, \widetilde{U}) - \tau_k^{-1}\widetilde{U}(p, t_{k-1}) \ge 0, \\
&\mathcal{L}\widehat{U}(p, t_k) + f(p, t_k, \widehat{U}) + g(p, t_k, \widehat{U}) - \tau_k^{-1}\widehat{U}(p, t_{k-1}) \le 0, \\
&\quad (p, t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}), \\
&\widehat{U}(p, t_k) \le h(p, t_k) \le \widetilde{U}(p, t_k), \quad p \in \partial\omega^h, \\
&\widehat{U}(p, 0) \le \psi(p) \le \widetilde{U}(p, 0), \quad p \in \overline{\omega}^h.
\end{aligned} \tag{7}$$

For a given pair of ordered upper and lower solutions $\widetilde{U}, \widehat{U}$ and t_k fixed, we define the sector

$$\langle \widehat{U}(t_k), \widetilde{U}(t_k)\rangle = \{U(p, t_k) : \widehat{U}(p, t_k) \le U(p, t_k) \le \widetilde{U}(p, t_k),\ p \in \overline{\omega}^h\}.$$

We assume that f and g_0 satisfy the constraints

$$\begin{aligned}
&\frac{\partial f}{\partial u}(p, t_k, U) \le c(p, t_k) \quad \text{on } \langle \widehat{U}(t_k), \widetilde{U}(t_k)\rangle, \\
&0 \le -\frac{\partial g_0}{\partial u}(p, t_k, t_l, U), \quad 1 \le l \le k,\ \text{on } \langle \widehat{U}(t_l), \widetilde{U}(t_l)\rangle,
\end{aligned} \tag{8}$$

where $c(p, t_k)$ is a nonnegative bounded function in $\overline{\omega}^h \times \overline{\omega}^\tau$. The function g_0 is said to be nondecreasing.

Remark 1. We say that g_0 is a nonincreasing function if $-\partial g_0/\partial u \le 0$. When the function g_0 is nonincreasing, a transformation given by $u \to M - u$ for some constant $M > 0$ leads to a similar system where the function g_0 is nondecreasing.

We now construct an iterative method for solving (2) in the following way. On each time level t_k, $k \ge 1$, we calculate $U^{(n)}(p, t_k)$ as follows:

$$\begin{aligned}
&(\mathcal{L} + c)Z^{(n)}(p, t_k) = -\mathcal{R}(p, t_k, U^{(n-1)}), \quad p \in \omega^h, \\
&\mathcal{R}(p, t_k, U^{(n-1)}) = \mathcal{L}U^{(n-1)}(p, t_k) + f(p, t_k, U^{(n-1)}) + g(p, t_k, U^{(n-1)}) - \tau_k^{-1}U(p, t_{k-1}), \\
&Z^{(1)}(p, t_k) = h(p, t_k) - U^{(0)}(p, t_k), \qquad Z^{(n)}(p, t_k) = 0, \quad n \ge 2,\ p \in \partial\omega^h, \\
&U^{(n)}(p, t_k) = U^{(n-1)}(p, t_k) + Z^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \\
&U(p, t_k) = U^{(n(t_k))}(p, t_k), \qquad U(p, 0) = \psi(p), \quad p \in \overline{\omega}^h,
\end{aligned} \tag{9}$$

where $\mathcal{R}(p, t_k, U^{(n-1)})$ is the residual of the difference scheme (2) on $U^{(n-1)}$, $U(p, t_k)$ is an approximation of the exact solution on time level t_k, $n(t_k)$ is a number of iterative steps on time level t_k, and $c(p, t_k)$ is defined in (8).

3.1. Monotone convergence of the iterative method

We introduce the notation

$$F(p, t_k, U) = c(p, t_k)U(p, t_k) - f(p, t_k, U) - g(p, t_k, U), \tag{10}$$

and give a monotone property of F.

Lemma 2. *Let U, V be two functions in $\langle \widehat{U}(t_k), \widetilde{U}(t_k)\rangle$ such that $U(p, t_k) \ge V(p, t_k)$, and let* (8) *hold. Then*

$$F(p, t_k, U) \ge F(p, t_k, V), \quad p \in \overline{\omega}^h. \tag{11}$$

Proof. From (10), we have

$$F(p,t_k,U)-F(p,t_k,V)=c(p,t_k)[U(p,t_k)-V(p,t_k)]-[f(p,t_k,U)-f(p,t_k,V)]-[g(p,t_k,U)-g(p,t_k,V)].$$

By the mean-value theorem, we have

$$\begin{aligned}
&[f(p,t_k,U)-f(p,t_k,V)]=\frac{\partial f}{\partial u}(t_k)(U(p,t_k)-V(p,t_k)),\\
&g(p,t_k,U)-g(p,t_k,V)=\sum_{l=1}^{k}\tau_l\frac{\partial g_0}{\partial u}(t_l)[U(p,t_l)-V(p,t_l)],\\
&\frac{\partial f}{\partial u}(t_k)\equiv\frac{\partial f}{\partial u}(p,t_k,E),\qquad V(p,t_k)\le E(p,t_k)\le U(p,t_k).\\
&\frac{\partial g_0}{\partial u}(t_l)\equiv\frac{\partial g_0}{\partial u}(p,t_k,t_l,Q),\qquad V(p,t_l)\le Q(p,t_l)\le U(p,t_l).
\end{aligned}\tag{12}$$

Thus, from here and the assumptions of the lemma, we conclude (11). □

In the following theorem we prove the monotone property of the iterative method (9).

Theorem 1. *Let the coefficients of the difference operator $\mathcal{L}$ from (2) satisfy (3) and the computational mesh $\overline{\omega}^h$ be connected (4). Assume that $f(p,t_k,U)$ and $g_0(p,t_k,t_l,U)$ satisfy the inequalities from (8), where $\widetilde{U}$ and $\widehat{U}$ are ordered upper and lower solutions (7) of the nonlinear difference scheme (2). Then the sequences $\{U_\alpha^{(n)}\}$, $\alpha=1,-1$, generated by (9) with, respectively, $U_1^{(0)}=\widetilde{U}$ and $U_{-1}^{(0)}=\widehat{U}$, are ordered upper $\alpha=1$ and lower $\alpha=-1$ solutions to (2) and on each time level t_k, $k\ge 1$, converge monotonically*

$$U_{-1}^{(n-1)}(p,t_k)\le U_{-1}^{(n)}(p,t_k)\le U_1^{(n)}(p,t_k)\le U_1^{(n-1)}(p,t_k),\quad p\in\overline{\omega}^h,\tag{13}$$

where $n\ge 1$.

Proof. Since $U_1^{(0)}=\widetilde{U}$ is an upper solution, then from (7) and (9) we conclude that

$$(\mathcal{L}+c)Z_1^{(1)}(p,t_1)\le 0,\quad p\in\omega^h,\ Z_1^{(1)}(p,t_1)\le 0,\ p\in\partial\omega^h,$$

where $t_1=\tau_1$. From Lemma 1, it follows that

$$Z_1^{(1)}(p,t_1)\le 0,\quad p\in\overline{\omega}^h.\tag{14}$$

Similarly, for a lower solution $U_{-1}^{(0)}=\widehat{U}$, we conclude that

$$Z_{-1}^{(1)}(p,t_1)\ge 0,\quad p\in\overline{\omega}^h.\tag{15}$$

We now prove that

$$U_{-1}^{(1)}(p,t_1)\le U_1^{(1)}(p,t_1),\quad p\in\overline{\omega}^h.\tag{16}$$

By (9),

$$\begin{aligned}
&(\mathcal{L}+c)U_\alpha^{(1)}(p,t_1)=c(p,t_1)U_\alpha^{(0)}(p,t_1)-f(p,t_1,U_\alpha^{(0)})-g(p,t_1,U_\alpha^{(0)})+\tau_1^{-1}U_\alpha(p,0),\quad p\in\omega^h,\\
&U_\alpha^{(1)}(p,t_1)=h(p,t_1),\quad p\in\partial\omega^h,\ \alpha=1,-1.
\end{aligned}$$

From here, taking into account that $U_\alpha(p,0)=\psi(p)$, $\alpha=1,-1$, in the notation $W^{(n)}=U_1^{(n)}-U_{-1}^{(n)}$, $n\ge 0$, we have

$$\begin{aligned}
&(\mathcal{L}+c)W^{(1)}(p,t_1)=F(p,t_1,U_1^{(0)})-F(p,t_1,U_{-1}^{(0)}),\quad p\in\omega^h,\\
&W^{(1)}(p,t_1)=0,\quad p\in\partial\omega^h,
\end{aligned}$$

where F is defined in (10). Since $U_1^{(0)}(p,t_1)\ge U_{-1}^{(0)}(p,t_1)$, by Lemma 2, we conclude that the right hand side in the difference equation is nonnegative. The positivity property in Lemma 1 implies $W^{(1)}(p,t_1)\ge 0$, and this leads to (16).

We now prove that $U_1^{(1)}(p,t_1)$ and $U_{-1}^{(1)}(p,t_1)$ are upper and lower solutions (7), respectively. Let $n_\alpha(t_k)$ be numbers of iterative steps implemented on time level t_k, $k\ge 1$ for the upper $\alpha=1$ and lower $\alpha=-1$ sequences. Taking into account that

$$g(\cdot,U_\alpha^{(n)})-g(\cdot,U_\alpha^{(n-1)})=\tau_k[g_0(\cdot,t_k,U_\alpha^{(n)})-g_0(\cdot,t_k,U_\alpha^{(n-1)})],$$

where $g(\cdot, U)$ and $g_0(\cdot, t_k, U)$ stand for $g(p, t_k, U)$ and $g_0(p, t_k, t_k, U(p, t_k))$, respectively. From here, by the mean-value theorem, we obtain

$$g(\cdot, U_\alpha^{(n)}) - g(\cdot, U_\alpha^{(n-1)}) = \tau_k \frac{\partial g_0}{\partial u}(t_k) Z_\alpha^{(n)}(p, t_k),$$
$$\frac{\partial g_0}{\partial u}(t_k) \equiv \frac{\partial g_0}{\partial u}(p, t_k, t_k, Q_\alpha^{(n)}),$$

where $Q_\alpha^{(n)}(p, t_k)$ lies between $U_\alpha^{(n)}(p, t_k)$ and $U_\alpha^{(n-1)}(p, t_k)$. From here, (9), by the mean-value theorem for $f(p, t_1, U_1^{(1)})$, we obtain

$$\mathcal{R}(p, t_1, U_1^{(1)}) = -\left(c - \frac{\partial f}{\partial u}(p, t_1, E_1^{(1)})\right) Z_1^{(1)}(p, t_1) + \tau_1 \frac{\partial g_0}{\partial u}(t_1) Z_1^{(1)}(p, t_1), \tag{17}$$

where the partial derivatives are calculated at intermediate points $E_1^{(1)}$ and $Q_1^{(1)}$, which lie in the sector $\langle U_1^{(1)}(t_1), U_1^{(0)}(t_1)\rangle$. From here, (14)–(16), it follows that the partial derivatives satisfy (8). From (8), (14) and (17), we conclude that

$$\mathcal{R}(p, t_1, U_1^{(1)}) \geq 0, \quad p \in \omega^h,\ U_1^{(1)}(p, t_1) = h(p, t_1),\ p \in \partial\omega^h.$$

Thus, $U_1^{(1)}(p, t_1)$ is an upper solution. Similarly, we can prove that $U_{-1}^{(1)}(p, t_1)$ is a lower solution, that is,

$$\mathcal{R}(p, t_1, U_{-1}^{(1)}) \leq 0, \quad p \in \omega^h,\ U_{-1}^{(1)}(p, t_1) = h(p, t_1),\ p \in \partial\omega^h.$$

By induction on n, we can prove that $\{U_1^{(n)}(p, t_1)\}$ is a monotonically decreasing sequence of upper solutions and $\{U_{-1}^{(n)}(p, t_1)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (13) for t_1.

From (13) with t_1, it follows that

$$\widehat{U}(p, t_1) \leq U_{-1}^{(n_{-1})}(p, t_1) \leq U_1^{(n_1)}(p, t_1) \leq \widetilde{U}(p, t_1), \quad p \in \overline{\omega}^h. \tag{18}$$

From here and by the assumption of the theorem that $\widetilde{U}(p, t_2)$ and $\widehat{U}(p, t_2)$ are, respectively, upper and lower solutions (7), we conclude that $\widetilde{U}(p, t_2)$ and $\widehat{U}(p, t_2)$ are upper and lower solutions with respect to $U_1^{(n_1)}(p, t_1)$ and $U_{-1}^{(n_{-1})}(p, t_1)$, that is,

$$\begin{aligned} &\mathcal{L}\widetilde{U}(p, t_2) + f(p, t_2, \widetilde{U}) + g(p, t_2, \widetilde{U}) - \tau_2^{-1} U_1^{(n_1)}(p, t_1) \geq 0, \quad p \in \omega^h, \\ &\mathcal{L}\widehat{U}(p, t_2) + f(p, t_2, \widehat{U}) + g(p, t_2, \widehat{U}) - \tau_2^{-1} U_{-1}^{(n_{-1})}(p, t_1) \leq 0, \quad p \in \omega^h. \end{aligned} \tag{19}$$

By (9) with t_2, we have

$$(\mathcal{L} + c) U_\alpha^{(1)}(p, t_2) = c(p, t_2) U_\alpha^{(0)}(p, t_2) - f(p, t_2, U_\alpha^{(0)}) - g(p, t_2, U_\alpha^{(0)}) + \tau_2^{-1} U_\alpha^{(n_\alpha)}(p, t_1), \quad p \in \omega^h,$$
$$U_\alpha^{(1)}(p, t_2) = h(p, t_2), \quad p \in \partial\omega^h,\ \alpha = 1, -1.$$

From here, we conclude that $W^{(1)}(p, t_2) = U_1^{(1)}(p, t_2) - U_{-1}^{(1)}(p, t_2)$ satisfies the difference problem

$$(\mathcal{L} + c) W^{(1)}(p, t_2) = F(p, t_2, U_1^{(0)}) - F(p, t_2, U_{-1}^{(0)}) + \tau_2^{-1}[U_1^{(n_1)}(p, t_1) - U_{-1}^{(n_{-1})}(p, t_1)], \quad p \in \omega^h,$$
$$W^{(1)}(p, t_2) = 0, \quad p \in \partial\omega^h.$$

Since $U_1^{(0)}(p, t_2) \geq U_{-1}^{(0)}(p, t_2)$ and taking into account (18), by Lemma 2, we conclude that the right hand side in the difference equation is nonnegative. The positivity property in Lemma 1 implies $W^{(1)}(p, t_2) \geq 0$, and this leads to

$$U_{-1}^{(1)}(p, t_2) \leq U_1^{(1)}(p, t_2), \quad p \in \overline{\omega}^h.$$

The proof that $U_1^{(1)}(p, t_2)$ and $U_{-1}^{(1)}(p, t_2)$ are, respectively, upper and lower solutions is similar to the proof of this result on time level t_1. By induction on n, we can prove that $\{U_1^{(n)}(p, t_2)\}$ is a monotonically decreasing sequence of upper solutions and $\{U_{-1}^{(n)}(p, t_2)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (13) for t_2.

By induction on k, $k \geq 1$, we can prove that $\{U_1^{(n)}(p, t_k)\}$ is a monotonically decreasing sequence of upper solutions and $\{U_{-1}^{(n)}(p, t_k)\}$ is a monotonically increasing sequence of lower solutions, which satisfy (13). Thus, we prove the theorem. □

3.2. Existence and uniqueness of a solution to difference scheme (2)

Applying Theorem 1, we investigate existence and uniqueness of a solution to the nonlinear difference scheme (2).

Lemma 3. *Let the coefficients of the difference operator $\mathcal{L}$ from (2) satisfy (3) and the computational mesh $\overline{\omega}^h$ be connected (4). Assume that $f(p, t_k, U)$ and $g_0(p, t_k, t_l, U)$ satisfy the inequalities from (8), where $\widetilde{U}$ and $\widehat{U}$ are ordered upper and lower solutions (7) of (2). Then a solution to the nonlinear difference scheme (2) exists.*

Proof. From (13), it follows that $\lim U_1^{(n)}(p, t_1) = V_1(p, t_1)$, $p \in \overline{\omega}^h$ as $n \to \infty$ exists, and

$$V_1(p, t_1) \le U_1^{(n)}(p, t_1), \qquad \lim_{n\to\infty} Z_1^{(n)}(p, t_1) = 0, \quad p \in \overline{\omega}^h. \tag{20}$$

Similar to (17), we can prove that

$$\mathcal{R}(p, t_1, U_1^{(n)}) = -\left(c - \frac{\partial f}{\partial u}(p, t_1, E_1^{(n)})\right) Z_1^{(n)}(p, t_1) + \tau_1 \frac{\partial g_0}{\partial u}(t_1) Z_1^{(n)}(p, t_1). \tag{21}$$

From here and (20), we conclude that $V_1(p, t_1)$ solves (2) at t_1. By the assumption of the lemma that $\widetilde{U}(p, t_2)$ is an upper solution and from (20), it follows that $\widetilde{U}(p, t_2)$ is an upper solution with respect to $V_1(p, t_1)$. Using a similar argument, we can prove that the following limit

$$\lim_{n\to\infty} U_1^{(n)}(p, t_2) = V_1(p, t_2), \quad p \in \overline{\omega}^h,$$

exists and solves (2) at t_2, where according to Theorem 1, $\{U_1^{(n)}(p, t_2)\}$ is a sequence of upper solutions with respect to $V_1(p, t_1)$.

By induction on k, $k \ge 1$, we can prove that

$$V_1(p, t_k) = \lim_{n\to\infty} U_1^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \ k \ge 1,$$

is a solution of the nonlinear difference scheme (2). Similarly, we can prove that the mesh function $V_{-1}(p, t_k)$ defined by

$$V_{-1}(p, t_k) = \lim_{n\to\infty} U_{-1}^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \ k \ge 1,$$

is a solution of the nonlinear difference scheme (2). $\square$

We now impose the two-sided constraints on f and g_0 (cf. (8)). We assume that f and g_0 satisfy the constraints

$$\underline{c}(p, t_k) \le \frac{\partial f}{\partial u}(p, t_k, U) \le c(p, t_k), \quad \text{on } \langle \widehat{U}(t_k), \widetilde{U}(t_k) \rangle, \tag{22}$$

$$0 \le -\frac{\partial g_0}{\partial u}(p, t_k, t_l, U) \le q(p, t_k, t_l), \quad 1 \le l \le k, \ \text{on } \langle \widehat{U}(t_l), \widetilde{U}(t_l) \rangle, \tag{23}$$

where $\widetilde{U}, \widehat{U}$ is a pair of ordered upper and lower solutions to (2), $c(p, t_k)$, $\underline{c}(p, t_k)$ and $q(p, t_k, t_l)$ are, respectively, nonnegative bounded, bounded and positive bounded functions in $\overline{\omega}^h \times \overline{\omega}^\tau$. We also assume that time step τ_k satisfies the inequality

$$\tau_k < \frac{|\gamma_k|}{2\rho_k} + \sqrt{\left(\frac{|\gamma_k|}{2\rho_k}\right)^2 + \frac{1}{\rho_k}}, \quad k \ge 1, \tag{24}$$

$$\underline{c}_k = \min_{p\in\overline{\omega}^h} \underline{c}(p, t_k), \qquad \gamma_k = \min(0, \underline{c}_k), \qquad \rho_k = \max_{1\le l\le k}\{\max_{p\in\overline{\omega}^h}[q(p, t_k, t_l)]\}.$$

We mention here that if $\underline{c}(p, t_k) \ge 0$, then $\tau_k < \sqrt{1/\rho_k}$.

Lemma 4. *Let the coefficients of the difference operator $\mathcal{L}$ from (2) satisfy (3), the mesh $\overline{\omega}^h$ be connected (4) and the mesh $\overline{\omega}^\tau$ satisfy (24). Assume that $f(p, t_k, U)$ and $g_0(p, t_k, t_l, U)$ satisfy (22), (23), where $\widetilde{U}$ and $\widehat{U}$ are ordered upper and lower solutions (7) of (2). Then the nonlinear difference scheme (2) has a unique solution.*

Proof. It suffices to show that

$$V_1(p, t_k) = V_{-1}(p, t_k), \quad p \in \overline{\omega}^h, \ k \ge 1,$$

where $V_1(p, t_k)$ and $V_{-1}(p, t_k)$ are solutions to the difference scheme (2), which are defined in Lemma 3. From (13) and Lemma 3, it follows that

$$U_{-1}^{(n)}(p, t_k) \le V_{-1}(p, t_k) \le V_1(p, t_k) \le U_1^{(n)}(p, t_k), \quad p \in \overline{\omega}^h, \ k \ge 1. \tag{25}$$

Letting $W(p, t_k) = V_1(p, t_k) - V_{-1}(p, t_k)$, from (2), we have

$$\mathcal{L}W(p, t_1) + [f(p, t_1, V_1) - f(p, t_1, V_{-1})] + [g(p, t_1, V_1) - g(p, t_1, V_{-1})] = 0, \quad p \in \omega^h,$$
$$W(p, t_1) = 0, \quad p \in \partial\omega^h.$$

Using the mean-value theorem, we obtain

$$\left(\mathcal{L}+\frac{\partial f}{\partial u}\right)W(p,t_1)=-\tau_1\frac{\partial g_0}{\partial u}(t_1)W(p,t_1),\quad p\in\omega^h,$$
$$W(p,t_1)=0,\quad p\in\partial\omega^h,$$

where the partial derivatives are calculated at intermediate points which lie in $\langle V_{-1}(t_1),V_1(t_1)\rangle$. From (25), it follows that the partial derivatives satisfy (22), (23). From here, (22), (23) and (6) with $\bar{c}(p,t_k)=0$, we obtain the estimate

$$w(t_1)\le\frac{\tau_1\rho_1}{\tau_1^{-1}+|\gamma_1|}w(t_1),$$

where ρ_1 is defined in (24), and we use the notation

$$w(t_k)\equiv\|W(\cdot,t_k)\|_{\overline{\omega}^h}. \tag{26}$$

By the assumption on τ_1 in (24) and $w(t_1)\ge 0$, we conclude that $w(t_1)=0$. From here, using the mean-value theorem, we get

$$\left(\mathcal{L}+\frac{\partial f}{\partial u}\right)W(p,t_2)=-\tau_2\frac{\partial g_0}{\partial u}(t_2)W(p,t_2),\quad p\in\omega^h,$$
$$W(p,t_2)=0,\quad p\in\partial\omega^h.$$

Similar to the proof that $w(t_1)=0$, we conclude that $w(t_2)=0$. Now by induction on k, $k\ge 1$, we can prove that $w(t_k)=0$, $k\ge 1$. Thus, we prove the lemma. □

3.3. Convergence of the monotone iterative method to the solution of the nonlinear difference scheme

We now choose the stopping criterion of the iterative method (9) in the form

$$\|\mathcal{R}(\cdot,t_k,U_\alpha^{(n)})\|_{\omega^h}\le\delta,\quad \alpha=1,-1, \tag{27}$$

where δ is a prescribed accuracy, and set up $U_\alpha(p,t_k)=U^{(n_\alpha)}(p,t_k)$, $p\in\overline{\omega}^h$, such that $n_\alpha(t_k)$ is minimal subject to (27).

We now assume that in (22)

$$\underline{c}(p,t_k)\ge\underline{c}_k=\text{const}>0,\quad \text{on } \langle\widehat{U}(t_k),\widetilde{U}(t_k)\rangle,\ k\ge 1. \tag{28}$$

Remark 2. We mention that the assumption $\partial f/\partial u\ge\underline{c}_k>0$ in (28) can always be obtained via a change of variables. Indeed, introduce the following function $z(x,t)=e^{-\lambda t}u(x,t)$, where λ is a constant. Now, $z(x,t)$ satisfies (1) with

$$\widetilde{f}=\lambda z+e^{-\lambda t}f(x,t,e^{\lambda t}z),\qquad \widetilde{g}_0=e^{-\lambda t}g_0(x,t,s,e^{\lambda s}z(x,s)),$$

instead of f and g_0, and we have

$$\frac{\partial\widetilde{f}}{\partial z}=\lambda+\frac{\partial f}{\partial u},\qquad -\frac{\partial\widetilde{g}_0}{\partial z}=e^{-\lambda(t-s)}\left(-\frac{\partial g_0}{\partial u}\right).$$

Thus, if $\lambda\ge\max_{k\ge 1}|\gamma_k|$, where γ_k is defined in (24), then from here and (22), we conclude that $\partial\widetilde{f}/\partial z$ satisfies (28). Since $0<e^{-\lambda(t-s)}\le 1$, $\lambda>0$, then $\partial\widetilde{g}_0/\partial z$ still satisfies (23).

We assume that time step τ_k satisfies the inequality

$$\tau_k<\min\left(\sqrt{\frac{1}{\rho_k}},\frac{\underline{c}_k}{\rho_k}\right),\quad k\ge 1, \tag{29}$$

and prove the following convergence result for the iterative method (9), (27).

Theorem 2. *Let the coefficients of the difference operator $\mathcal{L}$ from (2) satisfy (3), the mesh $\overline{\omega}^h$ be connected (4) and the mesh $\overline{\omega}^\tau$ satisfy (29). Assume that $f(p,t_k,U)$ and $g_0(p,t_k,t_l,U)$ satisfy (22), (23) and (28), where $\widetilde{U}$ and $\widehat{U}$ are ordered upper and lower solutions (7) of (2). Then for the sequences $\{U_\alpha^{(n)}\}$, $\alpha=1,-1$, generated by (9), (27) with, respectively, $U_1^{(0)}=\widetilde{U}$ and $U_{-1}^{(0)}=\widehat{U}$, the following estimate holds:*

$$\max_{t_k\in\overline{\omega}^\tau}\|U_\alpha(\cdot,t_k)-U^*(\cdot,t_k)\|_{\overline{\omega}^h}\le T\delta,\quad \alpha=1,-1, \tag{30}$$

where $U^*(p,t_k)$ is the unique solution to (2). Furthermore, on each time level the sequences converge monotonically (13).

Proof. The monotone convergence of the sequences $\{U_\alpha^{(n)}(p,t_k)\}$, $\alpha=1,-1$ follows from Theorem 1. The existence of the solution to (2) has been proved in Lemma 3. By (28), we conclude that $\gamma_k=0,\ k\geq 1$ in (24), and, thus, from (29), τ_k satisfies (24) as well. The uniqueness of the solution to (2) follows now from Lemma 4.

The difference problem for $U_\alpha(p,t_k)=U^{(n_\alpha)}(p,t_k),\ k\geq 1$, can be represented in the form

$$\mathcal{L}U_\alpha(p,t_k)+f(p,t_k,U_\alpha)+g(p,t_k,U_\alpha)-\frac{1}{\tau_k}U_\alpha(p,t_{k-1})=\mathcal{R}(p,t_k,U_\alpha^{(n_\alpha)}),$$
$$p\in\omega^h,\quad U_\alpha(p,t_k)=h(p,t_k),\ p\in\partial\omega^h,\ \alpha=1,-1.$$

From here, (2), by the mean-value theorem, for $W_\alpha(p,t_k)=U_\alpha(p,t_k)-U^*(p,t_k)$, $\alpha=1,-1$, we get the difference problems

$$\left(\mathcal{L}+\frac{\partial f}{\partial u}\right)W_\alpha(p,t_k)=\mathcal{R}(p,t_k,U_\alpha)+\frac{1}{\tau_k}W_\alpha(p,t_{k-1})-\tau_k\frac{\partial g_0}{\partial u}(t_k)W_\alpha(p,t_k),\tag{31}$$
$$p\in\omega^h,\quad W_\alpha(p,t_k)=0,\ p\in\partial\omega^h,\ \alpha=1,-1,$$

where the partial derivatives are calculated at intermediate points, which lie between $U^*(p,t_k)$ and $U_\alpha(p,t_k)$. Thus, the partial derivatives satisfy (22), (23) and (28). From here, (22), (23), (28), using (6) and taking into account that according to Theorem 1 the stopping criterion (27) can always be satisfied, in the notation (26), we obtain

$$w_\alpha(t_k)\leq\frac{1}{\tau_k^{-1}+\underline{c}_k}\left(\delta+\tau_k^{-1}w_\alpha(t_{k-1})+\tau_k\rho_k w_\alpha(t_k)\right),\quad \alpha=1,-1.$$

From here and (29), it follows that

$$w_\alpha(t_k)\leq\delta\tau_k+w_\alpha(t_{k-1}),\quad \alpha=1,-1.$$

Taking into account that $w_\alpha(t_0)=0$, $\alpha=1,-1$, by induction on k, we conclude that

$$w_\alpha(t_k)\leq\delta\sum_{l=1}^{k}\tau_l\leq T\delta,\quad k\geq 1,\ \alpha=1,-1.$$

Thus, we prove the theorem. $\square$

3.4. Convergence of the nonlinear difference scheme (2) to the solution of problem (1)

In [11], the analysis of convergence of a nonlinear difference scheme to one dimensional nonlinear integro-parabolic problem is based on the energy method, and in [12], the analysis for a linear integro-parabolic problem is based on the discrete version of Gronwall's inequality. In our analysis, we employ the approach, based on Gronwall's inequality, so error estimations are given in the maximum norms.

We now state Gronwall's inequality from [12] in the following form.

Lemma 5. *Let $\{w_k\}$ be a sequence on nonnegative real numbers satisfying*

$$w_k\leq a_k+\sum_{l=1}^{k}b_l w_l,\quad k\geq 1,$$

where $\{a_k\}$ is a nondecreasing sequence of nonnegative numbers, and $b_l\geq 0$. Then

$$w_k\leq a_k\exp\left(\sum_{l=1}^{k}b_l\right),\quad k\geq 1.$$

To simplify our analysis, we assume that $\tau_k=\tau,\ k\geq 1$, and with the aid of this lemma, prove the following theorem.

Theorem 3. *Let all the conditions in Theorem 2 be satisfied. Then the error in the nonlinear difference scheme (2) satisfies the inequality*

$$e(t_k)\leq C(T)\xi,\quad \xi=\max_{k\geq 1}\xi(t_k),\tag{32}$$
$$E(p,t_k)=U(p,t_k)-u(p,t_k),\qquad e(t_k)=\|E(\cdot,t_k)\|_{\overline{\omega}^h},\qquad \xi(t_k)=\|\Xi(\cdot,t_k)\|_{\omega^h},$$

where $U(p,t_k)$ and $u(p,t_k)$ are unique solutions to, respectively (2) and (1), and $\Xi(p,t_k)$ is the local truncation error of $u(x,t)$ on the nonlinear difference scheme (2).

Proof. Under the assumptions of Theorem 2 on f and g_0, the nonlinear integro-parabolic problem (1) has a unique solution (see Theorem 6.1 on p. 73 in [1], for details). From (2), by the mean-value theorem, we get the difference problem for the error $E(p,t_k)$

$$\left(\mathcal{L}+\frac{\partial f}{\partial u}\right)E(p,t_k)=\frac{1}{\tau}E(p,t_{k-1})-\sum_{l=1}^{k}\tau\frac{\partial g_0}{\partial u}(t_l)E(p,t_l)-\Xi(p,t_k),\quad p\in\omega^h,$$
$$E(p,t_k)=0,\quad p\in\partial\omega^h,\qquad E(p,0)=0,\quad p\in\overline{\omega}^h,$$

where the partial derivatives are calculated at intermediate points, which lie between $U(p,t_k)$ and $u(p,t_k)$. From here, (23), (24) and (28), by using (6), we get

$$e(t_k)\le\frac{1}{\tau+\underline{c}_k}\left(\tau^{-1}e(t_{k-1})+\sum_{l=1}^{k}\tau\rho_l e(t_l)+\xi(t_k)\right).$$

From here and (29), in the notation $\max_{k\ge1}\rho_k=\rho$, it follows that

$$e(t_k)\le e(t_{k-1})+\tau^2\rho\sum_{l=1}^{k}e(t_l)+\tau\xi.$$

From here and taking into account that $e(t_0)=0$, by induction on k, we prove the following inequality:

$$e(t_k)\le k\tau\xi+\tau^2\rho\left(\sum_{l=1}^{k}(k-l+1)e(t_l)\right).$$

By Lemma 5 with $a_k=k\tau\xi,\ k\ge1$ and $b_l=\tau^2\rho(k-l+1),\ 1\le l\le k$, we get

$$e(t_k)\le(k\tau\xi)\exp\left(\tau^2\rho\sum_{l=1}^{k}l\right).$$

From here and taking into account that $\sum_{l=1}^{k}l\le k^2/2,\ k\tau\le T$, we prove (32) with $C(T)=T\exp(\rho T^2/2)$. $\square$

3.5. Convergence analysis of the monotone iterative method

We now establish convergence properties of the iterative method (9) on each time level $t_k,\ k\ge1$.
We assume that time step τ_k satisfies the inequality

$$\tau_k<\sqrt{\left(\frac{\overline{c}_k}{2\rho_k}\right)^2+\frac{1}{\rho_k}}-\frac{\overline{c}_k}{2\rho_k},\quad \overline{c}_k=\max_{p\in\overline{\omega}^h}c(p,t_k),\ k\ge1,\tag{33}$$

where ρ_k is defined in (24), and introduce the notation

$$z_\alpha^{(n)}(t_k)=\|Z_\alpha^{(n)}(\cdot,t_k)\|_{\overline{\omega}^h}.\tag{34}$$

Lemma 6. *Let the coefficients of the difference operator $\mathcal{L}$ in (2) satisfy (3), the mesh $\overline{\omega}^h$ be connected with (4) and the mesh $\overline{\omega}^\tau$ satisfy (33). Assume that $f(p,t_k,U)$ and $g_0(p,t_k,t_l,U)$ satisfy (22), (23) and (28), where $\widetilde{U}$ and $\widehat{U}$ be ordered upper and lower solutions (7) of (2). Then for the sequences $\{U_\alpha^{(n)}\}$, $\alpha=1,-1$, generated by (9) with $U_1^{(0)}=\widetilde{U}$ and $U_{-1}^{(0)}=\widehat{U}$, the following estimate holds:*

$$z_\alpha^{(n)}(t_k)\le r_k^{n-1}z_\alpha^{(1)}(t_k),\quad r_k=\tau_k(\overline{c}_k+\tau_k\rho_k)<1,\ \alpha=1,-1.\tag{35}$$

Proof. Using (6), from (9), we have

$$z_\alpha^{(n)}(t_k)\le\tau_k\|\mathcal{R}(\cdot,t_k,U_\alpha^{(n-1)})\|_{\omega^h},\quad\alpha=1,-1.\tag{36}$$

Similar to (17), we can prove that

$$\mathcal{R}(p,t_k,U_\alpha^{(n)})=-\left(c-\frac{\partial f}{\partial u}(p,t_k,E_k^{(n)})\right)Z_\alpha^{(n)}(p,t_k)+\tau_k\frac{\partial g_0}{\partial u}(t_k)Z_\alpha^{(n)}(p,t_k).\tag{37}$$

From here, (22), (23) and (28), we conclude that

$$\|\mathcal{R}(\cdot,t_k,U_\alpha^{(n-1)})\|_{\omega^h}\le(\overline{c}_k+\tau_k\rho_k)\,z_\alpha^{(n-1)}(t_k),\quad\alpha=1,-1.$$

From here, (36), by using (6), we have

$$z_\alpha^{(n)}(t_k) \le \tau_k \left(\bar{c}_k + \tau_k \rho_k\right) z_\alpha^{(n-1)}(t_k),$$

and prove the lemma. □

Theorem 4. *Let the coefficients of the difference operator $\mathcal{L}$ in (2) satisfy (3), the mesh $\overline{\omega}^h$ be connected with (4) and the mesh $\overline{\omega}^\tau$ satisfy (29) and (33). Assume that $f(p,t,U)$ and $g_0(p,t_k,t_l,U)$ satisfy (22), (23) and (28), where $\widetilde{U}$ and $\widehat{U}$ be ordered upper and lower solutions (7) of (2). Then for the sequences $\{U_\alpha^{(n)}\}$, $\alpha = 1,-1$ generated by (9) with $U_1^{(0)} = \widetilde{U}$ and $U_{-1}^{(0)} = \widehat{U}$, the following estimate holds:*

$$\max_{t_k \in \overline{\omega}^\tau} \|U_\alpha(\cdot,t_k) - U^*(\cdot,t_k)\|_{\overline{\omega}^h} \le C r^{n_\alpha^*(t_k)-1}, \tag{38}$$

$$r = \max_{1 \le s \le k} r_k < 1, \qquad n_\alpha^*(t_k) = \min_{1 \le s \le k} n_\alpha(t_s), \quad \alpha = 1,-1,$$

where $U^*(p,t_k)$ is the unique solution to (2), r_k is defined in (35), constant C is independent of τ_k, and the number of iterative steps on each time level $n_\alpha(t_k) \ge 2$. Furthermore, on each time level the sequences converge monotonically (13).

Proof. From (31) and (37), for $k \ge 1$, $p \in \omega^h$, we conclude that

$$\begin{aligned} (\mathcal{L} + f_u)W_\alpha(p,t_k) &= -(c - f_u)Z_\alpha(p,t_k) + \frac{\partial g_0}{\partial u}(\cdot,t_k,Q_1)Z_\alpha(p,t_k) \\ &\quad - \frac{\partial g_0}{\partial u}(\cdot,t_k,Q_2)W_\alpha(p,t_k) + \frac{1}{\tau_k}W_\alpha(p,t_{k-1}), \end{aligned} \tag{39}$$

$$W_\alpha(p,t_k) = U_\alpha(p,t_k) - U^*(p,t_k), \qquad Z_\alpha(p,t_k) = Z_\alpha^{(n_\alpha(t_k))}(p,t_k), \quad \alpha = 1,-1,$$

where $\partial g_0(\cdot,t_k,Q)/\partial u$ stands for $\partial g_0(p,t_k,t_k,Q)/\partial u$ and $f_u \equiv \partial f/\partial u$. Taking into account that $W_\alpha(p,t_0) = 0$, $\alpha = 1,-1$, from (22), (23) and (28), by using (6), in the notation of (26) and (34), we have

$$w_\alpha(t_1) \le \frac{1}{\tau_1^{-1} + \underline{c}_1}\left[(\bar{c}_1 + \tau_1\rho_1)\, z_\alpha(t_1) + \tau_1\rho_1 w_\alpha(t_1)\right], \quad \alpha = 1,-1, \tag{40}$$

where $\underline{c}_k$, ρ_k and $\bar{c}_k$ are defined in (24) and (33), respectively. From here, (29) and (35), we obtain the estimate

$$w_\alpha(t_1) \le \tau_1 \left(\bar{c}_1 + \tau_1\rho_1\right) r_1^{n_\alpha(t_1)-1} z_\alpha^{(1)}(t_1), \quad \alpha = 1,-1. \tag{41}$$

From (9) by (6),

$$z_\alpha^{(1)}(t_1) \le \tau_1 \|\mathcal{L}U_\alpha^{(0)}(\cdot,t_1) + f(\cdot,t_1,U_\alpha^{(0)}) + g(\cdot,t_1,U_\alpha^{(0)}) - \tau_1^{-1}U_\alpha(\cdot,t_0)\|_{\omega^h}.$$

Since $U_\alpha^{(0)}(p,t_1)$, $U_\alpha(p,t_0)$ $\alpha = 1,-1$, are independent of τ_1, then for sufficiently small τ_1, $z_\alpha^{(1)}(t_1)$, $\alpha = 1,-1$, are independent of τ_1, that is,

$$z_\alpha^{(1)}(t_1) \le A_1, \quad \alpha = 1,-1,$$

where constant A_1 is independent of τ_1. Thus, from here and (41), we conclude that

$$w_\alpha(t_1) \le B_1 \tau_1 r_1^{n_\alpha(t_1)-1}, \quad \alpha = 1,-1, \tag{42}$$

where constant B_1 is independent of τ_1.

Similar to (40), from (39) with $k = 2$, we get

$$w_\alpha(t_2) \le \frac{1}{\tau_2^{-1} + \underline{c}_2}\left[(\bar{c}_2 + \tau_2\rho_2)\, z_\alpha(t_2) + \tau_2\rho_2 w_\alpha(t_2) + \tau_2^{-1} w_\alpha(t_1)\right].$$

From here, (29) and (35), we get the estimate

$$w_\alpha(t_2) \le \tau_2 \left(\bar{c}_2 + \tau_2\rho_2\right) r_2^{n_\alpha(t_2)-1} z_\alpha^{(1)}(t_2) + w_\alpha(t_1), \quad \alpha = 1,-1. \tag{43}$$

From (9) by (6), we obtain

$$z_\alpha^{(1)}(t_2) \le \tau_2 \|\mathcal{L}U_\alpha^{(0)}(\cdot,t_2) + f(\cdot,t_2,U_\alpha^{(0)}) + g(\cdot,t_2,U_\alpha^{(0)}) - \tau_2^{-1}U_\alpha(t_1)\|_{\omega^h}.$$

Since $U_\alpha^{(0)}(p,t_2)$, $U_\alpha(p,t_1)$, $\alpha = 1,-1$, are independent of τ_2, then for sufficiently small τ_2, $z_\alpha^{(1)}(t_2)$, $\alpha = 1,-1$, are independent of τ_2, that is,

$$z_\alpha^{(1)}(t_2) \le A_2, \quad \alpha = 1,-1,$$

where constant A_2 is independent of τ_2. Thus, from here, (42) and (43), we conclude that

$$w_\alpha(t_2) \le B_1\tau_1 r_1^{n_\alpha(t_1)-1} + B_2\tau_2 r_2^{n_\alpha(t_2)-1}, \quad \alpha = 1, -1,$$

where constant B_2 is independent of τ_2.

By induction on k, we can prove

$$w_k = \sum_{s=1}^{k} B_s\tau_s r_s^{n_\alpha(t_s)-1}, \quad \alpha = 1, -1,\ k \ge 1,$$

where constants B_s are independent of τ_s. Denoting

$$B = \max_{k\ge 1} B_k,$$

and taking into account that $\sum_{s=1}^{k} \tau_s \le T$, we prove the estimate in the theorem with $C = BT$. □

Remark 3. The implicit two-level difference scheme (2) is of first order with respect to time steps. As follows from (35), if $\bar{c}_k = \mathcal{O}(1)$ and $\rho_k = \mathcal{O}(1)$ $k \ge 1$, then $r_k = \mathcal{O}(\tau_k)$. To guarantee the consistency of the global errors in the implicit difference scheme and in the monotone iterative method (9), we can choose $n_\alpha(t_k) = 2, \alpha = 1, -1$, in (38). Thus, instead of using the stopping criterion (27), we can implement only two iterative steps on each time level $k \ge 1$.

4. Construction of initial upper and lower solutions

One of main ingredients in the implementation of the monotone iterative method (9) is the construction of initial upper $\widetilde{U}$ and lower $\widehat{U}$ solutions. Here, we give some conditions on functions f and g_0, for the existence of upper and lower solutions, which are used as the initial iterations in the monotone iterative method (9).

4.1. Bounded functions

Let functions f, g_0, h and ψ from (1) satisfy the following conditions:

$$\begin{aligned} &f(x,t,0) \le 0, \qquad g_0(x,t,s,0) \le 0, \qquad h(x,t) \ge 0, \qquad \psi(x) \ge 0, \\ &f(x,t,u) \ge -d_1, \qquad g_0(x,t,s,u) \ge -d_2, \qquad u \ge 0, \end{aligned} \tag{44}$$

where $d_i,\ i = 1, 2$, are positive constants.

From here and (7), it follows that the function

$$\widehat{U}(p,t_k) = \begin{cases} \psi(p), & k = 0, \\ 0, & k \ge 1, \end{cases} \quad p \in \overline{\omega}^h, \tag{45}$$

is a lower solution of (2).

Introduce the linear problem

$$\begin{aligned} &\mathcal{L}(p,t_k)\widetilde{U}(p,t_k) = \tau_k^{-1}\widetilde{U}(p,t_{k-1}) + d_1 + d_2 t_k, \quad p \in \omega^h,\ k \ge 1, \\ &\widetilde{U}(p,t_k) = h(p,t_k), \quad p \in \partial\omega^h,\ k \ge 1, \qquad \widetilde{U}(p,0) = \psi(p), \quad p \in \overline{\omega}^h. \end{aligned} \tag{46}$$

Lemma 7. *Let conditions in (44) be satisfied. Then $\widehat{U}$ and $\widetilde{U}$ from, respectively, (45) and (46) are ordered lower and upper solutions to (2), such that*

$$0 \le \widehat{U}(p,t_k) \le \widetilde{U}(p,t_k), \quad p \in \overline{\omega}^h,\ k \ge 0. \tag{47}$$

Proof. From (44) and (46), by the maximum principle in Lemma 1, we conclude (47) for $k = 1$

$$\widetilde{U}(p,t_1) \ge 0, \quad p \in \overline{\omega}^h.$$

By induction on k, we prove (47) for $k \ge 1$. We now show that $\widetilde{U}$ is an upper solution (7) to (2). From (7), (9), (44) and (46), we have

$$\begin{aligned} \mathcal{R}(p,t_k,\widetilde{U}) &= \mathcal{L}(p,t_k)\widetilde{U}(p,t_k) + f(p,t_k,\widetilde{U}) + g(p,t_k,\widetilde{U}) - \tau_k^{-1}\widetilde{U}(p,t_{k-1}) \\ &= [d_1 + f(p,t_k,\widetilde{U})] + [d_2 t_k + g(p,t_k,\widetilde{U})] \ge 0, \quad p \in \omega^h. \end{aligned}$$

Since $\widetilde{U}$ satisfies the boundary–initial conditions, we prove that $\widetilde{U}$ is an upper solution to (2). From here and (47), we conclude that $\widehat{U}$ and $\widetilde{U}$ from, respectively, (45) and (46), are ordered lower and upper solutions to (2). □

4.2. Constant upper and lower solutions

Let functions f, g_0, h and ψ from (1) satisfy the following conditions:

$$f(p,t_k,0)+\sum_{l=1}^{k}\tau_l g_0(p,t_k,t_l,0)\le 0,\quad 1\le l\le k,\ h\ge 0,\ \psi\ge 0. \tag{48}$$

It is clear that the function from (45) is a lower solution of (2). We assume that there exists a positive constant M, such that

$$f(p,t_k,M)+\sum_{l=1}^{k}\tau_l g_0(p,t_k,t_l,M)\ge 0,\quad 1\le l\le k,\ h\le M,\ \psi\le M, \tag{49}$$

and introduce the function

$$\widetilde{U}(p,t_k)=\begin{cases}\psi(p), & k=0,\\ M, & k\ge 1,\end{cases}\quad k\ge 1,\quad p\in\overline{\omega}^h. \tag{50}$$

Lemma 8. *Let conditions (48) and (49) be satisfied. Then $\widehat{U}$ and $\widetilde{U}$ from, respectively, (45) and (50), are ordered lower and upper solutions to (2) and satisfy (47).*

Proof. The proof of the lemma repeats the proof of Lemma 7 with the following modification:

$$\begin{aligned}\mathcal{R}(p,t_k,\widetilde{U}) &= \mathcal{L}(p,t_k)\widetilde{U}(p,t_k)+f(p,t_k,\widetilde{U})+g(p,t_k,\widetilde{U})-\tau_k^{-1}\widetilde{U}(p,t_{k-1})\\ &\ge f(p,t_k,M)+g(p,t_k,M)\ge 0,\quad p\in\omega^h.\quad\square\end{aligned}$$

5. Numerical experiments

In this section, we give applications of the monotone iterative method (9) for numerical solutions of two test problems. For the first test problem, the true continuous solution is explicitly known and is used to compare to a numerical solution, obtained by the monotone iterative method. In the case of the second test problem, the exact solution is unknown, and a numerical solution, obtained by the monotone iterative method, is compared to a corresponding reference solution.

We choose the stopping criterion in the form (27) with $\delta=10^{-5}$. In all numerical experiments, the monotone property of upper and lower solutions is observed at every mesh point of the computational domain.

Example 1. We consider the test problem with an internal source $q(x,t)$ in $\omega=\{0<x_1<1,\ 0<x_2<1\}$. This is given by

$$\begin{aligned}&u_t-\varepsilon(u_{x_1x_1}+u_{x_2x_2})+au^2-\int_0^t u(x,s)ds=q(x,t),\quad (x,t)\in\omega\times(0,T],\\ &u(x,t)=0,\quad (x,t)\in\partial\omega\times(0,T],\quad u(x,0)=0,\ x\in\overline{\omega},\\ &q(x,t)=\left(b+\varepsilon\pi^2(bt)+a(bt)^2\psi(x)-\frac{bt^2}{2}\right)\psi(x),\\ &\psi(x)=\sin(\pi x_1)\sin(\pi x_2),\end{aligned}$$

where a, b and ε are positive constants. It is easy to verify that the function

$$u(x,t)=(bt)\psi(x) \tag{51}$$

is the exact solution of the test problem. We assume that $f=au^2-q$ and $g_0=-u$, where

$$f_u=2au\ge 0,\quad u\ge 0,\qquad -\frac{\partial g_0}{\partial u}=1>0.$$

To satisfy (48), we choose the parameters ε and T, such that $q(x,t)\le 0$, that is,

$$T\le 2\varepsilon\pi^2.$$

To guarantee (49), we assume that M satisfies the inequality

$$aM^2-MT\ge q^*,\quad q^*=b+\varepsilon\pi^2 bT+ab^2T^2\ge\max_{x,t}q(x,t).$$

From here, we choose

$$M\ge M^*,\quad M^*=\frac{T}{2a}+\sqrt{\left(\frac{T}{2a}\right)^2+\frac{q^*}{2}}.$$

Table 1
Numerical results for Example 1 with $\tau = h^2$.

N	4	8	16	32	64
	$a = 1, b = 1, \varepsilon = 1, M = 1$				
Error	4.751e−2	1.149e−2	2.813e−3	6.932e−4	1.733e−4
Order	2.048	2.030	2.021	2.000	
# of iterations	4	4	3	3	3
	$a = 10, b = 1, \varepsilon = 1, M = 5$				
Error	2.104e−2	4.951e−2	1.222e−3	3.053e−4	7.665e−5
Order	2.087	2.018	2.001	1.994	
# of iterations	8	6	4	3	3

Table 2
Numerical results for Example 1 with $\tau = h$.

N	32	64	128	256	512
	$a = 1, b = 1, k = 1, M = 1$				
Error	1.903e−3	8.069e−4	3.625e−4	1.704e−4	8.265e−5
Order	1.238	1.154	1.089	1.043	
# of iterations	4	4	4	3	3
	$a = 10, b = 1, k = 1, M = 5$				
Error	8.593e−4	3.573e−4	1.603e−4	7.643e−5	3.683e−5
Order	1.266	1.156	1.069	1.053	
# of iterations	7	6	5	4	4

By Lemma 8, we conclude that $\widehat{U}$ and $\widetilde{U}$ from, respectively, (45) and (50) are ordered lower and upper solutions and satisfy (47). Thus,

$$0 \le \frac{\partial f}{\partial u}(p, t_k, U) \le 2aM \quad \text{on } \langle \widehat{U}(t_k), \widetilde{U}(t_k) \rangle.$$

From here, we choose $c = 2aM^*$ in the monotone iterative method (9).

We discretize the differential problem by the finite difference approximation on a uniform space mesh with the step size $h_1 = h_2 = h\,(N = 1/h)$.

In Table 1, for the two sets of parameters $a = 1, b = 1, \varepsilon = 1, M = 1$ and $a = 10, b = 1, \varepsilon = 1, M = 5$, we present the numerical error

$$\text{error}(h) = \|U_{-1}(\cdot, T) - u(\cdot, T)\|_{\overline{\omega}^h}, \quad T = 1,$$

where $u(x, t)$ is the exact solution (51), the order of the numerical error

$$\text{order}(h) = \log_2 \left(\frac{\text{error}(h)}{\text{error}(h/2)} \right),$$

and numbers of monotone iterations on each time level for different mesh sizes h and $\tau = h^2$. The data in the table show that the numerical solution has the second-order accuracy in the space variables, and numbers of iterations decrease as N increases.

In Table 2, for the same two sets of parameters as in Table 1, we present the numerical error, the order of the numerical error and numbers of monotone iterations on each time level for different mesh sizes h and $\tau = h$. The data in the table show that the numerical solution has the first-order accuracy in the time variable. Numbers of iterations decrease as N increases, and for the second data set, numbers of iterations are approximately twice as many as for $\tau = h^2$ and the same values of N (cf., Table 1).

Example 2. We consider a reaction–diffusion model with an unknown exact solution in $\omega = \{0 < x_1 < 1,\ 0 < x_2 < 1\}$:

$$u_t - \varepsilon(u_{x_1x_1} + u_{x_2x_2}) + au^2 - \int_0^t \frac{bu(x, s)}{1 + \varsigma u(x, s)} ds = 0, \quad (x, t) \in \omega \times (0, T],$$
$$u(x, t) = 0, \quad (x, t) \in \partial\omega \times (0, T], \qquad u(x, 0) = \psi(x), \quad x \in \overline{\omega},$$
$$\psi(x) = \sin(\pi x_1) \sin(\pi x_2),$$

where a, b, ε and ς are positive constants.

Table 3
Numerical results for Example 2 with $\tau = h^2$, the case of constant lower and upper solutions.

N	4	8	16	32	64
	$a = 10, b = 1, \varepsilon = 1, \varsigma = 1, M^* = 1$				
	Lower solutions				
Error	1.706e−3	4.372e−4	1.109e−4	2.686e−5	4.832e−6
Order	1.964	1.979	2.046	2.474	
# of iterations	4	3	3	3	2
	Upper solutions				
Error	1.790e−3	4.372e−4	1.101e−4	2.656e−5	5.048e−6
Order	2.034	1.990	2.052	2.396	
# of iterations	7	5	4	4	3
	$a = 1, b = 10, \varepsilon = 1, \varsigma = 1, M^* = 2.702$				
	Lower solutions				
Error	2.403e−2	6.077e−3	1.514e−3	3.561e−4	7.201e−5
Order	1.983	2.005	2.088	2.300	
# of iterations	7	5	4	3	3
	Upper solutions				
Error	2.405e−2	6.086e−3	1.523e−3	3.619e−4	7.693e−6
Order	1.983	1.999	2.073	2.234	
# of iterations	8	6	4	4	3

The case of constant upper and lower solutions. Condition (48) holds true without any restrictions on the parameters of the test problem. To guarantee (49), we assume that M satisfies the inequality $M \geq M^*$, where

$$M^* = \max[\overline{\psi}, \overline{M}], \quad \overline{\psi} = \|\psi(x)\|_{\overline{\omega}^h}, \quad \overline{M} = -\frac{1}{2\varsigma} + \sqrt{\left(\frac{1}{2\varsigma}\right)^2 + \frac{bT}{a\varsigma}}.$$

We now have

$$0 \leq f_u = 2au \leq 2aM, \quad -\frac{\partial g_0}{\partial u} = \frac{b}{(1+\varsigma u)^2} > 0, \; 0 \leq u \leq M.$$

By Lemma 8, we conclude that $\widehat{U}$ and $\widetilde{U}$ from, respectively, (45) and (50) are ordered lower and upper solutions and satisfy (47). Thus,

$$0 \leq \frac{\partial f}{\partial u}(p, t_k, U) \leq 2aM \quad \text{on } \langle \widehat{U}(t_k), \widetilde{U}(t_k) \rangle.$$

From here, we choose $c = 2aM^*$ in the monotone iterative method (9).

In Table 3, for the two sets of parameters $a = 10, b = 1, \varepsilon = 1, \varsigma = 1, M^* = 1$ and $a = 1, b = 10, \varepsilon = 1, \varsigma = 1, M^* = 2.702$, we present the numerical error

$$\text{error}(h) = \|U_\alpha(\cdot, T) - U_\alpha^{\text{ref}}(\cdot, T)\|_{\overline{\omega}^h}, \quad T = 1, \; \alpha = 1, -1,$$

where $U_\alpha^{\text{ref}}(p, t_k)$, $\alpha = 1, -1$, are reference solutions with $N = 128$, the order of the numerical error and numbers of monotone iterations on each time level for different mesh sizes h and $\tau = h^2$. The data in the table indicate that the numerical solution has the second-order accuracy in the space variables, and numbers of iterations decrease as N increases. We mention here, that numerical experiments show that if in the reference solution N increases, then the order of the numerical error tends to the second one.

In Table 4, for the same two sets of parameters as in Table 3, we present the numerical error, the order of the numerical error and numbers of monotone iterations on each time level for different mesh sizes of h and $\tau = h$, where $U_\alpha^{\text{ref}}(p, t_k)$, $\alpha = 1, -1$ are reference solutions with $N = 1024$. The data in the table show that the numerical solution has the first-order accuracy in the time variable, and numbers of iterations decrease as N increases. Similar to Example 1, for the second data set, numbers of iterations are approximately twice as many as for $\tau = h^2$ and the same values of N (cf., Table 3).

The case of bounded functions. On each time level t_k, $k \geq 1$, we now calculate an initial upper solution $\widetilde{U}(p, t_k)$ by solving the linear problem (46). All the conditions in (44) hold if $d_1 = 0, d_2 = b/\varsigma$. By Lemma 7, $\widehat{U}$ and $\widetilde{U}$ from, respectively, (45) and (46) are lower and upper solutions. From here, we choose $c(p, t_k) = 2a\widetilde{U}(p, t_k)$ in the monotone iterative method (9).

In Table 5, for the same two sets of parameters as in Tables 3 and 4, we present the numerical error

$$\text{error}(h) = \|U_1(\cdot, T) - U_1^{\text{ref}}(\cdot, T)\|_{\overline{\omega}^h}, \quad T = 1,$$

where $U_1^{\text{ref}}(p, t_k)$ are reference solutions with $N = 128$ for $\tau = h^2$ and with $N = 1024$ for $\tau = h$, the order of the numerical error and numbers of monotone iterations on each time level for different mesh sizes h. The data in the table show that the numerical solution has the second-order and the first-order accuracy in the time variable, and numbers of iterations

Table 4
Numerical results for Example 2 with $\tau = h$, the case of constant lower and upper solutions.

N	32	64	128	256	512
	$a = 10, b = 1, \varepsilon = 1, \varsigma = 1, M^* = 1$				
	Lower solutions				
Error	5.581e−4	2.786e−4	1.338e−4	5.840e−5	1.966e−5
Order	1.002	1.058	1.196	1.570	
# of iterations	4	3	3	3	3
	Upper solutions				
Error	5.579e−4	2.788e−4	1.340e−4	5.793e−5	1.962e−5
Order	1.001	1.057	1.210	1.562	
# of iterations	6	5	5	4	4
	$a = 1, b = 10, \varepsilon = 1, \varsigma = 1, M^* = 2.702$				
	Lower solutions				
Error	5.183e−3	2.564e−3	1.210e−3	5.282e−4	1.749e−4
Order	1.015	1.083	1.196	1.594	
# of iterations	6	5	4	4	4
	Upper solutions				
Error	5.187e−3	2.567e−3	1.218e−3	5.337e−4	1.780e−4
Order	1.015	1.076	1.190	1.584	
# of iterations	7	6	5	4	4

Table 5
Numerical results for Example 2 with $\tau = h^2$ and $\tau = h$, the case of bounded functions.

N	4	8	16	32	64
	$\tau = h^2, a = 10, b = 1, \varepsilon = 1, \varsigma = 1$				
Error	1.791e−3	4.383e−4	1.107e−4	2.656e−5	5.323e−6
Order	2.031	1.985	2.059	2.319	
# of iterations	4	3	2	2	2
	$\tau = h^2, a = 1, b = 10, \varepsilon = 1, \varsigma = 1$				
Error	2.404e−2	6.083e−3	1.519e−3	3.619e−4	7.245e−5
Order	1.983	2.002	2.070	2.321	
# of iterations	5	3	2	2	2
N	32	64	128	256	512
	$\tau = h, a = 10, b = 1, \varepsilon = 1, \varsigma = 1$				
Error	5.585e−4	2.799e−4	1.344e−4	5.852e−5	1.969e−5
Order	0.997	1.058	1.120	1.572	
# of iterations	3	3	2	2	2
	$\tau = h, a = 1, b = 10, \varepsilon = 1, \varsigma = 1$				
Error	5.184e−3	2.564e−3	1.215e−3	5.276e−4	1.759e−4
Order	1.016	1.077	1.203	1.585	
# of iterations	4	3	3	2	2

decrease as N increases. As follows from Tables 3–5, for both data sets, numbers of iterations for the monotone method, based on calculation of initial upper solutions by solving linear problems (46), are approximately twice as less as for the monotone method, based on constant initial upper solutions (50).

References

[1] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.
[2] I. Boglaev, Monotone algorithms for solving nonlinear monotone difference schemes of parabolic type in the canonical form, Numer. Math. 14 (2006) 247–266.
[3] I. Boglaev, On modified accelerated monotone iterates for solving semilinear parabolic problems, Appl. Numer. Math. 62 (2012) 1849–1863.
[4] C.V. Pao, Numerical methods of semilinear parabolic equations, SIAM J. Numer. Anal. 24 (1987) 24–35.
[5] C.V. Pao, Positive solutions and dynamics of a finite difference reaction–diffusion system, Numer. Methods Partial Differential Equations 9 (1993) 285–311.
[6] C.V. Pao, Accelerated iterative methods for finite difference equations of reacttion-diffusion, Numer. Math. 79 (1998) 261–281.
[7] C.V. Pao, X. Lu, Block monotone iterative method for semilinear parabolic equations with nonlinear boundary conditions, SIAM J. Numer. Anal. 47 (2010) 4581–4606.
[8] Y.-M. Wang, X.-L. Lan, Higher-order monotone iterative methods for finite difference systems of nonlinear reaction–diffusion–convection equations, Appl. Numer. Math. 59 (2009) 2677–2693.
[9] C.V. Pao, Numerical methods for nonlinear integro-parabolic equations of Fredholm type, Comput. Math. Appl. 41 (2001) 857–877.
[10] A. Samarskii, The Theory of Difference Schemes, Marcel Dekker, New York-Basel, 2001.
[11] J. Douglas, B.F. Jones, Numerical methods for integro-differential equation of parabolic and hyperbolic types, Numer. Math. 4 (1962) 96–102.
[12] I.H. Sloan, V. Thomee, Time distretization of an integro-differential equation of parabolic type, SIAM J. Numer. Anal. 23 (1986) 1052–1061.