



Numerical methods for fourth-order elliptic equations with nonlocal boundary conditions



C.V. Pao^a, Yuan-Ming Wang^{b,c,*}

^a Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

^b Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai 200241, People's Republic of China

^c Scientific Computing Key Laboratory of Shanghai Universities, Division of Computational Science, E-Institute of Shanghai Universities, Shanghai Normal University, Shanghai 200234, People's Republic of China

ARTICLE INFO

Article history:
Received 26 July 2012

MSC:
35J40
35J61
65N06
65N12

Keywords:
Fourth-order elliptic equation
Nonlocal boundary condition
Finite difference system
Method of upper and lower solutions
Monotone iterations
Convergence

ABSTRACT

This paper is concerned with some numerical methods for a fourth-order semilinear elliptic boundary value problem with nonlocal boundary condition. The fourth-order equation is formulated as a coupled system of two second-order equations which are discretized by the finite difference method. Three monotone iterative schemes are presented for the coupled finite difference system using either an upper solution or a lower solution as the initial iteration. These sequences of monotone iterations, called maximal sequence and minimal sequence respectively, yield not only useful computational algorithms but also the existence of a maximal solution and a minimal solution of the finite difference system. Also given is a sufficient condition for the uniqueness of the solution. This uniqueness property and the monotone convergence of the maximal and minimal sequences lead to a reliable and easy to use error estimate for the computed solution. Moreover, the monotone convergence property of the maximal and minimal sequences is used to show the convergence of the maximal and minimal finite difference solutions to the corresponding maximal and minimal solutions of the original continuous system as the mesh size tends to zero. Three numerical examples with different types of nonlinear reaction functions are given. In each example, the true continuous solution is constructed and is used to compare with the computed solution to demonstrate the accuracy and reliability of the monotone iterative schemes.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

There are extensive discussions on fourth-order elliptic equations especially in the area of two-point boundary value problems which arise from the static deflection of a bending beam (cf. [1–3]). However, most of the discussions in the literature are for homogeneous Dirichlet boundary conditions. In a recent article [4] the authors treated a general class of fourth-order elliptic equations with nonlocal boundary conditions by the method of upper and lower solutions and its associated monotone iterations. In this paper, we extend the above method to a corresponding discrete system of the problem and develop various monotone iterative schemes for the computation of numerical solutions, including the

* Corresponding author at: Department of Mathematics, Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, East China Normal University, Shanghai 200241, People's Republic of China.

E-mail addresses: cvpao@ncsu.edu (C.V. Pao), ymwang@math.ecnu.edu.cn (Y.-M. Wang).

existence of maximal and minimal solutions, the uniqueness of the solution, and the convergence of the discrete solution to the continuous solution. The basic problem under consideration is given by

$$\begin{aligned}\Delta^2 u - b_0 \Delta u + c_0 u &= f(x, u) \quad (x \in \Omega), \\ u(x') &= \int_{\Omega} \beta(x', x) u(x) dx + g^{(1)}(x') \quad (x' \in \partial\Omega), \\ (\Delta u)(x') &= \int_{\Omega} \beta(x', x) (\Delta u)(x) dx - g^{(0)}(x') \quad (x' \in \partial\Omega),\end{aligned}\tag{1.1}$$

where Ω is a bounded domain in \mathbf{R}^n with boundary $\partial\Omega$ ($n = 1, 2, \dots$), b_0 and c_0 are constants with $b_0 \geq 0$, and $f(x, u)$, $\beta(x', x)$ and $g^{(l)}(x')$ ($l = 0, 1$) are continuous functions in their respective domain. The function $\beta(x', x)$ is nonnegative on $\partial\Omega \times \Omega$ and is continuous in x' and is piecewise continuous in x for all $x' \in \partial\Omega$ and $x \in \Omega$. This implies that problem (1.1) is reduced to the standard fourth-order boundary value problem with Dirichlet boundary condition if $\beta(x', x) = 0$ on $\partial\Omega \times \Omega$. By allowing

$$\beta(x', x) = \beta_1 \delta(x - x_1) + \dots + \beta_p \delta(x - x_p) \quad ((x', x) \in \partial\Omega \times \Omega)$$

for a set of positive constants β_k and points $x_k \in \Omega$, where $k = 1, 2, \dots, p$ and $\delta(x - x_k)$ is the Dirac function, the boundary condition in (1.1) becomes the multi-point boundary condition

$$u(x') = \sum_{k=1}^p \beta_k u(x_k) + g^{(1)}(x'), \quad (\Delta u)(x') = \sum_{k=1}^p \beta_k (\Delta u)(x_k) - g^{(0)}(x').\tag{1.1a}$$

In our discussion for problem (1.1) we include the above boundary conditions as special cases of the problem.

In addition to the boundary condition in (1.1) we also consider the boundary condition

$$u(x') = \int_{\Omega} \beta(x', x) u(x) dx + g^{(1)}(x'), \quad (\Delta u)(x') = -g^{(0)}(x').\tag{1.1b}$$

The above boundary condition has been considered in [3,5–9] for the one-dimensional domain $\Omega = (0, 1)$ with $g^{(1)}(x') = g^{(0)}(x') = 0$. The inclusion of possible boundary sources $g^{(1)}(x')$ and $g^{(0)}(x')$ is for the convenience of construction of explicitly known solutions which are used to compare with the computed numerical solutions in the final section.

Fourth-order elliptic boundary value problems have been received considerable attention in recent years, and most of the discussions in the current literature are for the existence, uniqueness and multiplicity of solutions with homogeneous Dirichlet boundary conditions (cf. [10–14]). The works in [15–18] gave also some treatment on the numerical solutions of the corresponding finite difference system, including a second-order elliptic equation with nonlocal boundary condition. On the other hand, extensive attention has been given to ordinary differential equations with either multi-point boundary conditions or integral type of boundary conditions (cf. [7–9,19–25]). The purpose of this paper is to develop some monotone iterative schemes for the computation of numerical solutions of the nonlocal problem (1.1), including the boundary condition (1.1b), using the method of upper and lower solutions. The monotone iterative schemes, called Picard, Gauss–Seidel and Jacobi iterations, yield not only computational algorithms but also existence of maximal and minimal solutions of the corresponding discrete system. These two solutions can be computed by any one of the three iterative schemes using an upper solution or a lower solution as the initial iteration.

The plan of the paper is as follows: In Section 2, we formulate problem (1.1) as a coupled system of two second-order elliptic equations and then discretize it into a finite difference system. A Picard type of monotone iteration is developed for the coupled finite difference system, including the system with boundary condition (1.1b). The Gauss–Seidel and Jacobi monotone iterations as well as a comparison theorem among the three monotone iterations are given in Section 3. In Section 4, we show the convergence of the finite difference solution to the corresponding solution of the continuous problem as the mesh size tends to zero. Section 5 is devoted to the construction of positive upper and lower solutions which are used as initial iterations in the various monotone iterative schemes. Finally in Section 6, we give three examples as applications of the monotone iterations. In each example, an explicitly known continuous solution is constructed and is used to compare with the computed numerical solution to demonstrate the accuracy and reliability of the monotone iterative schemes.

2. Monotone iterations

To develop monotone iterative schemes for numerical solutions of problem (1.1), including the boundary conditions (1.1a) and (1.1b), we discretize the fourth-order equation in (1.1) by the finite difference method. Let h_v be the spatial increment in the x_v -direction. Let $i = (i_1, i_2, \dots, i_n)$ be a multiple index with $i_v = 1, 2, \dots, M_v$ for $v = 1, 2, \dots, n$, and let $x_i = (x_{i_1}, x_{i_2}, \dots, x_{i_n})$ be an interior point in Ω and $x'_j = (x'_{j_1}, x'_{j_2}, \dots, x'_{j_n})$ a boundary point on $\partial\Omega$, where M_v is the total number of interior points in the x_{i_v} -direction. Denote by Ω_h , $\overline{\Omega}_h$ and $\partial\Omega_h$ the set of mesh points in Ω , $\overline{\Omega}$ and $\partial\Omega$ respectively, and assume that Ω is connected. When no confusion arises we write $i \in \Omega_h$ and $j \in \partial\Omega_h$ instead of $x_i \in \Omega_h$

and $x'_j \in \partial\Omega_h$, respectively. For clarity of presentation, we use u_i to represent the approximation of $u(x_i)$ for any mesh point x_i , and define

$$f_i(u_i) = f(x_i, u_i), \quad g_j^{(l)} = g^{(l)}(x'_j) \quad (l = 0, 1).$$

Then by the central difference approximation

$$\Delta_h u_i \equiv \sum_{v=1}^n \Delta_h^{(v)} u_i \equiv \sum_{v=1}^n h_v^{-2} [u_{i+e_v} - 2u_i + u_{i-e_v}],$$

where e_v is the unit vector in \mathbf{R}^n with the v th component one and zero elsewhere, we approximate the differential equation in (1.1) by

$$\Delta_h(\Delta_h u_i) - b_0 \Delta_h u_i + c_0 u_i = f_i(u_i) \quad (i \in \Omega_h). \quad (2.1)$$

To approximate the boundary condition in (1.1) we let $\{\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_n}\}$ be any set of quadrature weights that possesses the property

$$0 < \omega_{i_v} \leq 1 \quad (v = 1, 2, \dots, n), \quad \sum_{i_1=1}^{M_1} \sum_{i_2=1}^{M_2} \cdots \sum_{i_n=1}^{M_n} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_n} \beta(x'_j, \xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n}) < \infty,$$

where $\xi_i = (\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n})$ denotes an arbitrary mesh point in Ω_h . This property is possessed by most commonly used quadrature rules such as Simpson's rule (cf. [26,27]). We then approximate the integrals in (1.1) by

$$J[x'_j, w] = \sum_{i_1=1}^{M_1} \sum_{i_2=1}^{M_2} \cdots \sum_{i_n=1}^{M_n} \omega_{i_1} \omega_{i_2} \cdots \omega_{i_n} \beta(x'_j, \xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n}) w(\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n})$$

for $w = u$ or $w = \Delta_h u$. The above approximation gives a Riemann sum of the integral if $w_{i_v} = h_v$ for all v . Using the above function $J[x'_j, w]$, we approximate the boundary condition in (1.1) by

$$u_j = J[x'_j, u] + g_j^{(1)}, \quad \Delta_h u_j = J[x'_j, \Delta_h u] - g_j^{(0)} \quad (x'_j \in \partial\Omega_h). \quad (2.2)$$

When the boundary condition is given by (1.1_a) we choose x_k as mesh points and approximate it by (2.2) with

$$J[x'_j, w] = \sum_{k=1}^p \beta_k w(x_k) \quad (x'_j \in \partial\Omega_h). \quad (2.2_a)$$

In case the boundary condition is given by (1.1_b) we approximate it by

$$u_j = J[x'_j, u] + g_j^{(1)}, \quad \Delta_h u_j = -g_j^{(0)} \quad (x'_j \in \partial\Omega_h). \quad (2.2_b)$$

To compute numerical solutions of the finite difference system (2.1), (2.2), including the boundary condition (2.2_b), we use the method of upper and lower solutions (cf. [28]).

Definition 2.1. A function \tilde{u}_i is called an upper solution of (2.1), (2.2) if it satisfies the inequalities

$$\begin{aligned} \Delta_h(\Delta_h \tilde{u}_i) - b_0 \Delta_h \tilde{u}_i + c_0 \tilde{u}_i &\geq f_i(\tilde{u}_i) \quad (i \in \Omega_h), \\ \tilde{u}_j &\geq J[x'_j, \tilde{u}] + g_j^{(1)}, \quad \Delta_h \tilde{u}_j \leq J[x'_j, \Delta_h \tilde{u}] - g_j^{(0)} \quad (x'_j \in \partial\Omega_h). \end{aligned} \quad (2.3)$$

Similarly, \hat{u}_i is called a lower solution if it satisfies (2.3) with all the inequalities reversed.

It is clear from the above definition that every solution of (2.1), (2.2) is an upper solution as well as a lower solution. The pair of upper and lower solutions is said to be ordered if $\tilde{u}_i \geq \hat{u}_i$ and $\Delta_h \tilde{u}_i \leq \Delta_h \hat{u}_i$. For a given pair of ordered upper and lower solutions \tilde{u}_i and \hat{u}_i , we set

$$\langle \hat{u}_i, \tilde{u}_i \rangle = \{u_i; \hat{u}_i \leq u_i \leq \tilde{u}_i, \Delta_h \hat{u}_i \geq \Delta_h u_i \geq \Delta_h \tilde{u}_i\}.$$

Let \bar{c} be any constant satisfying

$$\bar{c} \geq \max \left\{ -\frac{\partial f_i}{\partial u}(u_i); \hat{u}_i \leq u_i \leq \tilde{u}_i \right\}, \quad (2.4)$$

and let

$$c^* = \bar{c} + c_0, \quad f_i^*(u_i) = \bar{c} u_i + f_i(u_i). \quad (2.5)$$

Then by (2.4), $f_i^*(u_i)$ is monotone nondecreasing in u_i for $\widehat{u}_i \leq u_i \leq \widetilde{u}_i$, and Eq. (2.1) is equivalent to

$$\Delta_h(\Delta_h u_i) - b_0 \Delta_h u_i + c^* u_i = f_i^*(u_i) \quad (i \in \Omega_h). \quad (2.6)$$

Throughout the paper we assume that

$$b_0 \geq 0, \quad c^* \geq 0, \quad b_0^2 \geq 4c^*. \quad (2.7)$$

Define

$$\mu = \left(b_0 - \sqrt{b_0^2 - 4c^*} \right) / 2, \quad \mu^+ = \left(b_0 + \sqrt{b_0^2 - 4c^*} \right) / 2. \quad (2.8)$$

Then the relations $\mu + \mu^+ = b_0$ and $\mu\mu^+ = c^*$ imply that

$$(-\Delta_h + \mu^+)(-\Delta_h u_i + \mu u_i) = \Delta_h(\Delta_h u_i) - b_0 \Delta_h u_i + c^* u_i,$$

and Eq. (2.6) is equivalent to

$$(-\Delta_h + \mu^+)(-\Delta_h u_i + \mu u_i) = f_i^*(u_i) \quad (i \in \Omega_h).$$

Hence we may write (2.6) (or (2.1)) as the coupled system

$$-\Delta_h u_i + \mu u_i = v_i, \quad -\Delta_h v_i + \mu^+ v_i = f_i^*(u_i) \quad (i \in \Omega_h), \quad (2.9)$$

and the boundary condition (2.2) as

$$u_j = J[x'_j, u] + g_j^{(1)}, \quad v_j = J[x'_j, v] + g_j^{(2)} \quad (x'_j \in \partial\Omega_h), \quad (2.10)$$

where $g_j^{(2)} = g_j^{(0)} + \mu g_j^{(1)}$. It is clear that u_i is a solution of (2.1), (2.2) if and only if (u_i, v_i) is a solution of (2.9), (2.10). Our monotone iterative schemes are developed with respect to the above coupled system by the method of upper and lower solutions.

Definition 2.2. A function $(\widetilde{u}_i, \widetilde{v}_i)$ is called an upper solution of (2.9), (2.10) if

$$\begin{aligned} -\Delta_h \widetilde{u}_i + \mu \widetilde{u}_i &\geq \widetilde{v}_i, & -\Delta_h \widetilde{v}_i + \mu^+ \widetilde{v}_i &\geq f_i^*(\widetilde{u}_i) \quad (i \in \Omega_h), \\ \widetilde{u}_j &\geq J[x'_j, \widetilde{u}] + g_j^{(1)}, & \widetilde{v}_j &\geq J[x'_j, \widetilde{v}] + g_j^{(2)} \quad (x'_j \in \partial\Omega_h). \end{aligned} \quad (2.11)$$

Similarly, $(\widehat{u}_i, \widehat{v}_i)$ is called a lower solution if it satisfies (2.11) with all the inequalities reversed. The above pair of upper and lower solutions is said to be ordered if $(\widetilde{u}_i, \widetilde{v}_i) \geq (\widehat{u}_i, \widehat{v}_i)$.

It is obvious from the above definition that every solution of (2.9), (2.10) is an upper solution as well as a lower solution. For a given pair of upper and lower solutions $(\widetilde{u}_i, \widetilde{v}_i)$, $(\widehat{u}_i, \widehat{v}_i)$ of (2.9), (2.10), we set

$$\mathcal{S} \equiv \{(u_i, v_i); (\widehat{u}_i, \widehat{v}_i) \leq (u_i, v_i) \leq (\widetilde{u}_i, \widetilde{v}_i)\}. \quad (2.12)$$

Using either $(\widetilde{u}_i, \widetilde{v}_i)$ or $(\widehat{u}_i, \widehat{v}_i)$ as the initial iteration $(u_i^{(0)}, v_i^{(0)})$ we can construct a sequence $\{u_i^{(m)}, v_i^{(m)}\}$ from the linear iteration process

$$\begin{aligned} -\Delta_h u_i^{(m)} + \mu u_i^{(m)} &= v_i^{(m-1)}, & -\Delta_h v_i^{(m)} + \mu^+ v_i^{(m)} &= f_i^*(u_i^{(m-1)}) \quad (i \in \Omega_h), \\ u_j^{(m)} &= J[x'_j, u^{(m-1)}] + g_j^{(1)}, & v_j^{(m)} &= J[x'_j, v^{(m-1)}] + g_j^{(2)} \quad (x'_j \in \partial\Omega_h), \end{aligned} \quad (2.13)$$

where $m = 1, 2, \dots$. It is clear that the sequence $\{u_i^{(m)}, v_i^{(m)}\}$ is well-defined and can be computed by solving linear second order finite difference equations with Dirichlet boundary conditions. Denote the sequence by $\{\overline{u}_i^{(m)}, \overline{v}_i^{(m)}\}$ if $(u_i^{(0)}, v_i^{(0)}) = (\widetilde{u}_i, \widetilde{v}_i)$ and by $\{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ if $(u_i^{(0)}, v_i^{(0)}) = (\widehat{u}_i, \widehat{v}_i)$, and refer to them as maximal and minimal sequences, respectively. In the following theorem we show the monotone convergence of these sequences.

Theorem 2.1. Let condition (2.7) hold, and let $(\widetilde{u}_i, \widetilde{v}_i)$, $(\widehat{u}_i, \widehat{v}_i)$ be a pair of ordered upper and lower solutions of (2.9), (2.10). Then the following statements hold true:

- (i) Problem (2.9), (2.10) has a maximal solution $(\overline{u}_i, \overline{v}_i)$ and a minimal solution $(\underline{u}_i, \underline{v}_i)$ in \mathcal{S} .
- (ii) The maximal sequence $\{\overline{u}_i^{(m)}, \overline{v}_i^{(m)}\}$ converges to $(\overline{u}_i, \overline{v}_i)$, the minimal sequence $\{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ converges to $(\underline{u}_i, \underline{v}_i)$, and they satisfy the relation

$$\begin{aligned} (\widehat{u}_i, \widehat{v}_i) &\leq (\underline{u}_i^{(m)}, \underline{v}_i^{(m)}) \leq (\underline{u}_i^{(m+1)}, \underline{v}_i^{(m+1)}) \leq (\underline{u}_i, \underline{v}_i) \leq (\overline{u}_i, \overline{v}_i) \\ &\leq (\overline{u}_i^{(m+1)}, \overline{v}_i^{(m+1)}) \leq (\overline{u}_i^{(m)}, \overline{v}_i^{(m)}) \leq (\widetilde{u}_i, \widetilde{v}_i), \quad m = 1, 2, \dots \end{aligned} \quad (2.14)$$

- (iii) If $(\overline{u}_i, \overline{v}_i) = (\underline{u}_i, \underline{v}_i) (= (u_i^*, v_i^*))$ then (u_i^*, v_i^*) is the unique solution of (2.9), (2.10) in \mathcal{S} .

Proof. Let $\bar{w}^{(0)} = \bar{u}_i^{(0)} - \bar{u}_i^{(1)} = \tilde{u}_i - \bar{u}_i^{(1)}$, $\bar{z}^{(0)} = \bar{v}_i^{(0)} - \bar{v}_i^{(1)} = \tilde{v}_i - \bar{v}_i^{(1)}$. By (2.11), (2.13) and $(\bar{u}_i^{(0)}, \bar{v}_i^{(0)}) = (\tilde{u}_i, \tilde{v}_i)$ we have

$$\begin{aligned} -\Delta_h \bar{w}_i^{(0)} + \mu \bar{w}_i^{(0)} &= (-\Delta_h \tilde{u}_i + \mu \tilde{u}_i) - \tilde{v}_i \geq 0, \\ -\Delta_h \bar{z}_i^{(0)} + \mu \bar{z}_i^{(0)} &= (-\Delta_h \tilde{v}_i + \mu \tilde{v}_i) - f_i^*(\tilde{u}_i) \geq 0 \quad (i \in \Omega_h), \\ \bar{w}_j^{(0)} &= \tilde{u}_j - (J[x'_j, \tilde{u}] + g_j^{(1)}) \geq 0, \quad \bar{z}_j^{(0)} = \tilde{v}_j - (J[x'_j, \tilde{v}] + g_j^{(2)}) \geq 0 \quad (x'_j \in \partial\Omega_h). \end{aligned} \quad (2.15)$$

The positivity lemma for second order finite difference elliptic boundary value problems implies that $\bar{w}_i^{(0)} \geq 0$ and $\bar{z}_i^{(0)} \geq 0$ (cf. [16,29]). This yields $\bar{u}_i^{(0)} \geq \bar{u}_i^{(1)}$ and $\bar{v}_i^{(0)} \geq \bar{v}_i^{(1)}$. A similar argument using the property of a lower solution gives $\underline{u}_i^{(1)} \geq \underline{u}_i^{(0)}$ and $\underline{v}_i^{(1)} \geq \underline{v}_i^{(0)}$ where $(\underline{u}_i^{(0)}, \underline{v}_i^{(0)}) = (\hat{u}_i, \hat{v}_i)$. We next let $w_i^{(1)} = \bar{u}_i^{(1)} - \underline{u}_i^{(1)}$, $z_i^{(1)} = \bar{v}_i^{(1)} - \underline{v}_i^{(1)}$. Then by (2.13) and the non-decreasing property of $f_i^*(u_i)$ and $J[\cdot, u]$, we have

$$\begin{aligned} -\Delta_h w_i^{(1)} + \mu w_i^{(1)} &= \bar{v}_i^{(0)} - \underline{v}_i^{(0)} \geq 0, \\ -\Delta_h z_i^{(1)} + \mu z_i^{(1)} &= f_i^*(\bar{u}_i^{(0)}) - f_i^*(\underline{u}_i^{(0)}) \geq 0 \quad (i \in \Omega_h), \\ w_j^{(1)} &= (J[x'_j, \bar{u}^{(0)}] + g_j^{(1)}) - (J[x'_j, \underline{u}^{(0)}] + g_j^{(1)}) \geq 0, \\ z_j^{(1)} &= (J[x'_j, \bar{v}^{(0)}] + g_j^{(2)}) - (J[x'_j, \underline{v}^{(0)}] + g_j^{(2)}) \geq 0 \quad (x'_j \in \Omega_h). \end{aligned} \quad (2.16)$$

This leads to $w_i^{(1)} \geq 0$ and $z_i^{(1)} \geq 0$ which is equivalent to $\bar{u}_i^{(1)} \geq \underline{u}_i^{(1)}$ and $\bar{v}_i^{(1)} \geq \underline{v}_i^{(1)}$. The above conclusions show that

$$(\underline{u}_i^{(0)}, \underline{v}_i^{(0)}) \leq (\underline{u}_i^{(1)}, \underline{v}_i^{(1)}) \leq (\bar{u}_i^{(1)}, \bar{v}_i^{(1)}) \leq (\bar{u}_i^{(0)}, \bar{v}_i^{(0)}).$$

It follows by an induction argument that

$$(\underline{u}_i^{(m)}, \underline{v}_i^{(m)}) \leq (\underline{u}_i^{(m+1)}, \underline{v}_i^{(m+1)}) \leq (\bar{u}_i^{(m+1)}, \bar{v}_i^{(m+1)}) \leq (\bar{u}_i^{(m)}, \bar{v}_i^{(m)})$$

for every $m = 1, 2, \dots$. This monotone property ensures that the limits

$$\lim_{m \rightarrow \infty} (\bar{u}_i^{(m)}, \bar{v}_i^{(m)}) = (\bar{u}_i, \bar{v}_i), \quad \lim_{m \rightarrow \infty} (\underline{u}_i^{(m)}, \underline{v}_i^{(m)}) = (\underline{u}_i, \underline{v}_i) \quad (2.17)$$

exist and the relation (2.14) holds. Letting $m \rightarrow \infty$ in (2.13) show that both (\bar{u}_i, \bar{v}_i) and $(\underline{u}_i, \underline{v}_i)$ are solutions of (2.9), (2.10). Now if (u_i, v_i) is any solution of (2.9), (2.10) in \mathcal{S} then since it is an upper solution as well as a lower solution, a consideration of $(\tilde{u}_i, \tilde{v}_i)$ and (u_i, v_i) (resp. (u_i, v_i) and (\hat{u}_i, \hat{v}_i)) as the pair of upper and lower solutions in the above discussion we obtain $(\bar{u}_i, \bar{v}_i) \geq (u_i, v_i)$ (resp. $(u_i, v_i) \geq (\underline{u}_i, \underline{v}_i)$). This shows the maximal and minimal properties of the solutions (\bar{u}_i, \bar{v}_i) and $(\underline{u}_i, \underline{v}_i)$ which gives the conclusions in (i) and (ii). It is clear from the maximal and minimal properties of these solutions that (u_i^*, v_i^*) is the unique solution of (2.9), (2.10) in \mathcal{S} if $(\bar{u}_i, \bar{v}_i) = (\underline{u}_i, \underline{v}_i) \equiv (u_i^*, v_i^*)$. This proves the theorem. \square

We next show the uniqueness of the solution under the following additional condition:

$$(\partial f_i / \partial u)(u_i) \leq c_0 \quad \text{for } \hat{u}_i \leq u_i \leq \tilde{u}_i, \quad \text{and } J[x'_j, 1] < 1 \quad \text{for } x'_j \in \partial\Omega_h. \quad (2.18)$$

Theorem 2.2. Let the conditions in Theorem 2.1 be satisfied, and let (\bar{u}_i, \bar{v}_i) , $(\underline{u}_i, \underline{v}_i)$ be the maximal and minimal solutions of (2.9), (2.10). If, in addition, condition (2.18) holds then $(\bar{u}_i, \bar{v}_i) = (\underline{u}_i, \underline{v}_i) \equiv (u_i^*, v_i^*)$ and (u_i^*, v_i^*) is the unique solution of (2.9), (2.10) in \mathcal{S} .

Proof. Let $w_i^* = \underline{u}_i - \bar{u}_i \leq 0$, $z_i^* = \underline{v}_i - \bar{v}_i - \mu(\underline{u}_i - \bar{u}_i)$. Since by (2.9) and the mean-value theorem

$$-\Delta_h z_i^* + b_0 z_i^* = -c_0(\underline{u}_i - \bar{u}_i) + f_i(\underline{u}_i) - f_i(\bar{u}_i) = (-c_0 + (\partial f_i / \partial u)(\xi_i))w_i^*,$$

where ξ_i is an intermediate value between \underline{u}_i and \bar{u}_i , we see from condition (2.18), relation (2.10), and $w_i^* \leq 0$ that

$$-\Delta_h z_i^* + b_0 z_i^* \geq 0 \quad (i \in \Omega_h), \quad z_j^* = J[x'_j, z^*] \quad (x'_j \in \partial\Omega_h). \quad (2.19)$$

Assume, by contradiction, that $z_{j_0}^*$ has a negative minimum on $\bar{\Omega}_h$. Then since $b_0 \geq 0$, this negative minimum must be attained at some boundary point $x'_{j_0} \in \partial\Omega_h$ (cf. [29]). This implies that

$$z_{j_0}^* = J[x'_{j_0}, z^*] \geq J[x'_{j_0}, 1]z_{j_0}^*.$$

The negative property of $z_{j_0}^*$ implies that $J[x'_{j_0}, 1] \geq 1$ which contradicts the condition in (2.18). This contradiction shows that $z_i^* \geq 0$ on $\bar{\Omega}_h$ and therefore by (2.9) and (2.10),

$$-\Delta_h w_i^* = z_i^* \geq 0 \quad (i \in \Omega_h), \quad w_j^* = J[x'_j, w^*] \quad (x'_j \in \partial\Omega_h).$$

It follows from the same reasoning as that for (2.19) that $w_i^* \geq 0$ which yields $\bar{u}_i = \underline{u}_i$. Using this result in (2.9) we obtain $\bar{v}_i = \underline{v}_i$. This proves $(\bar{u}_i, \bar{v}_i) = (\underline{u}_i, \underline{v}_i)$ and thus the uniqueness of the solution of (2.9), (2.10) in \mathcal{S} . \square

It is easy to verify that if \tilde{u}_i and \hat{u}_i are ordered upper and lower solutions of (2.1), (2.2) then the pair

$$(\tilde{u}_i, \tilde{v}_i) = (\tilde{u}_i, \mu\tilde{u}_i - \Delta_h\tilde{u}_i), \quad (\hat{u}_i, \hat{v}_i) = (\hat{u}_i, \mu\hat{u}_i - \Delta_h\hat{u}_i) \quad (2.20)$$

is ordered upper and lower solutions of (2.9), (2.10). (The converse is not necessarily true.) This leads to the following conclusion for (2.1), (2.2).

Corollary 2.1. Let \tilde{u}_i, \hat{u}_i be a pair of ordered upper and lower solutions of (2.1), (2.2), and let condition (2.7) hold. Then

- (i) Problem (2.1), (2.2) has a maximal solution \bar{u}_i and a minimal solution \underline{u}_i in (\hat{u}_i, \tilde{u}_i) .
- (ii) The component $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}$ of $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}$ governed by (2.13) converges to \bar{u}_i while the component $\{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ converges to \underline{u}_i , where the initial iterations are the pair given in (2.20). Moreover, they satisfy the relation

$$\hat{u}_i \leq \underline{u}_i^{(m)} \leq \underline{u}_i^{(m+1)} \leq \underline{u}_i \leq \bar{u}_i \leq \bar{u}_i^{(m+1)} \leq \bar{u}_i^{(m)} \leq \tilde{u}_i, \quad m = 1, 2, \dots \quad (2.21)$$

- (iii) If, in addition, condition (2.18) holds then $\bar{u}_i = \underline{u}_i (= u_i^*)$ and u_i^* is the unique solution of (2.1), (2.2) in (\hat{u}_i, \tilde{u}_i) .

We next consider the finite difference system (2.1), (2.2_b). For this system, upper and lower solutions \tilde{u}_i and \hat{u}_i are defined by Definition 2.1 except with the boundary requirement for $\Delta_h\tilde{u}_j$ and $\Delta_h\hat{u}_j$ replaced, respectively, by

$$\Delta_h\tilde{u}_j \leq -g_j^{(0)}, \quad \Delta_h\hat{u}_j \geq -g_j^{(0)}. \quad (2.22)$$

When the boundary condition (2.2) is replaced by (2.2_b), the corresponding boundary condition (2.10) is equivalent to

$$u_j = J[x'_j, u] + g_j^{(1)}, \quad v_j = \mu J[x'_j, u] + g_j^{(2)}. \quad (2.23)$$

We define ordered upper and lower solutions $(\tilde{u}_i, \tilde{v}_i)$ and (\hat{u}_i, \hat{v}_i) of (2.9), (2.23) as that in Definition 2.2 where the boundary requirement for \tilde{v}_j and \hat{v}_j are replaced, respectively, by

$$\tilde{v}_j \geq \mu J[x'_j, \tilde{u}] + g_j^{(2)}, \quad \hat{v}_j \leq \mu J[x'_j, \hat{u}] + g_j^{(2)}. \quad (2.24)$$

For the system (2.9), (2.23), the sequence of iterations for the maximal and minimal sequences $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}, \{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ are governed by (2.13) except with the boundary condition replaced by

$$u_j^{(m)} = J[x'_j, u^{(m-1)}] + g_j^{(1)}, \quad v_j^{(m)} = \mu J[x'_j, u^{(m-1)}] + g_j^{(2)}. \quad (2.25)$$

In relation to this system we have the following conclusion.

Theorem 2.3. Let condition (2.7) hold, and let $(\tilde{u}_i, \tilde{v}_i), (\hat{u}_i, \hat{v}_i)$ be ordered upper and lower solutions of (2.9), (2.23). Then the following statements hold true:

- (i) All the conclusions in (i)–(iii) of Theorem 2.1 hold true for problem (2.9), (2.23) where the maximal and minimal sequences $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}, \{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ are governed by (2.13) except with the boundary condition replaced by (2.25).
- (ii) If, in addition, condition (2.18) holds then the conclusion in Theorem 2.2 holds true for problem (2.9), (2.23).

Proof. The proof is a slight modification of the proofs for Theorems 2.1 and 2.2 and we give a sketch for the present coupled boundary condition as follows:

Proof of (i). It is obvious from (2.25) and (2.24) that the function $(\bar{w}_i^{(0)}, \bar{z}_i^{(0)}) = (\bar{u}_i^{(0)} - \bar{u}_i^{(1)}, \bar{v}_i^{(0)} - \bar{v}_i^{(1)})$ satisfies the inequalities in (2.15) except that the relation for $\bar{z}_j^{(0)}$ is replaced by

$$\bar{z}_j^{(0)} = \tilde{v}_j - (\mu J[x'_j, \tilde{u}] + g_j^{(2)}) \geq 0.$$

This leads to $\bar{u}_i^{(0)} \geq \bar{u}_i^{(1)}$ and $\bar{v}_i^{(0)} \geq \bar{v}_i^{(1)}$. A similar argument gives $\underline{u}_i^{(1)} \geq \underline{u}_i^{(0)}$ and $\underline{v}_i^{(1)} \geq \underline{v}_i^{(0)}$. Moreover, the function $(w_i^{(1)}, z_i^{(1)}) = (\bar{u}_i^{(1)} - \underline{u}_i^{(1)}, \bar{v}_i^{(1)} - \underline{v}_i^{(1)})$ satisfies the inequalities in (2.16) except with the relation for $z_j^{(1)}$ replaced by

$$z_j^{(1)} = (\mu J[x'_j, \bar{u}^{(0)}] + g_j^{(2)}) - (\mu J[x'_j, \underline{u}^{(0)}] + g_j^{(2)}) \geq 0.$$

This leads to $\bar{u}_i^{(1)} \geq \underline{u}_i^{(1)}$ and $\bar{v}_i^{(1)} \geq \underline{v}_i^{(1)}$. The remaining proof is the same as that in the proof of Theorem 2.1.

Proof of (ii). Let $(\bar{u}_i, \bar{v}_i), (\underline{u}_i, \underline{v}_i)$ be the maximal and minimal solutions of (2.9), (2.23), and let $w_i^* = \underline{u}_i - \bar{u}_i \leq 0$, $z_i^* = \bar{v}_i - \underline{v}_i - \mu(\underline{u}_i - \bar{u}_i)$. Then the same proof as that for (2.19) shows that z_i^* satisfies

$$-\Delta_h z_i^* + b_0 z_i^* \geq 0 \quad (i \in \Omega_h), \quad z_j^* = 0 \quad (x'_j \in \partial\Omega_h). \quad (2.26)$$

This implies that $z_i^* \geq 0$ on $\bar{\Omega}_h$. The remaining proof follows from the same reasoning as that in the proof of Theorem 2.2. \square

As a consequence of Theorem 2.3 we have the following conclusion for (2.1), (2.2_b).

Corollary 2.2. Let \tilde{u}_i, \hat{u}_i be a pair of ordered upper and lower solutions of (2.1), (2.2_b), and let condition (2.7) hold. Then all the conclusions in (i)–(iii) of Corollary 2.1 hold true for problem (2.1), (2.2_b) where the maximal and minimal sequences $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}, \{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ are governed by (2.13) except with the boundary condition replaced by (2.25).

Remark 2.1. (a) It is easy to show that for every $m \geq 1$ the iterates $(\bar{u}_i^{(m)}, \bar{v}_i^{(m)})$ and $(\underline{u}_i^{(m)}, \underline{v}_i^{(m)})$ in Theorem 2.1 are ordered upper and lower solutions of problem (2.9), (2.10). The same is true for the iterates in Theorem 2.3 for problem (2.9), (2.23). Hence the monotone iterations in these theorems generate a sequence of ordered upper and lower solutions for the corresponding boundary value problem. (b) A slightly improved iterations process for problem (2.9), (2.10) is given by (2.13) where the function $v_i^{(m-1)}$ in the equation for $u_i^{(m)}$ is replaced by $v_i^{(m)}$ (or the function $u_i^{(m-1)}$ in the equation for $v_i^{(m)}$ is replaced by $u_i^{(m)}$). It can be shown that the maximal and minimal sequences obtained in this way also possess the monotone property in (2.14). The same is true for problem (2.9), (2.23).

3. Gauss–Seidel and Jacobi monotone iterations

In the computation of the maximal and minimal sequences from (2.13) it is necessary to solve an algebraic system. When the domain Ω is of two or higher dimension the size of the algebraic system may be very large and the system may require another process of iterations for the solution. The same is true for the system where the boundary condition is replaced by (2.25). To obtain a more direct computational process without additional iterations we use the Gauss–Seidel and Jacobi methods to develop similar monotone iterative schemes. We focus our attention on the Gauss–Seidel and Jacobi monotone iterations for the finite difference system (2.9), (2.10). The same methods can be developed for the finite difference system (2.9), (2.23). For this purpose, we express the finite difference system (2.9), (2.10) in vector form.

Let $N = M_1 M_2 \cdots M_n$ be the total number of interior mesh points in Ω_h , and let $U = (u_1, u_2, \dots, u_N)^T$ and $V = (v_1, v_2, \dots, v_N)^T$ be the vector representations of the solutions u_i and v_i arranged in a suitable fashion (notice that the index i here is single-valued and is different from that in the previous sections), where $(\cdot)^T$ denotes a column vector in \mathbf{R}^N . Define

$$F^*(U) = (f_1^*(u_1), f_2^*(u_2), \dots, f_N^*(u_N))^T.$$

Then we may express (2.9), (2.10) in the vector form

$$(A + \mu I)U = V + J[U] + G^{(1)}, \quad (A + \mu^+ I)V = F^*(U) + J[V] + G^{(2)}, \quad (3.1)$$

where A is an N by N matrix associated with the difference operator $-\Delta_h$, I is the N by N identity matrix, and the vectors $J[U]$, $J[V]$ and $G^{(l)}$ ($l = 1, 2$) are associated with the boundary functions $J[x'_j, u]$, $J[x'_j, v]$ and $g_j^{(l)}$ ($l = 1, 2$), respectively. It is to be noted that the components of the vectors $J[U]$, $J[V]$ and $G^{(l)}$ ($l = 1, 2$) are mostly zero except at points which are related to the boundary points on $\partial\Omega_h$. For example, in the one-dimensional domain $\Omega = (0, L)$, we have $N = M_1$ and

$$J[U] = \frac{1}{h^2} (J[0, u], 0, \dots, 0, J[L, u])^T, \quad G^{(1)} = \frac{1}{h^2} (g_0^{(1)}, 0, \dots, 0, g_{M_1+1}^{(1)})^T,$$

where $h = L/(M_1 + 1)$ and

$$J[0, u] = \sum_{i=1}^{M_1} \omega_i \beta(0, \xi_i) u(\xi_i), \quad J[L, u] = \sum_{i=1}^{M_1} \omega_i \beta(L, \xi_i) u(\xi_i), \quad g_0^{(1)} = g^{(1)}(0), \quad g_{M_1+1}^{(1)} = g^{(1)}(L).$$

The vectors for $J[V]$ and $G^{(2)}$ are similar. Since our main concern is the mathematical structure of the finite difference system, detailed formulation of (3.1) is left to the interested readers (see [16,29–31] for some detailed discussions). In relation to the system (3.1) we have the following definition of upper and lower solutions.

Definition 3.1. A pair of vectors $(\tilde{U}, \tilde{V}), (\hat{U}, \hat{V})$ in $\mathbf{R}^N \times \mathbf{R}^N$ is called ordered upper and lower solutions of (3.1) if $(\tilde{U}, \tilde{V}) \geq (\hat{U}, \hat{V})$ and if

$$(A + \mu I)\tilde{U} \geq \tilde{V} + J[\tilde{U}] + G^{(1)}, \quad (A + \mu^+ I)\tilde{V} \geq F^*(\tilde{U}) + J[\tilde{V}] + G^{(2)}, \quad (3.2)$$

and (\hat{U}, \hat{V}) satisfies the above relation with inequalities reversed.

For a given pair of ordered upper and lower solutions $(\tilde{U}, \tilde{V}), (\hat{U}, \hat{V})$, we set

$$\mathcal{S}^* = \{(U, V) \in \mathbf{R}^N \times \mathbf{R}^N; (\hat{U}, \hat{V}) \leq (U, V) \leq (\tilde{U}, \tilde{V})\}.$$

To treat the problem (3.1) directly we make the following hypothesis:

(H)(i) The matrix $A \equiv (a_{k,l})$ is irreducible, and $a_{k,k} > 0$, $a_{k,l} \leq 0$ for $k \neq l$ and

$$\sum_{l=1}^N a_{k,l} \geq 0 \quad \text{for all } k \text{ with the strict inequality holds for at least one } k,$$

(ii) $\mu \geq 0$, $\mu^+ \geq 0$ and $F^*(U)$ is a C^1 -function of U and is nondecreasing in U for $(U, V) \in \mathcal{S}^*$.

The condition in (H)-(i) implies that A is an M -matrix and its inverse A^{-1} exists and is a positive matrix (cf. [32–34]). Moreover, the smallest eigenvalue ν_0 of A is positive and corresponding to it there is a positive (normalized) eigenvector Φ . This implies that for any $\mu \geq 0$ the inverse $(A + \mu I)^{-1}$ exists and is a positive matrix. It is easy to verify from the central difference approximation $\Delta_h u_i$ that all the conditions in (H)-(i) are satisfied, and the connectedness assumption of Ω ensures that A is irreducible.

To develop the Gauss–Seidel and Jacobi monotone iterations we write A in the split form $A = \mathcal{D} - \mathcal{L} - \mathcal{U}$, where \mathcal{D} , $-\mathcal{U}$ and $-\mathcal{L}$ are diagonal, upper tridiagonal and lower tridiagonal sub-matrices of A , respectively. It is obvious from hypothesis (H) that \mathcal{D} is a diagonal matrix with positive diagonal elements, and \mathcal{U} and \mathcal{L} are triangular matrices with nonnegative elements. Define matrices

$$\begin{aligned} \mathcal{P}^{(1)} &= A + \mu I, & \mathcal{P}^{(2)} &= A + \mu^+ I, & \mathcal{J}^{(1)} &= \mathcal{D} + \mu I, & \mathcal{J}^{(2)} &= \mathcal{D} + \mu^+ I, \\ \mathcal{G}^{(1)} &= \mathcal{D} + \mu I - \mathcal{L}, & \mathcal{G}^{(2)} &= \mathcal{D} + \mu^+ I - \mathcal{L}. \end{aligned} \quad (3.3)$$

Then for each $l = 1, 2$, $\mathcal{G}^{(l)}$ is a lower triangular matrix and $\mathcal{J}^{(l)}$ is a diagonal matrix. In view of hypothesis (H), the inverses $(\mathcal{G}^{(l)})^{-1}$ and $(\mathcal{J}^{(l)})^{-1}$ exist and are nonnegative matrices.

Using either (\tilde{U}, \tilde{V}) or (\hat{U}, \hat{V}) as the initial iteration $(U^{(0)}, V^{(0)})$ we construct a sequence $\{U^{(m)}, V^{(m)}\}$ from any one of the following iteration processes:

(a) Picard iteration:

$$\begin{aligned} \mathcal{P}^{(1)} U^{(m)} &= V^{(m-1)} + J[U^{(m-1)}] + G^{(1)}, \\ \mathcal{P}^{(2)} V^{(m)} &= F^*(U^{(m-1)}) + J[V^{(m-1)}] + G^{(2)}, \end{aligned} \quad (3.4)$$

(b) Gauss–Seidel iteration:

$$\begin{aligned} \mathcal{G}^{(1)} U^{(m)} &= \mathcal{U} U^{(m-1)} + V^{(m-1)} + J[U^{(m-1)}] + G^{(1)}, \\ \mathcal{G}^{(2)} V^{(m)} &= \mathcal{U} V^{(m-1)} + F^*(U^{(m-1)}) + J[V^{(m-1)}] + G^{(2)}, \end{aligned} \quad (3.5)$$

(c) Jacobi iteration:

$$\begin{aligned} \mathcal{J}^{(1)} U^{(m)} &= (\mathcal{U} + \mathcal{L}) U^{(m-1)} + V^{(m-1)} + J[U^{(m-1)}] + G^{(1)}, \\ \mathcal{J}^{(2)} V^{(m)} &= (\mathcal{U} + \mathcal{L}) V^{(m-1)} + F^*(U^{(m-1)}) + J[V^{(m-1)}] + G^{(2)}, \end{aligned} \quad (3.6)$$

where $m = 1, 2, \dots$. Since for each $l = 1, 2$, the inverses $(\mathcal{P}^{(l)})^{-1}$, $(\mathcal{G}^{(l)})^{-1}$ and $(\mathcal{J}^{(l)})^{-1}$ all exist, the sequence $\{U^{(m)}, V^{(m)}\}$ from each of the above iteration processes is well-defined. Denote the sequence by $\{\bar{U}^{(m)}, \bar{V}^{(m)}\}$ if $(U^{(0)}, V^{(0)}) = (\tilde{U}, \tilde{V})$ and by $\{\underline{U}^{(m)}, \underline{V}^{(m)}\}$ if $(U^{(0)}, V^{(0)}) = (\hat{U}, \hat{V})$, and refer to them as maximal and minimal sequences respectively. Then we have the following monotone convergence results.

Theorem 3.1. Let $(\tilde{U}, \tilde{V}), (\hat{U}, \hat{V})$ be a pair of ordered upper and lower solutions of (3.1), and let $\{\bar{U}^{(m)}, \bar{V}^{(m)}\}, \{\underline{U}^{(m)}, \underline{V}^{(m)}\}$ be the maximal and minimal sequences from any one of the iteration processes in (3.4)–(3.6). Assume that hypothesis (H) is satisfied. Then the following statements hold true:

- (i) Problem (3.1) has a maximal solution (\bar{U}, \bar{V}) and a minimal solution $(\underline{U}, \underline{V})$ in \mathcal{S}^* .
- (ii) The maximal sequence $\{\bar{U}^{(m)}, \bar{V}^{(m)}\}$ converges to (\bar{U}, \bar{V}) , the minimal sequence $\{\underline{U}^{(m)}, \underline{V}^{(m)}\}$ converges to $(\underline{U}, \underline{V})$, and they possess the monotone property

$$\begin{aligned} (\hat{U}, \hat{V}) &\leq (\underline{U}^{(m)}, \underline{V}^{(m)}) \leq (\underline{U}^{(m+1)}, \underline{V}^{(m+1)}) \leq (\underline{U}, \underline{V}) \leq (\bar{U}, \bar{V}) \\ &\leq (\bar{U}^{(m+1)}, \bar{V}^{(m+1)}) \leq (\bar{U}^{(m)}, \bar{V}^{(m)}) \leq (\tilde{U}, \tilde{V}), \quad m = 1, 2, \dots \end{aligned} \quad (3.7)$$

- (iii) If $(\bar{U}, \bar{V}) = (\underline{U}, \underline{V}) (= (U^*, V^*))$, then (U^*, V^*) is the unique solution of (3.1) in \mathcal{S}^* .

Proof. (a) Picard iteration. Since the Picard iteration (3.4) is simply a vector representation of (2.13), the conclusion of the theorem is a direct consequence of Theorem 2.1.

(b) Gauss–Seidel iteration. By (3.2), (3.3), (3.5) and $(\bar{U}^{(0)}, \bar{V}^{(0)}) = (\tilde{U}, \tilde{V})$,

$$\begin{aligned} \mathcal{G}^{(1)} (\bar{U}^{(0)} - \bar{U}^{(1)}) &= (\mathcal{D} + \mu I - \mathcal{L}) \tilde{U} - (\mathcal{U} \bar{U}^{(0)} + \bar{V}^{(0)} + J[\bar{U}^{(0)}] + G^{(1)}) \\ &= (A + \mu I) \tilde{U} - (\tilde{V} + J[\tilde{U}] + G^{(1)}) \geq 0, \\ \mathcal{G}^{(2)} (\bar{V}^{(0)} - \bar{V}^{(1)}) &= (\mathcal{D} + \mu^+ I - \mathcal{L}) \tilde{V} - (\mathcal{U} \bar{V}^{(0)} + F^*(\bar{U}^{(0)}) + J[\bar{V}^{(0)}] + G^{(2)}) \\ &= (A + \mu^+ I) \tilde{V} - (F^*(\tilde{U}) + J[\tilde{V}] + G^{(2)}) \geq 0. \end{aligned}$$

The nonnegative property of $(g^{(l)})^{-1}$ for $l = 1, 2$ ensures that $\bar{U}^{(0)} \geq \bar{U}^{(1)}$ and $\bar{V}^{(0)} \geq \bar{V}^{(1)}$. A similar argument using the property of a lower solution gives $\underline{U}^{(1)} \geq \underline{U}^{(0)}$ and $\underline{V}^{(1)} \geq \underline{V}^{(0)}$. Moreover, by (3.5), the nonnegative property of \mathcal{U} , and the nondecreasing property of $F^*(U)$ we have

$$\begin{aligned} g^{(1)}(\bar{U}^{(1)} - \underline{U}^{(1)}) &= (\mathcal{U}\bar{U}^{(0)} + \bar{V}^{(0)} + J[\bar{U}^{(0)}] + G^{(1)}) - (\mathcal{U}\underline{U}^{(0)} + \underline{V}^{(0)} + J[\underline{U}^{(0)}] + G^{(1)}) \\ &= \mathcal{U}(\bar{U}^{(0)} - \underline{U}^{(0)}) + (\bar{V}^{(0)} - \underline{V}^{(0)}) + (J[\bar{U}^{(0)}] - J[\underline{U}^{(0)}]) \geq 0, \\ g^{(2)}(\bar{V}^{(1)} - \underline{V}^{(1)}) &= \mathcal{U}(\bar{V}^{(0)} - \underline{V}^{(0)}) + (F^*(\bar{U}^{(0)}) - F^*(\underline{U}^{(0)})) + (J[\bar{U}^{(0)}] - J[\underline{U}^{(0)}]) \geq 0. \end{aligned}$$

It follows again from the nonnegative property of $(g^{(l)})^{-1}$ that $\bar{U}^{(1)} \geq \underline{U}^{(1)}$ and $\bar{V}^{(1)} \geq \underline{V}^{(1)}$. The above conclusions show that $(\underline{U}^{(0)}, \underline{V}^{(0)}) \leq (\underline{U}^{(1)}, \underline{V}^{(1)}) \leq (\bar{U}^{(1)}, \bar{V}^{(1)}) \leq (\bar{U}^{(0)}, \bar{V}^{(0)})$. An induction argument yields

$$(\underline{U}^{(m)}, \underline{V}^{(m)}) \leq (\underline{U}^{(m+1)}, \underline{V}^{(m+1)}) \leq (\bar{U}^{(m+1)}, \bar{V}^{(m+1)}) \leq (\bar{U}^{(m)}, \bar{V}^{(m)})$$

for every $m = 1, 2, \dots$. The above monotone property ensures that the limits

$$\lim_{m \rightarrow \infty} (\bar{U}^{(m)}, \bar{V}^{(m)}) = (\bar{U}, \bar{V}), \quad \lim_{m \rightarrow \infty} (\underline{U}^{(m)}, \underline{V}^{(m)}) = (\underline{U}, \underline{V}) \quad (3.8)$$

exist and satisfy the relation (3.7). Letting $m \rightarrow \infty$ in (3.5) shows that both (\bar{U}, \bar{V}) and $(\underline{U}, \underline{V})$ are solutions of the system

$$g^{(1)}U = \mathcal{U}U + V + J[U] + G^{(1)}, \quad g^{(2)}V = \mathcal{U}V + F^*(U) + J[V] + G^{(2)}$$

which is equivalent to the system (3.1). This proves the existence of solutions in (i) and the monotone property (3.7) in (ii). To show the maximal and minimal properties of (\bar{U}, \bar{V}) and $(\underline{U}, \underline{V})$ we observe that every solution (U^*, V^*) of (3.1) is an upper solution as well as a lower solution. Hence by considering (\tilde{U}, \tilde{V}) , (U^*, V^*) and (U^*, V^*) , (\hat{U}, \hat{V}) as the pair of ordered upper and lower solutions in the above discussion, we obtain the relation $(\bar{U}, \bar{V}) \geq (U^*, V^*) \geq (\underline{U}, \underline{V})$. Finally, the uniqueness result in (iii) is a direct consequence of this relation.

(c) Jacobi iteration. The proof for the Jacobi iteration is similar to the proof for Gauss–Seidel iteration and we omit the details. \square

We next give a comparison relation among the three iterative schemes in (3.4)–(3.6). Denote their respective maximal and minimal sequences by $(\{\bar{U}_p^{(m)}, \bar{V}_p^{(m)}\}, \{\underline{U}_p^{(m)}, \underline{V}_p^{(m)}\})$, $(\{\bar{U}_G^{(m)}, \bar{V}_G^{(m)}\}, \{\underline{U}_G^{(m)}, \underline{V}_G^{(m)}\})$ and $(\{\bar{U}_J^{(m)}, \bar{V}_J^{(m)}\}, \{\underline{U}_J^{(m)}, \underline{V}_J^{(m)}\})$. Then we have the following result.

Theorem 3.2. Let the conditions in Theorem 3.1 be satisfied. Then

$$\begin{aligned} (\bar{U}_p^{(m)}, \bar{V}_p^{(m)}) &\leq (\bar{U}_G^{(m)}, \bar{V}_G^{(m)}) \leq (\bar{U}_J^{(m)}, \bar{V}_J^{(m)}), \\ (\underline{U}_p^{(m)}, \underline{V}_p^{(m)}) &\geq (\underline{U}_G^{(m)}, \underline{V}_G^{(m)}) \geq (\underline{U}_J^{(m)}, \underline{V}_J^{(m)}), \quad m = 1, 2, \dots \end{aligned} \quad (3.9)$$

Proof. By (3.3), (3.5) and (3.7), we have

$$\begin{aligned} \mathcal{P}^{(1)}\bar{U}_G^{(m)} &= (g^{(1)} - \mathcal{U})\bar{U}_G^{(m)} = \mathcal{U}(\bar{U}_G^{(m-1)} - \bar{U}_G^{(m)}) + \bar{V}_G^{(m-1)} + J[\bar{U}_G^{(m-1)}] + G^{(1)} \\ &\geq \bar{V}_G^{(m-1)} + J[\bar{U}_G^{(m-1)}] + G^{(1)}, \\ \mathcal{P}^{(2)}\bar{V}_G^{(m)} &= (g^{(2)} - \mathcal{U})\bar{V}_G^{(m)} = \mathcal{U}(\bar{V}_G^{(m-1)} - \bar{V}_G^{(m)}) + F^*(\bar{U}_G^{(m-1)}) + J[\bar{V}_G^{(m-1)}] + G^{(2)} \\ &\geq F^*(\bar{U}_G^{(m-1)}) + J[\bar{V}_G^{(m-1)}] + G^{(2)}. \end{aligned} \quad (3.10)$$

Since $(\bar{U}_p^{(m)}, \bar{V}_p^{(m)})$ satisfies (3.4), a subtraction of (3.4) from (3.10) gives

$$\begin{aligned} \mathcal{P}^{(1)}(\bar{U}_G^{(m)} - \bar{U}_p^{(m)}) &\geq \bar{V}_G^{(m-1)} - \bar{V}_p^{(m-1)} + J[\bar{U}_G^{(m-1)}] - J[\bar{U}_p^{(m-1)}], \\ \mathcal{P}^{(2)}(\bar{V}_G^{(m)} - \bar{V}_p^{(m)}) &\geq F^*(\bar{U}_G^{(m-1)}) - F^*(\bar{U}_p^{(m-1)}) + J[\bar{V}_G^{(m-1)}] - J[\bar{V}_p^{(m-1)}] \end{aligned} \quad (3.11)$$

for every $m = 1, 2, \dots$. Consider the case $m = 1$. In view of $(\bar{U}_G^{(0)}, \bar{V}_G^{(0)}) = (\bar{U}_p^{(0)}, \bar{V}_p^{(0)}) = (\tilde{U}, \tilde{V})$, we see that $\mathcal{P}^{(1)}(\bar{U}_G^{(1)} - \bar{U}_p^{(1)}) \geq 0$ and $\mathcal{P}^{(2)}(\bar{V}_G^{(1)} - \bar{V}_p^{(1)}) \geq 0$. This leads to $\bar{U}_G^{(1)} \geq \bar{U}_p^{(1)}$ and $\bar{V}_G^{(1)} \geq \bar{V}_p^{(1)}$. Assume, by induction, that $(\bar{U}_G^{(m-1)}, \bar{V}_G^{(m-1)}) \geq (\bar{U}_p^{(m-1)}, \bar{V}_p^{(m-1)})$ for some $m \geq 1$. Then by (3.11) and the nondecreasing property of $F^*(U)$ and $J[W]$ (for $W = U$ or V) we obtain $\mathcal{P}^{(1)}(\bar{U}_G^{(m)} - \bar{U}_p^{(m)}) \geq 0$ and $\mathcal{P}^{(2)}(\bar{V}_G^{(m)} - \bar{V}_p^{(m)}) \geq 0$ which yield $(\bar{U}_G^{(m)}, \bar{V}_G^{(m)}) \geq (\bar{U}_p^{(m)}, \bar{V}_p^{(m)})$. It follows from the induction principle that the first inequality in (3.9) for the maximal sequence holds.

To show the second inequality between $(\bar{U}_G^{(m)}, \bar{V}_G^{(m)})$ and $(\bar{U}_J^{(m)}, \bar{V}_J^{(m)})$ we observe from (3.3), (3.6) and (3.7) that

$$\begin{aligned}\mathcal{G}^{(1)}\bar{U}_J^{(m)} &= \mathcal{U}\bar{U}_J^{(m-1)} + \mathcal{L}(\bar{U}_J^{(m-1)} - \bar{U}_J^{(m)}) + \bar{V}_J^{(m-1)} + J[\bar{U}_J^{(m-1)}] + G^{(1)} \\ &\geq \mathcal{U}\bar{U}_J^{(m-1)} + \bar{V}_J^{(m-1)} + J[\bar{U}_J^{(m-1)}] + G^{(1)}, \\ \mathcal{G}^{(2)}\bar{V}_J^{(m)} &= \mathcal{U}\bar{V}_J^{(m-1)} + \mathcal{L}(\bar{V}_J^{(m-1)} - \bar{V}_J^{(m)}) + F^*(\bar{U}_J^{(m-1)}) + J[\bar{V}_J^{(m-1)}] + G^{(2)} \\ &\geq \mathcal{U}\bar{V}_J^{(m-1)} + F^*(\bar{U}_J^{(m-1)}) + J[\bar{V}_J^{(m-1)}] + G^{(2)}.\end{aligned}\quad (3.12)$$

Since $(\bar{U}_G^{(m)}, \bar{V}_G^{(m)})$ satisfies (3.5), a subtraction of (3.5) from (3.12) gives

$$\begin{aligned}\mathcal{G}^{(1)}(\bar{U}_J^{(m)} - \bar{U}_G^{(m)}) &\geq \mathcal{U}(\bar{U}_J^{(m-1)} - \bar{U}_G^{(m-1)}) + \bar{V}_J^{(m-1)} - \bar{V}_G^{(m-1)} + J[\bar{U}_J^{(m-1)}] - J[\bar{U}_G^{(m-1)}], \\ \mathcal{G}^{(2)}(\bar{V}_J^{(m)} - \bar{V}_G^{(m)}) &\geq \mathcal{U}(\bar{V}_J^{(m-1)} - \bar{V}_G^{(m-1)}) + F^*(\bar{U}_J^{(m-1)}) - F^*(\bar{U}_G^{(m-1)}) + J[\bar{V}_J^{(m-1)}] - J[\bar{V}_G^{(m-1)}].\end{aligned}$$

By the relation $(\bar{U}_G^{(0)}, \bar{V}_G^{(0)}) = (\bar{U}_J^{(0)}, \bar{V}_J^{(0)}) = (\tilde{U}, \tilde{V})$, the above relation for $m = 1$ yields $\mathcal{G}^{(1)}(\bar{U}_J^{(1)} - \bar{U}_G^{(1)}) \geq 0$ and $\mathcal{G}^{(2)}(\bar{V}_J^{(1)} - \bar{V}_G^{(1)}) \geq 0$. This leads to $(\bar{U}_J^{(1)}, \bar{V}_J^{(1)}) \geq (\bar{U}_G^{(1)}, \bar{V}_G^{(1)})$. An induction argument shows that $(\bar{U}_J^{(m)}, \bar{V}_J^{(m)}) \geq (\bar{U}_G^{(m)}, \bar{V}_G^{(m)})$ for every m . This proves the relation in (3.9) for the maximal sequences. The proof for the minimal sequences is similar and is omitted. \square

Remark 3.1. (a) Let $f^*(x, u) = \bar{c}u + f(x, u)$. As for the finite difference system (2.9), (2.10) the continuous problem (1.1) is equivalent to the coupled system

$$\begin{aligned}-\Delta u + \mu u &= v, \quad -\Delta v + \mu^+ v = f^*(x, u) \quad \text{in } \Omega, \\ u(x') &= \int_{\Omega} \beta(x', x)u(x)dx + g^{(1)}(x'), \quad v(x') = \int_{\Omega} \beta(x', x)v(x)dx + g^{(2)}(x') \quad \text{on } \partial\Omega,\end{aligned}\quad (3.13)$$

where μ and μ^+ are defined by (2.8). It can be shown that the above system can also be discretized into the form (3.1) by the finite element method. In fact, a suitable choice of the basis functions ensures that the matrix A associated with the Laplacian $-\Delta$ possesses all the properties in (H)-(i) and the function $F^*(U)$ satisfies the condition in (H)-(ii). This implies that all the conclusions in Theorems 3.1 and 3.2 are directly applicable to a suitable discrete system by the finite element method. (b) The comparison results in Theorem 3.2 imply that with the same initial iteration, which is either an upper solution or a lower solution, the sequence of Picard iteration converges faster than the sequence of Gauss–Seidel iteration which in turn converges faster than the sequence of Jacobi iteration. However, in practical computations, the Jacobi iteration is very straightforward, and the Picard iteration may require another process of iteration if the size N is large. Our numerical results in the final section indicate that the number of iterations by the Jacobi iteration is about twice the number of the Gauss–Seidel iterations.

4. Convergence of finite difference solutions

In this section, we show the convergence of the maximal and minimal solutions of the finite difference problem (2.9), (2.10) to the corresponding maximal and minimal solutions of the continuous problem (3.13) as $|h| \rightarrow 0$ where $|h| = h_1 + h_2 + \dots + h_n$. The same treatment can be given for the finite difference problem (2.9), (2.23).

Recall that a pair of smooth functions $(\tilde{u}(x), \tilde{v}(x))$, $(\hat{u}(x), \hat{v}(x))$ is called ordered upper and lower solutions of the problem (3.13) if $(\tilde{u}, \tilde{v}) \geq (\hat{u}, \hat{v})$ and if they satisfy (3.13) with the equality sign “=” replaced, respectively, by the inequality signs “ \geq ” and “ \leq ” (cf. [4,28]). Using either (\tilde{u}, \tilde{v}) or (\hat{u}, \hat{v}) as the initial iteration $(u^{(0)}, v^{(0)})$ we construct a sequence $\{u^{(m)}, v^{(m)}\}$ from the linear iteration process:

$$\begin{aligned}-\Delta u^{(m)} + \mu u^{(m)} &= v^{(m-1)}, \quad -\Delta v^{(m)} + \mu^+ v^{(m)} = f^*(x, u^{(m-1)}) \quad (x \in \Omega), \\ u^{(m)}(x') &= \int_{\Omega} \beta(x', x)u^{(m-1)}(x)dx + g^{(1)}(x'), \\ v^{(m)}(x') &= \int_{\Omega} \beta(x', x)v^{(m-1)}(x)dx + g^{(2)}(x') \quad (x' \in \partial\Omega), \quad m = 1, 2, \dots\end{aligned}\quad (4.1)$$

Denote the sequence by $\{\bar{u}^{(m)}, \bar{v}^{(m)}\}$ if $(u^{(0)}, v^{(0)}) = (\tilde{u}, \tilde{v})$ and by $\{\underline{u}^{(m)}, \underline{v}^{(m)}\}$ if $(u^{(0)}, v^{(0)}) = (\hat{u}, \hat{v})$. It has been shown in [4] that if the function $f^*(x, u)$ is nondecreasing in u for $\hat{u} \leq u \leq \tilde{u}$ then $\{\bar{u}^{(m)}, \bar{v}^{(m)}\}$ converges to a maximal solution (\bar{u}, \bar{v}) , $\{\underline{u}^{(m)}, \underline{v}^{(m)}\}$ converges to a minimal solution $(\underline{u}, \underline{v})$, and they satisfy the relation

$$\begin{aligned}(\hat{u}, \hat{v}) &\leq (\underline{u}^{(m)}, \underline{v}^{(m)}) \leq (\underline{u}^{(m+1)}, \underline{v}^{(m+1)}) \leq (\underline{u}, \underline{v}) \leq (\bar{u}, \bar{v}) \\ &\leq (\bar{u}^{(m+1)}, \bar{v}^{(m+1)}) \leq (\bar{u}^{(m)}, \bar{v}^{(m)}) \leq (\tilde{u}, \tilde{v}), \quad m = 1, 2, \dots\end{aligned}\quad (4.2)$$

To show the convergence of the finite difference solutions (\bar{u}_i, \bar{v}_i) and $(\underline{u}_i, \underline{v}_i)$ as $|h| \rightarrow 0$, we consider a fixed partition $\bar{\Omega}_h^*$ and assume that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that when $|h| < \delta$,

$$|\tilde{u}(x_i) - \tilde{u}_i| + |\tilde{v}(x_i) - \tilde{v}_i| < \varepsilon, \quad |\hat{u}(x_i) - \hat{u}_i| + |\hat{v}(x_i) - \hat{v}_i| < \varepsilon \quad (x_i \in \bar{\Omega}_h^*), \quad (4.3)$$

where $(\tilde{u}(x), \tilde{v}(x))$ and $(\hat{u}(x), \hat{v}(x))$ are ordered upper and lower solutions of (3.13). We also assume that every refined partition of $\bar{\Omega}_h^*$ contains $\bar{\Omega}_h^*$ and $h_v = h$ for all $v = 1, 2, \dots, n$, when the mesh size is sufficiently small. Under the above conditions we have the following convergence result.

Theorem 4.1. Let $((\tilde{u}(x), \tilde{v}(x)), (\hat{u}(x), \hat{v}(x))), ((\tilde{u}_i, \tilde{v}_i), (\hat{u}_i, \hat{v}_i))$ be ordered upper and lower solutions of (3.13) and (2.9), (2.10), respectively, and assume that conditions (2.7) and (4.3) are satisfied. Then as $|h| \rightarrow 0$, the maximal solution (\bar{u}_i, \bar{v}_i) of (2.9), (2.10) converges to the maximal solution $(\bar{u}(x_i), \bar{v}(x_i))$ of (3.13) at every point $x_i \in \bar{\Omega}_h^*$, while the minimal solution $(\underline{u}_i, \underline{v}_i)$ of (2.9), (2.10) converges to the minimal solution $(\underline{u}(x_i), \underline{v}(x_i))$ of (3.13).

Proof. We prove the convergence of the maximal solution (\bar{u}_i, \bar{v}_i) to the corresponding maximal solution $(\bar{u}(x_i), \bar{v}(x_i))$ at every point $x_i \in \bar{\Omega}_h^*$ as $|h| \rightarrow 0$. This will be proven if given any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x_i \in \bar{\Omega}_h^*$,

$$|\bar{u}_i - \bar{u}(x_i)| + |\bar{v}_i - \bar{v}(x_i)| < \varepsilon \quad \text{when } |h| < \delta. \quad (4.4)$$

Let $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}$ and $\{\bar{u}^{(m)}(x_i), \bar{v}^{(m)}(x_i)\}$ be the respective maximal sequences of (2.13) and (4.1). Then the convergence of $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}$ to (\bar{u}_i, \bar{v}_i) and the uniform convergence of $\{\bar{u}^{(m)}(x_i), \bar{v}^{(m)}(x_i)\}$ to $\{\bar{u}(x_i), \bar{v}(x_i)\}$ as $m \rightarrow \infty$ implies that there exists $m^* = m^*(\varepsilon)$ such that for all $x_i \in \bar{\Omega}_h^*$,

$$|\bar{u}_i - \bar{u}_i^{(m)}| + |\bar{u}(x_i) - \bar{u}^{(m)}(x_i)| < \varepsilon/3, \quad |\bar{v}_i - \bar{v}_i^{(m)}| + |\bar{v}(x_i) - \bar{v}^{(m)}(x_i)| < \varepsilon/3 \quad \text{when } m \geq m^*.$$

Since

$$\begin{aligned} |\bar{u}_i - \bar{u}(x_i)| &\leq |\bar{u}_i - \bar{u}_i^{(m)}| + |\bar{u}_i^{(m)} - \bar{u}^{(m)}(x_i)| + |\bar{u}^{(m)}(x_i) - \bar{u}(x_i)|, \\ |\bar{v}_i - \bar{v}(x_i)| &\leq |\bar{v}_i - \bar{v}_i^{(m)}| + |\bar{v}_i^{(m)} - \bar{v}^{(m)}(x_i)| + |\bar{v}^{(m)}(x_i) - \bar{v}(x_i)| \end{aligned}$$

for every m , it suffices to show the existence of $m_0 \geq m^*$ and $\delta = \delta(\varepsilon) > 0$ such that for every $x_i \in \bar{\Omega}_h^*$,

$$|\bar{u}_i^{(m_0)} - \bar{u}^{(m_0)}(x_i)| + |\bar{v}_i^{(m_0)} - \bar{v}^{(m_0)}(x_i)| < \varepsilon/3 \quad \text{when } |h| < \delta. \quad (4.5)$$

By the finite difference approximation of (4.1), we have

$$\begin{aligned} -\Delta_h \bar{u}^{(m)}(x_i) + \mu \bar{u}^{(m)}(x_i) &= \bar{v}^{(m-1)}(x_i) + O^{(m)}(|h|^2), \\ -\Delta_h \bar{v}^{(m)}(x_i) + \mu^+ \bar{v}^{(m)}(x_i) &= f^*(x_i, \bar{u}^{(m-1)}(x_i)) + O^{(m)}(|h|^2) \quad (i \in \Omega_h), \\ \bar{u}^{(m)}(x'_j) &= J[x'_j, \bar{u}^{(m-1)}] + g^{(1)}(x'_j) + O^{(m)}(|h|), \\ \bar{v}^{(m)}(x'_j) &= J[x'_j, \bar{v}^{(m-1)}] + g^{(2)}(x'_j) + O^{(m)}(|h|) \quad (x'_j \in \partial\Omega_h), \quad m = 1, 2, \dots, \end{aligned} \quad (4.6)$$

where $O^{(m)}(|h|^\alpha) \rightarrow 0$ ($\alpha = 1, 2$) as $|h| \rightarrow 0$. Let $\bar{w}_i^{(m)} = \bar{u}^{(m)}(x_i) - \bar{u}_i^{(m)}$ and $\bar{z}_i^{(m)} = \bar{v}^{(m)}(x_i) - \bar{v}_i^{(m)}$. Then a subtraction of (2.13) from (4.6) and using the mean-value theorem yield

$$\begin{aligned} -\Delta_h \bar{w}_i^{(m)} + \mu \bar{w}_i^{(m)} &= \bar{z}_i^{(m-1)} + O^{(m)}(|h|^2), \\ -\Delta_h \bar{z}_i^{(m)} + \mu^+ \bar{z}_i^{(m)} &= f_u^*(x_i, \xi_i) \bar{w}_i^{(m-1)} + O^{(m)}(|h|^2) \quad (i \in \Omega_h), \\ \bar{w}^{(m)}(x'_j) &= J[x'_j, \bar{w}^{(m-1)}] + O^{(m)}(|h|), \\ \bar{z}^{(m)}(x'_j) &= J[x'_j, \bar{z}^{(m-1)}] + O^{(m)}(|h|) \quad (x'_j \in \partial\Omega_h), \end{aligned} \quad (4.7)$$

where $\xi_i \equiv \xi_i^{(m-1)}$ is an intermediate value between $\bar{u}^{(m-1)}(x_i)$ and $\bar{u}_i^{(m-1)}$. In vector form the relation in (4.7) can be written as

$$\begin{aligned} (A + \mu I) \bar{W}^{(m)} &= \bar{Z}^{(m-1)} + J[\bar{W}^{(m-1)}] + \mathcal{O}^{(m)}(|h|), \\ (A + \mu^+ I) \bar{Z}^{(m)} &= F_u^*(\xi) \bar{W}^{(m-1)} + J[\bar{Z}^{(m-1)}] + \mathcal{O}^{(m)}(|h|), \end{aligned} \quad (4.8)$$

where $(\bar{W}^{(m)}, \bar{Z}^{(m)})$ and $\mathcal{O}^{(m)}(|h|)$ are the respective vector representation of $(\bar{w}_i^{(m)}, \bar{z}_i^{(m)})$ and $O^{(m)}(|h|)$, the vectors $(J[\bar{W}^{(m-1)}], J[\bar{Z}^{(m-1)}])$ are defined by the same way as $(J[U], J[V])$ in (3.1), and $F_u^*(\xi)$ is a diagonal matrix with diagonal

elements $f_u^*(x_i, \xi_i)$. Since $(A + \mu^* I)^{-1}$ is positive and $(A + \mu^* I)^{-1} \leq A^{-1}$, where $\mu^* = \mu$ or $\mu^* = \mu^+$, relation (4.8) implies that

$$\begin{aligned} |\bar{W}^{(m)}| &\leq A^{-1} \left(|\bar{Z}^{(m-1)}| + |J[\bar{W}^{(m-1)}]| + |\mathcal{O}^{(m)}(|h|)| \right), \\ |\bar{Z}^{(m)}| &\leq A^{-1} \left(|F_u^*(\xi) \bar{W}^{(m-1)}| + |J[\bar{Z}^{(m-1)}]| + |\mathcal{O}^{(m)}(|h|)| \right), \end{aligned} \quad (4.9)$$

where $|Y| = (|y_1|, \dots, |y_N|)^T$ for any vector $Y = (y_1, \dots, y_N)^T \in \mathbf{R}^N$ (cf. [32,33]).

Let $A^{-1}E = S$, where $E = (1, 1, \dots, 1) \in \mathbf{R}^N$. Then $S > 0$ and $\|A^{-1}\|_\infty = \|S\|_\infty$. To estimate $\|S\|_\infty$, we observe that $AS = E$ which is equivalent to

$$-\Delta_h s_i = 1 \quad (i \in \Omega_h), \quad s_j = 0 \quad (j \in \partial\Omega_h), \quad (4.10)$$

where s_i denotes the component of S . Let

$$\tilde{s}(x) = \frac{1}{2n} \sum_{j=1}^n x_{ij}^2 \quad (x = (x_{i1}, x_{i2}, \dots, x_{in}) \in \bar{\Omega}), \quad s^* = \max_{x \in \bar{\Omega}} \tilde{s}(x), \quad w_i = s_i + \tilde{s}(x_i).$$

Since $-\Delta_h \tilde{s}(x_i) = -\Delta \tilde{s}(x_i) = -1$, we have $-\Delta_h w_i = -\Delta_h s_i - \Delta_h \tilde{s}(x_i) = 0$ for all $i \in \Omega_h$. The maximal principle for the difference operator $-\Delta_h$ implies

$$\max_{i \in \Omega_h} w_i \leq \max_{j \in \partial\Omega_h} w_j \leq \max_{j \in \partial\Omega_h} s_j + \max_{j \in \partial\Omega_h} \tilde{s}(x_j) \leq s^*.$$

We therefore obtain $\|A^{-1}\|_\infty = \|S\|_\infty = \max_{i \in \Omega_h} s_i \leq \max_{i \in \Omega_h} w_i \leq s^*$. Let M^* be the maximum of the elements $|f_u^*(x_i, \xi_i)|$, and let $\bar{\beta} = \max \beta(x', x)$ on $\partial\Omega \times \bar{\Omega}$. Then by (4.9) and the definition of $J[x'_j, w]$,

$$\begin{aligned} \|\bar{W}^{(m)}\|_\infty &\leq s^* \left(\|\bar{Z}^{(m-1)}\|_\infty + \bar{\beta} V_\Omega \|\bar{W}^{(m-1)}\|_\infty + \|\mathcal{O}^{(m)}(|h|)\|_\infty \right), \\ \|\bar{Z}^{(m)}\|_\infty &\leq s^* \left(M^* \|\bar{W}^{(m-1)}\|_\infty + \bar{\beta} V_\Omega \|\bar{Z}^{(m-1)}\|_\infty + \|\mathcal{O}^{(m)}(|h|)\|_\infty \right), \end{aligned} \quad (4.11)$$

where $V_\Omega = \int_\Omega dx$. By adding these two inequalities we see that

$$\|\bar{W}^{(m)}\|_\infty + \|\bar{Z}^{(m)}\|_\infty \leq s^* (\bar{\beta} V_\Omega + M^*) \|\bar{W}^{(m-1)}\|_\infty + s^* (1 + \bar{\beta} V_\Omega) \|\bar{Z}^{(m-1)}\|_\infty + \|\mathcal{O}^{(m)}(|h|)\|_\infty.$$

Let $\bar{a} = s^* \max\{\bar{\beta} V_\Omega + M^*, 1 + \bar{\beta} V_\Omega\}$ and $r^{(m)} = \|\bar{W}^{(m)}\|_\infty + \|\bar{Z}^{(m)}\|_\infty$. Then

$$r^{(m)} \leq \bar{a} r^{(m-1)} + \|\mathcal{O}^{(m)}(|h|)\|_\infty.$$

It follows by an induction argument that

$$r^{(m)} \leq \bar{a}^m r^{(0)} + R^{(m)}(|h|),$$

where $R^{(m)}(|h|) = \|\mathcal{O}^{(m)}(|h|)\|_\infty + \bar{a} \|\mathcal{O}^{(m-1)}(|h|)\|_\infty + \dots + \bar{a}^{m-1} \|\mathcal{O}^{(1)}(|h|)\|_\infty$. Let $m_0 \geq m^*$ be fixed. Then by (4.3) and $R^{(m_0)}(|h|) \rightarrow 0$ as $|h| \rightarrow 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$r^{(0)} \leq \varepsilon / (6\bar{a}^{m_0}), \quad R^{(m_0)}(|h|) < \varepsilon / 6 \quad \text{when } |h| < \delta.$$

This leads to

$$\|\bar{W}^{(m_0)}\|_\infty + \|\bar{Z}^{(m_0)}\|_\infty = r^{(m_0)} \leq \varepsilon / 6 + \varepsilon / 6 = \varepsilon / 3 \quad \text{when } |h| < \delta$$

which yields the relation in (4.5). The proof for the convergence of the minimal solution $(\underline{u}_i, \underline{v}_i)$ is similar. \square

5. Construction of upper and lower solutions

It is seen from the previous sections that in order to implement the monotone iterative schemes for the computation of maximal and minimal sequences it is necessary to find a pair of ordered upper and lower solutions which are used as the initial iterations. In this section, we give various conditions on the nonlinear function $f_i(u_i)$ for the construction of these functions.

The pair of ordered upper and lower solutions $(\tilde{u}_i, \tilde{v}_i)$ and (\hat{u}_i, \hat{v}_i) for problem (2.9), (2.10) can be constructed either directly from (2.11) or from (2.3) for \tilde{u}_i, \hat{u}_i and then use the relation $(\tilde{u}_i, \tilde{v}_i) = (\tilde{u}_i, \mu \tilde{u}_i - \Delta_h \tilde{u}_i)$ and $(\hat{u}_i, \hat{v}_i) = (\hat{u}_i, \mu \hat{u}_i - \Delta_h \hat{u}_i)$. The boundary functions in (2.10) are assumed to satisfy the following condition:

$$g_j^{(l)} \geq 0 \quad (l = 0, 1, 2), \quad J[x'_j, 1] < 1 \quad (x'_j \in \partial\Omega). \quad (5.1)$$

For notational convenience, we set

$$\gamma^{(l)} = \max\{g_j^{(l)} / (1 - J[x'_j, 1]); x'_j \in \partial\Omega\} \quad (l = 0, 1, 2). \quad (5.2)$$

In the following lemmas we always assume that conditions (2.7) and (5.1) are satisfied.

In the construction of upper and lower solutions we often make use of the smallest eigenvalue $\nu_0 > 0$ and its corresponding positive (normalized) eigenfunction ϕ_i of the discrete eigenvalue problem

$$\Delta_h \phi_i + \nu_0 \phi_i = 0 \quad (i \in \Omega_h), \quad \phi_j = 0 \quad (j \in \partial\Omega_h). \quad (5.3)$$

In vector form, ν_0 is the smallest eigenvalue of the matrix A in (3.1). Our first lemma is for the construction of a positive upper solution.

Lemma 5.1. Problem (2.9), (2.10) has a positive upper solution $(\tilde{u}_i, \tilde{v}_i)$ if one of the following conditions holds:

- (a) $g_j^{(0)} \leq 0$, $\mu > 0$ and there exists a constant $K^* \geq \gamma^{(1)}$ such that $f_i(K^*) \leq c_0 K^*$.
- (b) There exists a constant $K > 0$ such that $f_i(u_i) \leq c_0 u_i + K$ for $u_i \geq 0$.
- (c) $\mu > 0$ and there exists a small constant $\varepsilon_0 > 0$ such that

$$\limsup_{\rho \rightarrow +\infty} \frac{f_i(\rho(1 + \varepsilon \phi_i))}{\rho} < c_0 \quad \text{for } 0 < \varepsilon < \varepsilon_0 \quad (i \in \Omega_h). \quad (5.4)$$

Proof. (a) It is obvious that $\tilde{u}_i = K^*$ satisfies the inequalities in (2.3) if

$$c_0 K^* \geq f_i(K^*) \quad (i \in \Omega_h), \quad K^* \geq J[x'_j, K^*] + g_j^{(1)}, \quad 0 \leq 0 - g_j^{(0)} \quad (x'_j \in \partial\Omega_h).$$

The above inequalities are clearly satisfied by the conditions in (a). This shows that $(\tilde{u}_i, \tilde{v}_i) = (K^*, \mu K^*)$ is a positive upper solution of problem (2.9), (2.10).

(b) Consider the linear problem

$$\Delta_h(\Delta_h u_i) - b_0 \Delta_h u_i = K \quad (i \in \Omega_h)$$

with the same boundary condition as that in (2.2). If this problem has a solution $u_i^* \geq 0$ then from the hypothesis $f_i(u_i) \leq c_0 u_i + K$ for $u_i \geq 0$, $\tilde{u}_i = u_i^*$ is an upper solution of (2.1), (2.2). To show the existence of a positive solution to this problem we write it in the equivalent form:

$$\begin{aligned} -\Delta_h u_i &= v_i \quad (i \in \Omega_h), & u_j &= J[x'_j, u] + g_j^{(1)} \quad (x'_j \in \partial\Omega_h), \\ -\Delta_h v_i + b_0 v_i &= K \quad (i \in \Omega_h), & v_j &= J[x'_j, v] + g_j^{(0)} \quad (x'_j \in \partial\Omega_h). \end{aligned} \quad (5.5)$$

Since problem (5.5) is a special case of (2.9), (2.10) with $\mu = 0$, $\mu^+ = b_0$ and $f_i^*(u_i) = K$, it has a nonnegative solution (u_i, v_i) if it has a pair of ordered nonnegative upper and lower solutions. It is obvious from condition (5.1) that $(\hat{u}_i, \hat{v}_i) = (0, 0)$ is a lower solution. To find a positive upper solution we observe that the two nonlocal problems in (5.5) are not coupled, and each problem has a positive solution if it has a positive upper solution. This can be seen from the iteration process (2.13) (see also [17]). Consider the problem in (5.5) for v_i . If $b_0 > 0$ then for any positive constant \bar{K} satisfying $\bar{K} \geq \max\{K/b_0, \gamma^{(0)}\}$ the pair $\tilde{v}_i = \bar{K}$ and $\hat{v}_i = 0$ is ordered upper and lower solutions. This shows that the problem has a positive solution V_i and $0 < V_i \leq \bar{K}$. For the case $b_0 = 0$ we seek a positive upper solution in the form $\tilde{v}_i = w_i^{(0)} + w_i^{(1)}$, where $w_i^{(0)}$ and $w_i^{(1)}$ are the respective positive solutions of the linear problems

$$\begin{aligned} -\Delta_h w_i^{(0)} &= K \quad (i \in \Omega_h), & w_j^{(0)} &= 0 \quad (j \in \partial\Omega_h), \\ -\Delta_h w_i^{(1)} &= 0 \quad (i \in \Omega_h), & w_j^{(1)} &= K^{(0)} \quad (j \in \partial\Omega_h), \end{aligned} \quad (5.6)$$

where $K^{(0)} > 0$ is a constant to be chosen. Indeed, by the maximal principle for second order discrete boundary problems, the solution $w_i^{(1)}$ of the second problem in (5.6) is bounded above by $K^{(0)}$ on $\bar{\Omega}_h$. Since

$$-\Delta_h \tilde{v}_i = -(\Delta_h w_i^{(0)} + \Delta_h w_i^{(1)}) = K \quad (i \in \Omega_h), \quad \tilde{v}_j = K^{(0)} \quad (j \in \partial\Omega_h),$$

we see from $w_i^{(1)} \leq K^{(0)}$ that \tilde{v}_i is a positive upper solution if

$$K^{(0)} \geq J[x'_j, w^{(0)} + K^{(0)}] + g_j^{(0)} \quad (x'_j \in \partial\Omega_h)$$

which is equivalent to

$$K^{(0)}(1 - J[x'_j, 1]) \geq J[x'_j, w^{(0)}] + g_j^{(0)} \quad (x'_j \in \partial\Omega_h).$$

The above inequality is satisfied by any positive constant $K^{(0)}$ satisfying

$$K^{(0)} \geq (J[x'_j, w^{(0)}] + g_j^{(0)}) / (1 - J[x'_j, 1]) \quad (x'_j \in \partial\Omega_h). \quad (5.7)$$

With this choice of $K^{(0)}$ in (5.6), the function $\tilde{v}_i = w_i^{(0)} + w_i^{(1)}$ is a positive upper solution and so the problem in (5.5) (with $b_0 = 0$) for v_i has a positive solution V_i and $0 < V_i \leq \tilde{v}_i$. We next consider the problem in (5.5) for u_i with $v_i = V_i$. This problem has also a positive solution U_i if it has a positive upper solution \tilde{u}_i . By the above argument for the case $b_0 = 0$ this positive upper solution can be constructed in the form $\tilde{u}_i = w_i^{(0)} + w_i^{(1)}$, where $w_i^{(0)}$ and $w_i^{(1)}$ are the respective positive solutions of the linear problems in (5.6) with K replaced by V_i and $K^{(0)}$ replaced by $K^{(1)} \geq (J[x'_j, w^{(0)}] + g_j^{(1)}) / (1 - J[x'_j, 1])$ for all $x'_j \in \partial\Omega_h$. This ensures the existence of a positive solution (U_i, V_i) to (5.5), and therefore, $\tilde{u}_i = U_i$ is a positive upper solution of (2.1), (2.2). Finally, a positive upper solution for problem (2.9), (2.10) is given by $(\tilde{u}_i, \tilde{v}_i) = (U_i, \mu U_i + V_i)$.

(c) We seek a positive upper solution of (2.9), (2.10) in the form

$$(\tilde{u}_i, \tilde{v}_i) = (\rho(1 + \varepsilon\phi_i), \mu\rho + (v_0 + \mu)\rho\varepsilon\phi_i) \quad (5.8)$$

for a sufficiently large constant ρ . Since by (5.3),

$$\begin{aligned} -\Delta_h \tilde{u}_i + \mu \tilde{u}_i &= \rho \varepsilon v_0 \phi_i + \mu \rho (1 + \varepsilon \phi_i) = \mu \rho + (v_0 + \mu) \rho \varepsilon \phi_i = \tilde{v}_i, \\ -\Delta_h \tilde{v}_i + \mu^+ \tilde{v}_i &= (v_0 + \mu) v_0 \rho \varepsilon \phi_i + \mu^+ (\mu \rho + (v_0 + \mu) \rho \varepsilon \phi_i) \\ &= \mu \mu^+ \rho + (v_0 + \mu) (v_0 + \mu^+) \rho \varepsilon \phi_i \quad (i \in \Omega_h), \end{aligned}$$

and $\tilde{u}_j = \rho$, $\tilde{v}_j = \mu\rho$ on $\partial\Omega_h$, we see that $(\tilde{u}_i, \tilde{v}_i)$ satisfies the inequalities in (2.11) if

$$\begin{aligned} \mu \mu^+ \rho + (v_0 + \mu) (v_0 + \mu^+) \rho \varepsilon \phi_i &\geq f_i^*(\rho(1 + \varepsilon \phi_i)) \quad (i \in \Omega_h), \\ \rho &\geq \rho (J[x'_j, 1] + \varepsilon J[x'_j, \phi]) + g_j^{(1)} \quad (x'_j \in \partial\Omega_h), \\ \mu \rho &\geq \mu \rho J[x'_j, 1] + (v_0 + \mu) \rho \varepsilon J[x'_j, \phi] + g_j^{(2)} \quad (x'_j \in \partial\Omega_h). \end{aligned}$$

In view of $\mu \mu^+ = c^* = \bar{c} + c_0$, $f_i^*(u_i) = \bar{c} u_i + f_i(u_i)$ and $J[x'_j, 1] < 1$, the above inequalities hold if

$$\begin{aligned} \bar{c} + c_0 + (v_0 + \mu) (v_0 + \mu^+) \varepsilon \phi_i &\geq \frac{1}{\rho} (\bar{c} \rho (1 + \varepsilon \phi_i) + f_i(\rho(1 + \varepsilon \phi_i))) \quad (i \in \Omega_h), \\ \rho (1 - J[x'_j, 1] - \varepsilon J[x'_j, \phi]) &\geq g_j^{(1)} \quad (x'_j \in \partial\Omega_h), \\ \mu \rho \left(1 - J[x'_j, 1] - \left(1 + \frac{v_0}{\mu} \right) \varepsilon J[x'_j, \phi] \right) &\geq g_j^{(2)} \quad (x'_j \in \partial\Omega_h). \end{aligned} \quad (5.9)$$

Let $\varepsilon_0^* > 0$ be a sufficiently small constant such that $\varepsilon_0^* \leq \varepsilon_0$ and

$$\varepsilon_0^* \left(1 + \frac{v_0}{\mu} \right) J[x'_j, \phi] < 1 - J[x'_j, 1] \quad (x'_j \in \partial\Omega_h), \quad (5.10)$$

and set

$$\theta_j = 1 - J[x'_j, 1] - \varepsilon_0^* \left(1 + \frac{v_0}{\mu} \right) J[x'_j, \phi] \quad (x'_j \in \partial\Omega_h), \quad \bar{\rho} = \max_j \left\{ g_j^{(1)} / \theta_j, g_j^{(2)} / (\mu \theta_j) \right\}.$$

Then $\theta_j > 0$, $\bar{\rho} \geq 0$ and the boundary inequalities in (5.9) are satisfied by any constant $\rho \geq \bar{\rho}$ and $\varepsilon \leq \varepsilon_0^*$. The first inequality in (5.9) is also satisfied if

$$\frac{1}{\rho} f_i(\rho(1 + \varepsilon \phi_i)) \leq ((v_0 + \mu)(v_0 + \mu^+) - \bar{c}) \varepsilon \phi_i + c_0. \quad (5.11)$$

By the condition (5.4), the above inequality holds for sufficiently large $\rho \geq \bar{\rho}$ and sufficiently small $\varepsilon \leq \varepsilon_0^*$. With this choice of ρ and ε , the function $(\tilde{u}_i, \tilde{v}_i)$ in (5.8) is a positive upper solution of problem (2.9), (2.10). \square

We next give some sufficient conditions for the construction of a positive lower solution.

Lemma 5.2. Problem (2.9), (2.10) has a positive lower solution if one of the following conditions holds:

- (a') $f_i(0) \geq 0$ and $f_i(0) \neq 0$.
- (b') $g_j^{(0)} \geq 0$, $g_j^{(1)} > 0$, $\mu > 0$ and there exists a positive constant $k^* > 0$ such that

$$f_i(k^*) \geq c_0 k^* \quad (i \in \Omega_h), \quad k^* \leq \gamma^{(1)}.$$

$$(c') \quad \liminf_{\delta \rightarrow 0^+} \frac{f_i(\delta\phi_i)}{\delta\phi_i} > v_0^2 + b_0v_0 + c_0 \quad (i \in \Omega_h). \quad (5.12)$$

Proof. (a') It is obvious from $f_i^*(0) = f_i(0) \geq 0$ that $(\widehat{u}_i, \widehat{v}_i) = (0, 0)$ is a lower solution of (2.9), (2.10). To find a positive lower solution we observe from $f_i^*(0) = f_i(0) \neq 0$ that the uncoupled linear problem

$$\begin{aligned} -\Delta_h v_i + \mu^+ v_i &= \varepsilon f_i^*(0) \quad (i \in \Omega_h), & v_j &= 0 \quad (j \in \partial\Omega_h), \\ -\Delta_h u_i + \mu u_i &= v_i \quad (i \in \Omega_h), & u_j &= 0 \quad (j \in \partial\Omega_h) \end{aligned} \quad (5.13)$$

has a positive solution $(u_i^{(\varepsilon)}, v_i^{(\varepsilon)})$, where $\varepsilon \leq 1$ is a positive constant. Since $\varepsilon f_i^*(0) \leq f_i^*(u_i^{(\varepsilon)})$ and $u_j^{(\varepsilon)} = v_j^{(\varepsilon)} = 0$ on $\partial\Omega_h$, we see from (5.13) that $(\widehat{u}_i, \widehat{v}_i) = (u_i^{(\varepsilon)}, v_i^{(\varepsilon)})$ is a positive lower solution of (2.9), (2.10).

(b') It is obvious that $\widehat{u}_i = k^*$ is a positive lower solution of (2.1), (2.2) if

$$c_0 k^* \leq f_i(k^*) \quad (i \in \Omega_h), \quad k^* \leq k^* J[x_j', 1] + g_j^{(1)}, \quad 0 \geq -g_j^{(0)} \quad (x_j' \in \partial\Omega_h).$$

The above inequalities are fulfilled by the condition in (b'). This shows that $(\widehat{u}_i, \widehat{v}_i) = (k^*, \mu k^*)$ is a positive lower solution of (2.9), (2.10).

(c') Let $(\widehat{u}_i, \widehat{v}_i) = (\delta\phi_i, (v_0 + \mu)\delta\phi_i)$, where $\delta > 0$ is a small constant. Since $-\Delta_h \phi_i = v_0 \phi_i$, $\Delta_h(\Delta_h \phi_i) = v_0^2 \phi_i$ and $\phi_j = 0$ for $i \in \Omega_h$ and $j \in \partial\Omega_h$, we see that $(\widehat{u}_i, \widehat{v}_i)$ is a lower solution of (2.9), (2.10) if

$$(v_0^2 + b_0v_0 + c_0)\delta\phi_i \leq f_i(\delta\phi_i) \quad (i \in \Omega_h).$$

By the condition (5.12), there exists a small constant $\delta_0 > 0$ such that the above inequality holds for any $0 < \delta \leq \delta_0$ (including $\delta = 0$). This shows that $(\widehat{u}_i, \widehat{v}_i) = (\delta\phi_i, (v_0 + \mu)\delta\phi_i)$ is a positive lower solution of (2.9), (2.10). \square

It is seen from the above construction that under one of the conditions (a), (b) and (c) in Lemma 5.1, a positive upper solution of problem (2.9), (2.10) is given, respectively, by

$$(K^*, \mu K^*), \quad (U_i, \mu U_i + V_i), \quad (\rho(1 + \varepsilon\phi_i), \mu\rho + (v_0 + \mu)\rho\varepsilon\phi_i), \quad (5.14)$$

while under one of the conditions (a'), (b') and (c') in Lemma 5.2, a positive lower solution is given, respectively, by

$$(u_i^{(\varepsilon)}, v_i^{(\varepsilon)}), \quad (k^*, \mu k^*), \quad (\delta\phi_i, (v_0 + \mu)\delta\phi_i), \quad (5.15)$$

where (U_i, V_i) and $(u_i^{(\varepsilon)}, v_i^{(\varepsilon)})$ are the respective positive solutions of (5.5) and (5.13). Moreover, since the constants ρ and K (in (5.5)) can be chosen arbitrarily large, and δ and ε arbitrarily small, the pair of upper and lower solutions in (5.14) and (5.15) is ordered except possibly the constant pairs $(K^*, \mu K^*)$ and $(k^*, \mu k^*)$. As a consequence of Theorems 2.1 and 2.2 we have the following result.

Theorem 5.1. Let conditions (2.7) and (5.1) hold, and let one of the conditions (a), (b) and (c) in Lemma 5.1 and one of the conditions (a'), (b') and (c') in Lemma 5.2 be satisfied. Assume that the corresponding pair of upper and lower solutions $(\widehat{u}_i, \widehat{v}_i)$, $(\widetilde{u}_i, \widetilde{v}_i)$ from (5.14) and (5.15) is ordered. Then all the conclusions in Theorems 2.1 and 2.2 hold true with respect to the above pair of upper and lower solutions. In the vector from (3.1), the conclusions in Theorems 3.1 and 3.2 hold true also.

It is easy to verify by a slight modification in the proof of Lemma 5.1 that all the conclusions in Lemmas 5.1 and 5.2 remain true if the boundary condition (2.2) is replaced by (2.2_b). In fact, for the boundary condition (2.2_b) the construction of a lower solution given by (5.15) remains the same, while the construction of an upper solution in (5.14) requires that in the choice of positive constants \bar{K} and $K^{(0)}$ (for item (b)) the constant $\gamma^{(0)}$ be replaced by $\bar{g}^{(0)} \equiv \max\{g_j^{(0)}; x_j' \in \partial\Omega\}$ and $K^{(0)} \geq \bar{g}^{(0)}$, and in the definition of ε_0^* and θ_j (for item (c)) the constant v_0 be replaced by zero. This observation leads to the following result for (2.9), (2.23).

Theorem 5.2. Let the conditions in Theorem 5.1 be satisfied. Then all the conclusions in Theorem 2.3 hold true for problem (2.9), (2.23), where $(\widetilde{u}_i, \widetilde{v}_i)$ and $(\widehat{u}_i, \widehat{v}_i)$ are given by (5.14) and (5.15), respectively.

It is obvious that conditions (5.4) and (5.12) are satisfied if $c_0 > 0$ and

$$\limsup_{\rho \rightarrow +\infty} \frac{f_i(\rho(1 + \varepsilon\phi_i))}{\rho} = 0 \quad (0 < \varepsilon < \varepsilon_0), \quad \liminf_{\delta \rightarrow 0^+} \frac{f_i(\delta\phi_i)}{\delta\phi_i} = +\infty. \quad (5.16)$$

This implies that for a sufficiently large constant ρ and two sufficiently small positive constants ε, δ the pair

$$(\widehat{u}_i, \widehat{v}_i) = (\rho(1 + \varepsilon\phi_i), \mu\rho + (v_0 + \mu)\rho\varepsilon\phi_i), \quad (\widetilde{u}_i, \widetilde{v}_i) = (\delta\phi_i, (v_0 + \mu)\delta\phi_i) \quad (5.17)$$

is ordered upper and lower solutions of problems (2.9), (2.10) and (2.9), (2.23). As a consequence of the above theorems we have the following corollary which is quite useful in practical applications.

Corollary 5.1. Let conditions (2.7) and (5.1) be satisfied, and let $c_0 > 0$, $\mu > 0$. If condition (5.16) holds, then all the conclusions in Theorems 5.1 and 5.2 hold true, where $(\widetilde{u}_i, \widetilde{v}_i)$ and $(\widehat{u}_i, \widehat{v}_i)$ are given by (5.17). In particular, condition (5.16) is satisfied if $f_i(0) > 0$ and $f_i(u_i)$ is bounded as $u_i \rightarrow +\infty$.

6. Numerical results

To compute numerical solutions of (2.9), (2.10) by the monotone iterative schemes we consider three examples where the true continuous solutions are explicitly known by suitable choices of the functions $f(x, u) = f(u) + q(x)$ and $g^{(l)}(x')$ ($l = 0, 1$). In each example, the values of the true solution are used to compare with the computed solution at every mesh point to demonstrate the accuracy and reliability of the monotone iterations. Numerical results for the same problem with $q(x) = 0$ and/or $g^{(l)}(x') = 0$ are also given. In all the examples the weight functions $\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_n}$ in the approximation of the integrals in the boundary condition are chosen by the composite Simpson's rule (see [26,27]).

Example 1. In our first example we consider a one-dimensional problem in the form:

$$\begin{aligned} u'''' - 3u'' + 2u &= \sigma u/(1+u) + q(x) \quad (0 < x < 1), \\ u(0) &= \int_0^1 x^2 u(x) dx + g^{(1)}(0), \quad u(1) = \int_0^1 x^2 u(x) dx + g^{(1)}(1), \\ u''(0) &= \int_0^1 x^2 u''(x) dx - g^{(0)}(0), \quad u''(1) = \int_0^1 x^2 u''(x) dx - g^{(0)}(1), \end{aligned} \quad (6.1)$$

where σ is a positive constant, and $q(x)$, $g^{(1)}(x')$ and $g^{(0)}(x')$ ($x' = 0, 1$) are positive functions to be chosen.

Let $h = 1/(N+1)$ be the mesh size, where N is a positive odd number. Then the composite Simpson's rule (cf. [26,27]) for the integral in (6.1) is given by

$$\int_0^1 x^2 w(x) dx \approx J[x'_j, w] = \frac{h}{3} \left(2 \sum_{i=1}^{\frac{N-1}{2}} (2ih)^2 w(2ih) + 4 \sum_{i=1}^{\frac{N+1}{2}} ((2i-1)h)^2 w((2i-1)h) + w(1) \right) \quad (6.2)$$

for $w = u$ or $w = u''$. Since the function $f_i(u_i) = \sigma u_i/(1+u_i) + q_i$ is nondecreasing in u_i for $u_i \geq 0$, where $q_i = q(x_i)$, the constant \bar{c} in (2.4) is taken as $\bar{c} = 0$ whenever $\hat{u}_i \geq 0$. It is clear from $b_0 = 3$, $c_0 = 2$, $c^* = c_0$ and

$$J[x'_j, 1] = \int_0^1 x^2 dx = \frac{1}{3} \quad (x'_j = 0, 1)$$

that conditions (2.7) and (5.1) are satisfied. Moreover from (2.8), we have $\mu = 1$ and $\mu^+ = 2$. This implies that the corresponding problem (2.9), (2.10) for (6.1) is reduced to

$$\begin{aligned} -\Delta_h u_i + u_i &= v_i, \quad -\Delta_h v_i + 2v_i = \sigma u_i/(1+u_i) + q_i \quad (1 \leq i \leq N), \\ u_j &= J[x'_j, u] + g_j^{(1)}, \quad v_j = J[x'_j, v] + g_j^{(2)} \quad (x'_j = 0, 1), \end{aligned} \quad (6.3)$$

where $g_j^{(2)} = g_j^{(0)} + g_j^{(1)}$.

By the relation $f_i(0) = q_i > 0$ and

$$\limsup_{\rho \rightarrow +\infty} \frac{f_i(\rho(1+\varepsilon\phi_i))}{\rho} = \lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \left(\frac{\sigma\rho(1+\varepsilon\phi_i)}{1+\rho(1+\varepsilon\phi_i)} + q_i \right) = 0,$$

we see from Lemmas 5.1 and 5.2 that

$$(\tilde{u}_i, \tilde{v}_i) = (\rho(1+\varepsilon\phi_i), \rho + (v_0+1)\rho\varepsilon\phi_i), \quad (\hat{u}_i, \hat{v}_i) = (0, 0) \quad (6.4)$$

are ordered upper and lower solutions of (6.3), where ρ and ε are suitably large and small positive constants, respectively, and (v_0, ϕ_i) is the positive eigenpair of (5.3). This shows that problem (6.3) has a maximal solution (\bar{u}_i, \bar{v}_i) and a minimal solution $(\underline{u}_i, \underline{v}_i) > (0, 0)$. Moreover, in view of

$$\frac{\partial f_i}{\partial u}(u_i) = \frac{\sigma}{(1+u_i)^2} \leq c_0 \quad \text{for all } u_i \geq 0 \text{ whenever } \sigma \leq c_0,$$

Theorem 2.2 ensures that for any $\sigma \leq 2$, $(\bar{u}_i, \bar{v}_i) = (\underline{u}_i, \underline{v}_i) \equiv (u_i^*, v_i^*)$ and (u_i^*, v_i^*) is the unique solution of (6.3) between $(0, 0)$ and (\bar{u}_i, \bar{v}_i) . The arbitrariness of ρ implies that (u_i^*, v_i^*) is the unique positive solution. The above conclusions hold true also if $q_i = g_j^{(1)} = g_j^{(2)} = 0$.

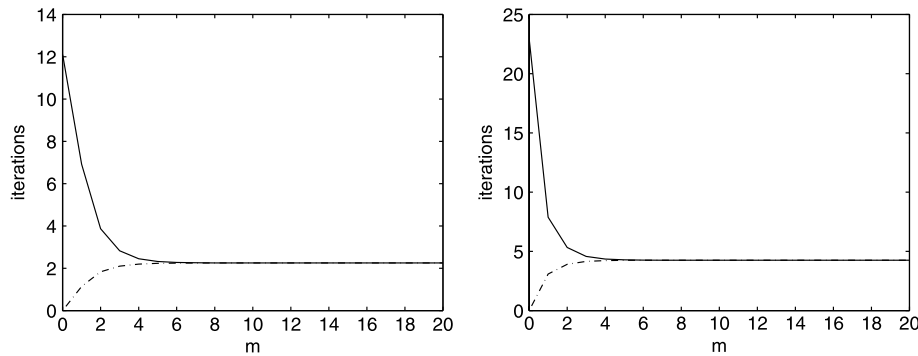
To construct an explicitly known solution of (6.1) we let $w^*(x) = \tilde{K} + \alpha x(1-x)$ and choose

$$\begin{aligned} q(x) &= 6\alpha + 2w^*(x) - \sigma w^*(x)/(1+w^*(x)), \\ g^{(1)}(x') &= \frac{2}{3}\tilde{K} - \frac{\alpha}{20}, \quad g^{(0)}(x') = \frac{4}{3}\alpha \quad (x' = 0, 1), \end{aligned} \quad (6.5)$$

Table 1(a)

The computed solutions of (6.3) by (2.13).

Solution	$x_i = 0$	$x_i = 1/4$	$x_i = 1/2$	$x_i = 3/4$	$x_i = 1$
\bar{u}_i	2.0000000022	2.1875000021	2.2500000021	2.1875000021	2.0000000022
\underline{u}_i	1.9999999968	2.1874999969	2.2499999969	2.1874999969	1.9999999968
$u^*(x_i)$	2	2.1875000000	2.2500000000	2.1875000000	2
\bar{v}_i	3.9999999990	4.1874999993	4.2499999993	4.1874999993	3.9999999990
\underline{v}_i	3.9999999978	4.1874999981	4.2499999982	4.1874999981	3.9999999978
$v^*(x_i)$	4	4.1875000000	4.2500000000	4.1875000000	4

**Fig. 1.** The monotone convergence of $\{\bar{u}_i^{(m)}, \underline{u}_i^{(m)}\}$ (left) and $\{\bar{v}_i^{(m)}, \underline{v}_i^{(m)}\}$ (right) at $x_i = 0.5$.

where σ , α and \tilde{K} are positive constants satisfying $\sigma \leq \max\{2, 6\alpha\}$ and $\alpha < 40\tilde{K}/3$. It is easy to verify that with the above choice of $q(x)$ and $g^{(l)}(x')$ ($l = 0, 1$) the corresponding solution of (6.1) is given by $u^*(x) = w^*(x)$.

To compute the solution (u_i^*, v_i^*) of (6.3) from the iteration process (2.13), we observe that for the present one-dimensional domain $\Omega = (0, 1)$ the eigenpair (v_0, ϕ_i) of (5.3) is given by (see [33])

$$v_0 = \frac{4}{h^2} \sin^2 \left(\frac{\pi}{2(N+1)} \right), \quad \phi_i = \sin \left(\frac{i\pi}{N+1} \right) \quad (i = 1, 2, \dots, N).$$

The constants ρ and ε in (6.4) can be chosen from (5.11) and $\varepsilon \leq \varepsilon_0^*$, where ε_0^* is determined by (5.10). For the present problem, these two conditions become

$$\sigma(1 + \varepsilon\phi_i)/(1 + \rho(1 + \varepsilon\phi_i)) + q_i/\rho \leq (v_0 + 1)(v_0 + 2)\varepsilon\phi_i + 2, \quad \varepsilon_0^*(1 + v_0)J[x'_j, \phi] < 1 - J[x'_j, 1].$$

The above inequalities are satisfied if $\rho \geq 1$, $\sigma + (13\alpha + 4\tilde{K})/(2\rho) < 2$ and $\varepsilon_0^* < 2/(1 + \pi^2)$. It is to be noted that since $f_i(u_i) \leq \sigma + q_i$ for $u_i \geq 0$, a different upper solution of (6.3) is given by $(U_i, U_i + V_i)$ which is the solution of (5.5) with $K = \sigma + \bar{q}$, where \bar{q} is the maximum of q_i .

Using the above parameters in the iteration process (2.13) with $(u_i^{(0)}, v_i^{(0)}) = (0, 0)$ and $(\bar{u}_i^{(0)}, \bar{v}_i^{(0)}) = (\tilde{u}_i, \tilde{v}_i)$ we compute the minimal and maximal solutions $(\underline{u}_i, \underline{v}_i)$ and (\bar{u}_i, \bar{v}_i) of (6.3) for various values of $\sigma, \tilde{K}, \alpha, \rho, \varepsilon$ and various mesh size h . Since the solution (u_i^*, v_i^*) is unique we use the stopping criterion in the iteration process by

$$|\bar{u}_i^{(m)} - \underline{u}_i^{(m)}| + |\bar{v}_i^{(m)} - \underline{v}_i^{(m)}| < \varepsilon^* \quad (0 \leq i \leq N+1) \quad (6.6)$$

for various $\varepsilon^* > 0$. Numerical values of (\bar{u}_i, \bar{v}_i) and $(\underline{u}_i, \underline{v}_i)$ together with the values of the true solution $(u^*(x_i), v^*(x_i)) = (\tilde{K} + \alpha x_i(1 - x_i), \tilde{K} + 2\alpha + \alpha x_i(1 - x_i))$ at the quarter points for the case

$$\sigma = 1, \quad \tilde{K} = 2, \quad \alpha = 1, \quad \rho = 11, \quad \varepsilon = 0.1, \quad h = 0.01, \quad \varepsilon^* = 10^{-8}$$

are given in Table 1(a). It is seen from this table that the computed maximal and minimal solutions and the true continuous solution coincide in the first nine effective digits. In the numerical computations, the basic feature of monotone convergence of the maximal and minimal sequences $\{\bar{u}_i^{(m)}, \bar{v}_i^{(m)}\}$ and $\{\underline{u}_i^{(m)}, \underline{v}_i^{(m)}\}$ was observed at every mesh point x_i . Numerical result for the monotone convergence of these sequences at $x_i = 0.5$ is sketched in Fig. 1, where the solid and dashed-dotted lines represent the maximal and minimal sequences, respectively.

In addition to the above problem with known continuous solution, we also consider the case $q(x) = 0$ with the same $g^{(1)}(x')$ and $g^{(0)}(x')$ in (6.5). Numerical results of the maximal and minimal solutions with the same parameters and initial iterations $(\tilde{u}_i, \tilde{v}_i)$ and (\bar{u}_i, \bar{v}_i) in (6.4) are given in Table 1(b).

Table 1(b)The computed solutions of (6.3) with $q_i = 0$ by (2.13).

Solution	$x_i = 0$	$x_i = 1/4$	$x_i = 1/2$	$x_i = 3/4$	$x_i = 1$
\bar{u}_i	1.9656605447	2.0674472920	2.0971755178	2.0674472920	1.9656605447
\underline{u}_i	1.9656605391	2.0674472867	2.0971755125	2.0674472867	1.9656605391
\bar{v}_i	3.7135704591	3.1847591243	3.0156762698	3.1847591243	3.7135704591
\underline{v}_i	3.7135704578	3.1847591231	3.0156762686	3.1847591231	3.7135704578

Example 2. Our second example is for the two-dimensional rectangular domain $\Omega = \{(x, y); 0 < x < L_1, 0 < y < L_2\}$ and is given in the form

$$\begin{aligned} \Delta^2 u - b_0 \Delta u + c_0 u &= q(x, y) - u^\gamma \quad ((x, y) \in \Omega), \\ u(x', y') &= \beta_0 \int_0^{L_1} \int_0^{L_2} xy u(x, y) dx dy + g^{(1)}(x', y'), \\ (\Delta u)(x', y') &= \beta_0 \int_0^{L_1} \int_0^{L_2} xy (\Delta u)(x, y) dx dy - g^{(0)}(x', y') \quad ((x', y') \in \partial\Omega), \end{aligned} \quad (6.7)$$

where γ and β_0 are positive constants satisfying $\gamma \geq 1$, $\beta_0 < 4/(L_1 L_2)^2$. The above problem is a special case of (1.1) with $f(x, y, u) = q(x, y) - u^\gamma$ and $\beta(x', y', x, y) = \beta_0 xy$. The functions $q(x, y)$, $g^{(l)}(x', y')$ ($l = 0, 1$) are positive functions which will be chosen so that an explicit solution can be constructed. To achieve this we let

$$\lambda_0 = (\pi/L_1)^2 + (\pi/L_2)^2, \quad \phi(x, y) = \sin(\pi x/L_1) \sin(\pi y/L_2), \quad (6.8)$$

and choose

$$\begin{aligned} q(x, y) &= (1 - \alpha \phi(x, y))^\gamma + c_0 - \alpha(\lambda_0^2 + b_0 \lambda_0 + c_0) \phi(x, y), \\ g^{(1)}(x', y') &= 1 + \alpha \beta_0 (L_1 L_2 / \pi)^2 - \beta_0 (L_1 L_2)^2 / 4, \\ g^{(0)}(x', y') &= \alpha \beta_0 \lambda_0 (L_1 L_2 / \pi)^2, \end{aligned} \quad (6.9)$$

where α is a constant satisfying $0 < \alpha < 1$. Then an explicit solution of (6.7) is given by $u^*(x, y) = 1 - \alpha \phi(x, y)$, and $g^{(l)}(x', y') > 0$ ($l = 0, 1$) and $q(x, y) > 0$ for a suitably small α if $c_0 > -1$.

Let $h_x = L_1/(M_1 + 1)$ and $h_y = L_2/(M_2 + 1)$ be the mesh sizes in the x - and y -directions, respectively, where M_1 and M_2 are two positive odd numbers. To obtain the approximation $J[x'_i, y'_j, w]$ ($w = u$ or $w = \Delta u$) of the integral in (6.7) we use the composite Simpson's rule in the x - and y -directions as that in (6.2). It is obvious from $g^{(l)}(x', y') > 0$ ($l = 0, 1$) and

$$J[x'_i, y'_j, 1] = \beta_0 \int_0^{L_1} \int_0^{L_2} xy dx dy = \beta_0 (L_1 L_2)^2 / 4 < 1$$

that condition (5.1) is satisfied. Let $c_0 \geq 0$. Since for any constant $K \geq q(x, y)$ the function $f(x, y, u) = q(x, y) - u^\gamma$ possesses the property

$$f(x, y, 0) > 0, \quad f(x, y, u) \leq K < c_0 u + K \quad \text{for } u \geq 0, (x, y) \in \bar{\Omega},$$

we see that condition (b) of Lemma 5.1 and condition (a') of Lemma 5.2 are fulfilled. This implies that the linear system (5.5) for problem (6.7) has a positive solution $(U_{i,j}, V_{i,j})$, and therefore the pair $\tilde{u}_{i,j} = U_{i,j}$, $\hat{u}_{i,j} = 0$ is ordered upper and lower solutions of the corresponding problem (2.1), (2.2) of (6.7). In view of $-(\partial f_{i,j}/\partial u)(u_{i,j}) = \gamma u_{i,j}^{\gamma-1} \leq \gamma u_{i,j}^{\gamma-1}$ for $0 \leq u_{i,j} \leq U_{i,j}$, condition (2.4) becomes $\bar{c} \geq \gamma U_{i,j}^{\gamma-1}$. With this choice of \bar{c} in (2.8) and assume that condition (2.7) holds then the corresponding difference problem (2.9), (2.10) of (6.7) becomes

$$\begin{aligned} -\Delta_h u_{i,j} + \mu u_{i,j} &= v_{i,j}, & -\Delta_h v_{i,j} + \mu^+ v_{i,j} &= f_{i,j}^*(u_{i,j}) \quad ((i, j) \in \Omega_h), \\ u_{i,j} &= J[x'_i, y'_j, u] + g_{i,j}^{(1)}, & v_{i,j} &= J[x'_i, y'_j, v] + g_{i,j}^{(2)} \quad ((x'_i, y'_j) \in \partial\Omega_h), \end{aligned} \quad (6.10)$$

where $g_{i,j}^{(2)} = g_{i,j}^{(0)} + \mu g_{i,j}^{(1)}$. By the proof of Lemmas 5.1 and 5.2 the above problem has a pair of ordered upper and lower solutions $(\bar{u}_{i,j}, \bar{v}_{i,j}) = (U_{i,j}, \mu U_{i,j} + V_{i,j})$ and $(\underline{u}_{i,j}, \underline{v}_{i,j}) = (0, 0)$. Hence if condition (2.7) is satisfied then by Theorem 2.1 (or Theorem 5.1), problem (6.10) has a maximal solution $(\bar{u}_{i,j}, \bar{v}_{i,j})$ and a minimal solution $(\underline{u}_{i,j}, \underline{v}_{i,j})$. Moreover by the relation

$$\frac{\partial f}{\partial u}(\cdot, u) = -\gamma u^{\gamma-1} \leq 0 \quad \text{for } u \geq 0,$$

Theorem 2.2 ensures that $(\bar{u}_{i,j}, \bar{v}_{i,j}) = (\underline{u}_{i,j}, \underline{v}_{i,j}) \equiv (u_{i,j}^*, v_{i,j}^*)$ and $(u_{i,j}^*, v_{i,j}^*)$ is the unique solution between $(0, 0)$ and $(U_{i,j}, \mu U_{i,j} + V_{i,j})$. Since $(U_{i,j}, V_{i,j})$ can be made arbitrarily large by taking K in (5.5) sufficiently large we conclude that $(u_{i,j}^*, v_{i,j}^*)$

Table 2(a)

The computed solutions of (6.10) by (2.13).

Solution	(0, 1/2)	(1/4, 1/2)	(1/2, 1/2)	(3/4, 1/2)	(1, 1/2)
$\bar{u}_{i,j}$	0.9999998306	0.9981917714	0.9974428519	0.9981917714	0.9999998306
$\underline{u}_{i,j}$	0.9999998275	0.9981917682	0.9974428486	0.9981917682	0.9999998275
$u^*(x_i, y_j)$	1	0.9981921509	0.9974433153	0.9981921509	1
$\bar{v}_{i,j}$	0.9999993533	0.9758872745	0.9658996901	0.9758872745	0.9999993533
$\underline{v}_{i,j}$	0.9999993508	0.9758872719	0.9658996874	0.9758872719	0.9999993508
$v^*(x_i, y_j)$	1	0.9758887069	0.9659014824	0.9758887069	1

is the unique positive solution of (6.10). It is an approximation of the solution $(u^*(x_i, y_j), v^*(x_i, y_j)) = (1 - \alpha\phi(x_i, y_j), \mu - \alpha(\mu + \lambda_0)\phi(x_i, y_j))$ of the equivalent problem (3.13) corresponding to (6.7).

To compute the solution $(u_{i,j}^*, v_{i,j}^*)$ we choose the parameters

$$b_0 = 10, \quad c_0 = 1, \quad \gamma = 4, \quad \beta_0 = 1/2, \quad L_1 = 1, \quad L_2 = 2, \quad \alpha = (\lambda_0^2 + b_0\lambda_0 + c_0)^{-1}.$$

Then

$$\begin{aligned} \lambda_0 &= 5\pi^2/4, \quad \phi(x, y) = \sin(\pi x) \sin(\pi y/2), \quad \alpha = (\lambda_0^2 + 10\lambda_0 + 1)^{-1}, \\ q(x, y) &= (1 - \alpha\phi(x, y))^4 + 1 - \phi(x, y), \\ g^{(1)}(x', y') &= 1/2 + 2\alpha/\pi^2, \quad g^{(0)}(x', y') = 5\alpha/2. \end{aligned} \quad (6.11)$$

This implies that $f(x, y, u) \leq q(x, y) \leq 2$ for $u \geq 0$. We use $K = 2$ and the above values of $g^{(1)}$ and $g^{(0)}$ in (5.5) to compute the solution $(U_{i,j}, V_{i,j})$. By a simple calculation, the constant \bar{c} in (2.4) can be chosen as $\bar{c} = 8$ which yields

$$c^* = 9, \quad \mu = 1, \quad \mu^+ = 9, \quad f_{i,j}^*(u_{i,j}) = 8u_{i,j} - u_{i,j}^4 + q_{i,j}, \quad (6.12)$$

where $q_{i,j} = q(x_i, y_j)$ and $q(x, y)$ is given by (6.11).

Using the values in (6.12) in the iteration process (2.13) (with $g_{i,j}^{(2)} = 1/2 + (2/\pi^2 + 5/2)\alpha$) we compute the maximal and minimal solutions $(\bar{u}_{i,j}, \bar{v}_{i,j})$, $(\underline{u}_{i,j}, \underline{v}_{i,j})$ with the respective initial iterations $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (U_{i,j}, U_{i,j} + V_{i,j})$ and $(u_{i,j}^{(0)}, v_{i,j}^{(0)}) = (0, 0)$. The stopping criterion for the iterations is given by the relation

$$|\bar{u}_{i,j}^{(m)} - \underline{u}_{i,j}^{(m)}| + |\bar{v}_{i,j}^{(m)} - \underline{v}_{i,j}^{(m)}| < \varepsilon^* \quad ((i, j) \in \bar{\Omega}_h) \quad (6.13)$$

for various ε^* . Numerical results for $(\bar{u}_{i,j}, \bar{v}_{i,j})$, $(\underline{u}_{i,j}, \underline{v}_{i,j})$ and $(u^*(x_i, y_j), v^*(x_i, y_j))$ at the quarter points in the x -direction and $y_j = 1/2$ are given in Table 2(a), where $h_x = h_y = 1/100$ and $\varepsilon^* = 10^{-8}$.

In addition to the above problem with known continuous solution we also compute the maximal and minimal solutions for the case $g^{(1)}(x', y') = g^{(0)}(x', y') = 0$ with the same $q(x, y)$ in (6.11). The initial iterations are the same except that $(U_{i,j}, V_{i,j})$ is given by (5.5) with $g_{i,j}^{(0)} = g_{i,j}^{(1)} = 0$. Numerical results of these solutions at the quarter points in the x -direction and $y_j = 1/2$ are given in Table 2(b). Moreover, for the present two-dimensional domain, we compute the solutions by the Gauss–Seidel iteration (3.5) and the Jacobi iteration (3.6). Numerical results of the solutions are almost the same as that by (2.13) and therefore are not given. However, a comparison for the number of iterations by the three iterative schemes are given in Table 2(c), where $h_x = h_y = h$ and the stopping criterion is given by (6.13) with $\varepsilon^* = 10^{-4}$. It is seen from this table that the rate of convergence of the Picard method is the fastest, and the number of iterations by the Jacobi method is about twice the number of iterations by the Gauss–Seidel method.

Example 3. As an application of problem (2.1), (2.2_b) we consider the following example in the rectangular domain $\Omega = \{(x, y); 0 < x < 1, 0 < y < 2\}$:

$$\begin{aligned} \Delta^2 u - b_0 \Delta u + c_0 u &= \sigma(1 - u)e^{-u} + q(x, y) \quad ((x, y) \in \Omega), \\ u(x', y') &= \int_0^1 \int_0^2 x^2 y u(x, y) dx dy, \quad (\Delta u)(x', y') = -g^{(0)}(x', y') \quad ((x', y') \in \partial\Omega), \end{aligned} \quad (6.14)$$

where b_0, c_0 and σ are positive constants satisfying $b_0^2 \geq 4(c_0 + 2\sigma)$, and $q(x, y)$ and $g^{(0)}(x', y')$ are nonnegative functions to be chosen.

Let $h_x = 1/(M_1 + 1)$ and $h_y = 2/(M_2 + 1)$ be the mesh sizes in the x - and y -directions, respectively, where M_1 and M_2 are two positive odd numbers. As in Example 2, we use the composite Simpson's rule in the x - and y -directions to obtain the approximation $J[x'_i, y'_j, u]$ of the integral in (6.14). By the relation

$$-\frac{\partial f}{\partial u}(x, y, u) = -\frac{\partial}{\partial u}(\sigma(1 - u)e^{-u} + q(x, y)) = \sigma(2 - u)e^{-u} \leq 2\sigma \quad \text{for } u \geq 0,$$

Table 2(b)The computed solutions of (6.10) with $g_{i,j}^{(1)} = g_{i,j}^{(2)} = 0$ by (2.13).

Solution	(0, 1/2)	(1/4, 1/2)	(1/2, 1/2)	(3/4, 1/2)	(1, 1/2)
$\bar{u}_{i,j}$	0.0045320765	0.0104306106	0.0124691972	0.0104306106	0.0045320765
$\underline{u}_{i,j}$	0.0045320737	0.0104306075	0.0124691940	0.0104306075	0.0045320737
$\bar{v}_{i,j}$	0.0396086619	0.0925667053	0.1031893996	0.0925667053	0.0396086619
$\underline{v}_{i,j}$	0.0396086581	0.0925667011	0.1031893952	0.0925667011	0.0396086581

Table 2(c)Comparison of the different methods for (6.10) with $g_{i,j}^{(1)} = g_{i,j}^{(2)} = 0$.

Method	Number of iterations		
	($h = 1/10$)	($h = 1/20$)	($h = 1/40$)
Picard	12	12	12
Gauss–Seidel	179	704	2811
Jacobi	353	1408	5630

the constant \bar{c} in (2.4) may be taken as $\bar{c} = 2\sigma$ whenever $\hat{u}_{i,j} \geq 0$. This implies that $c^* = c_0 + 2\sigma$ and $f_{i,j}^*(u_{i,j}) = 2\sigma u_{i,j} + \sigma(1 - u_{i,j})e^{-u_{i,j}} + q_{i,j}$, where $q_{i,j} = q(x_i, y_j)$. Since $b_0 > 0$, $c_0 > 0$, $\sigma > 0$ and $b_0^2 \geq 4(c_0 + 2\sigma)$, condition (2.7) is satisfied. Moreover, the corresponding difference system (2.9), (2.23) of (6.14) is given by

$$\begin{aligned} -\Delta_h u_{i,j} + \mu u_{i,j} &= v_{i,j}, \\ -\Delta_h v_{i,j} + \mu^+ v_{i,j} &= 2\sigma u_{i,j} + \sigma(1 - u_{i,j})e^{-u_{i,j}} + q_{i,j} \quad ((i, j) \in \Omega_h), \\ u_{i,j} &= J[x'_i, y'_j, u], \quad v_{i,j} = \mu J[x'_i, y'_j, u] + g_{i,j}^{(0)} \quad ((x'_i, y'_j) \in \partial\Omega_h). \end{aligned} \quad (6.15)$$

It is clear from $q(x, y) \geq 0$, $g^{(0)}(x', y') \geq 0$ and

$$J[x'_i, y'_j, 1] = \int_0^1 \int_0^2 x^2 y dx dy = \frac{2}{3}, \quad (x'_i, y'_j) \in \partial\Omega$$

that condition (5.1) is satisfied. In view of the property

$$f_{i,j}(0) > 0, \quad f_{i,j}(u_{i,j}) = \sigma(1 - u_{i,j})e^{-u_{i,j}} + q_{i,j} \leq \sigma + \bar{q} \quad \text{for all } u_{i,j} \geq 0,$$

where \bar{q} is the maximum of $q_{i,j}$, we see from Lemmas 5.1 and 5.2 that the pair $(\tilde{u}_{i,j}, \tilde{v}_{i,j}) = (U_{i,j}, \mu U_{i,j} + V_{i,j})$ and $(\hat{u}_{i,j}, \hat{v}_{i,j}) = (0, 0)$ is ordered upper and lower solutions of (6.15), where $(U_{i,j}, V_{i,j})$ is the positive solution of the corresponding system (5.5) of (6.14) with $K = \sigma + \bar{q}$, $g_{i,j}^{(1)} = 0$ and $V_{i,j} = g_{i,j}^{(0)}$ on $\partial\Omega_h$. By Theorem 2.3, problem (6.15) has a maximal solution $(\bar{u}_{i,j}, \bar{v}_{i,j})$ and a minimal solution $(\underline{u}_{i,j}, \underline{v}_{i,j})$ which can be computed by the maximal and minimal sequences $\{\bar{u}_{i,j}^{(m)}, \bar{v}_{i,j}^{(m)}\}$ and $\{\underline{u}_{i,j}^{(m)}, \underline{v}_{i,j}^{(m)}\}$ from the iteration process

$$\begin{aligned} -\Delta_h \bar{u}_{i,j}^{(m)} + \mu \bar{u}_{i,j}^{(m)} &= \bar{v}_{i,j}^{(m-1)}, \quad -\Delta_h \bar{v}_{i,j}^{(m)} + \mu^+ \bar{v}_{i,j}^{(m)} = f_{i,j}^*(\bar{u}_{i,j}^{(m-1)}) \quad ((i, j) \in \Omega_h), \\ \bar{u}_{i,j}^{(m)} &= J[x'_i, y'_j, \bar{u}_{i,j}^{(m-1)}], \quad \bar{v}_{i,j}^{(m)} = \mu J[x'_i, y'_j, \bar{u}_{i,j}^{(m-1)}] + g_{i,j}^{(0)} \quad ((x'_i, y'_j) \in \partial\Omega_h), \end{aligned} \quad (6.16)$$

where $(\bar{u}_{i,j}^{(0)}, \bar{v}_{i,j}^{(0)}) = (U_{i,j}, \mu U_{i,j} + V_{i,j})$ and $(\underline{u}_{i,j}^{(0)}, \underline{v}_{i,j}^{(0)}) = (0, 0)$.

To construct a known solution of (6.14) we let

$$w^* \equiv u^*(x, y) = 1/5 + xy(1 - x)(2 - y),$$

and choose

$$\begin{aligned} q(x, y) &= 8 + 2b_0(x(1 - x) + y(2 - y)) + c_0 w^* - \sigma(1 - w^*)e^{-w^*}, \\ g^{(0)}(x', y') &= 2(x'(1 - x') + y'(2 - y')). \end{aligned} \quad (6.17)$$

It is easy to verify that $u^*(x, y) = w^*(x, y)$ is a solution of (6.14). This implies that the solution $(u^*(x, y), v^*(x, y))$ of the corresponding continuous problem of (6.15) is given by

$$(u^*(x, y), v^*(x, y)) = (1/5 + xy(1 - x)(2 - y), \mu u^*(x, y) + 2(x(1 - x) + y(2 - y))).$$

This solution will be used to compare with the solution of (6.15).

To compute the maximal solution $(\bar{u}_{i,j}, \bar{v}_{i,j})$ and the minimal solution $(\underline{u}_{i,j}, \underline{v}_{i,j})$ of (6.15) we choose $b_0 = 5$, $c_0 = 2$ and $\sigma = 1$. Then $b_0^2 - 4(c_0 + 2\sigma) = 9$ and therefore, $\mu = 1$ and $\mu^+ = 4$. In this case, a simple calculation shows that condition (2.18) holds, and so by Theorem 2.3, $(\bar{u}_{i,j}, \bar{v}_{i,j}) = (\underline{u}_{i,j}, \underline{v}_{i,j}) \equiv (u_{i,j}^*, v_{i,j}^*)$ and $(u_{i,j}^*, v_{i,j}^*)$ is the unique solution of (6.15) between

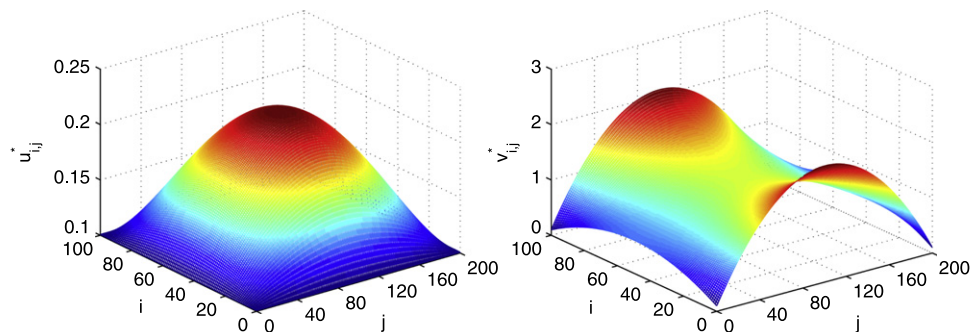
Table 3(a)

The computed solutions of (6.15) by (6.16).

Solution	(0, 1/2)	(1/4, 1/2)	(1/2, 1/2)	(3/4, 1/2)	(1, 1/2)
$\bar{u}_{i,j}$	0.1999999953	0.3406249953	0.3874999954	0.3406249953	0.1999999953
$\underline{u}_{i,j}$	0.1999999908	0.3406249908	0.3874999908	0.3406249908	0.1999999908
$u^*(x_i, y_j)$	0.2000000000	0.3406250000	0.3875000000	0.3406250000	0.2000000000
$\bar{v}_{i,j}$	1.6999999953	2.2156249961	2.3874999964	2.2156249961	1.6999999953
$\underline{v}_{i,j}$	1.6999999908	2.2156249924	2.3874999928	2.2156249924	1.6999999908
$v^*(x_i, y_j)$	1.7000000000	2.2156250000	2.3875000000	2.2156250000	1.7000000000

Table 3(b)The computed solutions of (6.15) with $q_{i,j} = g_{i,j}^{(0)} = 0$ by (6.16).

Solution	(0, 1/2)	(1/4, 1/2)	(1/2, 1/2)	(3/4, 1/2)	(1, 1/2)
$\bar{u}_{i,j}$	0.0054086025	0.0092684465	0.0107783552	0.0092684465	0.0054086025
$\underline{u}_{i,j}$	0.0054085991	0.0092684431	0.0107783517	0.0092684431	0.0054085991
$\bar{v}_{i,j}$	0.0054086025	0.0611429582	0.0772533564	0.0611429582	0.0054086025
$\underline{v}_{i,j}$	0.0054085991	0.0611429555	0.0772533540	0.0611429555	0.0054085991

**Fig. 2.** The computed solution $(u_{i,j}^*, v_{i,j}^*)$ of (6.15) with $q_{i,j} = 0$.

$(0, 0)$ and $(U_{i,j}, U_{i,j} + V_{i,j})$. Using $(U_{i,j}, U_{i,j} + V_{i,j})$ and $(0, 0)$ as the respective initial iteration $(u_{i,j}^{(0)}, v_{i,j}^{(0)})$ in (6.16), we compute the maximal solution $(\bar{u}_{i,j}, \bar{v}_{i,j})$ and the minimal solution $(\underline{u}_{i,j}, \underline{v}_{i,j})$. The stopping criterion for the iteration is again given by the relation (6.13). Numerical results for these solutions together with $(u^*(x_i, y_j), v^*(x_i, y_j))$ at the quarter points in the x -direction and $y_j = 1/2$ are given in Table 3(a), where $h_x = h_y = 1/100$ and $\varepsilon^* = 10^{-8}$.

We next compute the solution of (6.15) for the case $q_{i,j} = g_{i,j}^{(0)} = 0$. In this situation the constant pair $(\tilde{u}_{i,j}, \tilde{v}_{i,j}) = (K^*, K^*)$ and $(\hat{u}_{i,j}, \hat{v}_{i,j}) = (0, 0)$ is ordered upper and lower solutions of (6.15) for any $K^* \geq 1$. We compute the maximal and minimal solutions $(\bar{u}_{i,j}, \bar{v}_{i,j})$, $(\underline{u}_{i,j}, \underline{v}_{i,j})$ from (6.16) with the respective initial iterations $(\bar{u}_{i,j}^{(0)}, \bar{v}_{i,j}^{(0)}) = (1, 1)$ and $(\underline{u}_{i,j}^{(0)}, \underline{v}_{i,j}^{(0)}) = (0, 0)$. Numerical results of these solutions at the quarter points in the x -direction and $y_j = 1/2$ are given in Table 3(b), where $h_x = h_y = 1/100$ and $\varepsilon^* = 10^{-8}$. It is seen from this table that at least the first six effective digits of $(\bar{u}_{i,j}, \bar{v}_{i,j})$ and $(\underline{u}_{i,j}, \underline{v}_{i,j})$ coincide and it gives the uniqueness property of the solution. Finally, we compute the solution of (6.15) for the case $q_{i,j} = 0$ with the same $g^{(0)}(x', y')$ in (6.17). A geometrical presentation of the computed solution $(u_{i,j}^*, v_{i,j}^*)$ for this case with $h_x = h_y = 1/100$ and $\varepsilon^* = 10^{-8}$ is given in Fig. 2.

Acknowledgments

The work of second author was supported in part by E-Institutes of Shanghai Municipal Education Commission No. E03004, Science and Technology Commission of Shanghai Municipality (STCSM) No. 13dz2260400, the Natural Science Foundation of Shanghai No. 10ZR1409300 and Shanghai Leading Academic Discipline Project No. B407.

References

- [1] Z. Bai, H. Wang, On positive solutions of some nonlinear fourth-order beam equations, *J. Math. Anal. Appl.* 270 (2002) 357–368.
- [2] C.P. Gupta, Existence and uniqueness theorem for the bending of an elastic beam equation, *Appl. Anal.* 26 (1988) 289–304.
- [3] X. Zhang, L. Liu, Positive solutions of fourth-order multi-point boundary value problems with bending term, *Appl. Math. Comput.* 194 (2007) 321–332.
- [4] C.V. Pao, Y.M. Wang, Nonlinear fourth-order elliptic equations with nonlocal boundary conditions, *J. Math. Anal. Appl.* 372 (2010) 351–365.
- [5] W. Feng, On a m -point nonlinear boundary value problem, *Nonlinear Anal.* 30 (1997) 5369–5374.
- [6] W. Jiang, Y. Guo, Multiple positive solutions for second-order m -point boundary value problems, *J. Math. Anal. Appl.* 327 (2007) 415–424.

- [7] C.V. Pao, Y.M. Wang, Fourth-order boundary value problems with multi-point boundary conditions, *Commun. Appl. Nonlinear Anal.* 16 (2009) 1–22.
- [8] Z. Wei, C. Pan, The method of upper and lower solutions for fourth order singular m -point boundary value problems, *J. Math. Anal. Appl.* 322 (2006) 675–692.
- [9] Q. Zhang, S. Chen, J. Lü, Upper and lower solution method for fourth-order four-point boundary value problems, *J. Comput. Appl. Math.* 196 (2006) 387–393.
- [10] Q.-Huang Choi, Yinghau Jin, Nonlinearity and nontrivial solution of fourth-order elliptic equations, *J. Math. Anal. Appl.* 290 (2004) 224–234.
- [11] A.M. Micheletti, A. Pistoia, Nontrivial solutions for some fourth-order semilinear elliptic problems, *Nonlinear Anal.* 34 (1998) 509–523.
- [12] C.V. Pao, On fourth-order elliptic boundary value problems, *Proc. Amer. Math. Soc.* 128 (2000) 1023–1030.
- [13] Y.M. Wang, On fourth-order elliptic boundary value problems with nonmonotone nonlinear function, *J. Math. Anal. Appl.* 307 (2005) 1–11.
- [14] J. Zhang, Existence results for some fourth-order nonlinear elliptic problems, *Nonlinear Anal.* 45 (2001) 29–36.
- [15] J. Li, Full-order convergence of a mixed finite element method for fourth order nonlinear elliptic equations, *J. Math. Anal. Appl.* 230 (1999) 329–349.
- [16] C.V. Pao, Numerical methods for fourth-order nonlinear elliptic boundary value problems, *Numer. Methods Partial Differential Equations* 17 (2001) 347–368.
- [17] C.V. Pao, Numerical solutions of reaction–diffusion equations with nonlocal boundary conditions, *J. Comput. Appl. Math.* 136 (2001) 227–243.
- [18] C.V. Pao, Xin Lu, Block monotone iterations for numerical solutions of fourth-order nonlinear elliptic boundary value problems, *SIAM J. Sci. Comput.* 25 (2003) 164–185.
- [19] M. Akram, M.A. Pasha, A numerical method for the heat equation with a nonlocal boundary condition, *Int. J. Inf. Syst. Sci.* 1 (2005) 162–171.
- [20] Y. Li, Positive solutions of fourth-order boundary value problems with two parameters, *J. Math. Anal. Appl.* 281 (2003) 477–484.
- [21] X.L. Liu, W.T. Li, Existence and multiplicity of solutions for fourth-order boundary value problems with parameters, *J. Math. Anal. Appl.* 327 (2007) 362–375.
- [22] H. Ma, Symmetric positive solutions for nonlocal boundary value problems of fourth order, *Nonlinear Anal.* 68 (2008) 645–651.
- [23] R.Y. Ma, J.H. Zhang, S.M. Fu, The method of lower and upper solutions for fourth-order two-point boundary value problems, *J. Math. Anal. Appl.* 215 (1997) 415–422.
- [24] Y. Yang, Fourth-order two-point boundary value problem, *Proc. Amer. Math. Soc.* 104 (1988) 175–180.
- [25] Z. Yang, Existence and uniqueness of positive solutions for an integral boundary value problem, *Nonlinear Anal.* 69 (2008) 3910–3918.
- [26] A.H. Stroud, *Approximate Calculation of Multiple Integrals*, Printice Hall, Englewoods Cliffs, NJ, 1971.
- [27] E. Süli, D. Mayers, *An Introduction to Numerical Analysis*, Cambridge University Press, Cambridge, 2003.
- [28] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
- [29] C.V. Pao, Numerical solutions for some coupled systems of nonlinear boundary value problems, *Numer. Math.* 51 (1987) 381–394.
- [30] W.F. Ames, *Numerical Methods for Partial Differential Equations*, Academic Press, San Diego, 1992.
- [31] C.A. Hall, T.A. Porsching, *Numerical Analysis of Partial Differential Equations*, Prentice Hall, Englewood Cliffs, NJ, 1990.
- [32] A. Berman, R. Plemmons, *Nonnegative Matrix in the Mathematical Science*, Academic Press, New York, 1979.
- [33] R.S. Varga, *Matrix Iterative Analysis*, second ed., Springer-Verlag, Berlin, 2000.
- [34] D.M. Young, *Iterative Solution of Large Linear System*, Academic Press, New York, 1971.