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An extension of Heston's SV model to Stochastic Interest Rates

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September 5, 2018

Abstract

In [6], Heston proposes a Stochastic Volatility (SV) model with constant interest rate and derives a semi-explicit valuation formula. Heston also describes, in general terms, how the model could be extended to incorporate Stochastic Interest Rates (SIR). This paper is devoted to the construction of an extension of Heston's SV model with a particular stochastic bond model which, just increasing in one the number of parameters, allows to incorporate SIR and to derive a semi-explicit formula for option pricing.

Keywords: Stochastic Volatility, Stochastic Interest Rates, Option Pricing.

1 Introduction

In [6], Heston proposes a Stochastic Volatility (SV) model with constant interest rate and derives a semi-explicit valuation formula. Heston also describes, in general terms, how the model could be extended to incorporate Stochastic Interest Rates (SIR). We will see how, with a particular stochastic bond model and just increasing in one the number of parameters, we can incorporate SIR and derive a semi-explicit formula for option pricing.

The paper will be organized as follows. First, we will review Heston's original model with constant interest rates. In a second step, we will make the theoretical development of the extended model as presented in [6]. In a third step, we will search for a stochastic bond formula that can be nested within this framework, i.e., that fits with the specifications of the pricing model and it does not increase much the number of parameters.

Finally, we will assume that the market is composed by the stock and the discounted bond computed in the previous step. We will see that, under certain parameter restrictions, the resulting model is of the type proposed by Heston in [6]. We will derive a semi-explicit formula and a pricing model will be obtained with just one more parameter than the original Heston's SV. Thus, we will have incorporated stochastic interest rates without increasing much the number of parameters.

2 Heston SV model

In [6], Heston assumes that the asset $\bar{S}(t)$ follows a diffusion model with constant drift and where the volatility $\sqrt{\bar{v}(t)}$ of the asset follows an Ornstein-Uhlenbeck process (see [9]). After some calculus, the dynamics is:

$$\begin{cases} d\bar{S}(t) = \mu\bar{S}(t)dt + \sqrt{\bar{v}(t)}\bar{S}(t)d\bar{z}_1(t), \\ d\bar{v}(t) = k[\theta - \bar{v}(t)]dt + \sigma\sqrt{\bar{v}(t)}d\bar{z}_2(t), \end{cases} \tag{1}$$

where μ is the constant drift of the asset, σ is the volatility's volatility and $\bar{z}_1$ and $\bar{z}_2$ are Wiener processes. The variance drifts (in the physical measure), toward a long-run mean of θ with mean-reversion speed given by k.

Employing the notation of [1] or [6], we define the (instantaneous) correlation coefficient ρ by $\rho dt = \mathrm{Cov}(d\bar{z}_1, d\bar{z}_2)$, where $\mathrm{Cov}(.,.)$ stands for covariance.

We also assume that a constant rate risk-free bond exists and it is denoted by $B(t,T) = e^{-r_0(T-t)}$, where t denotes today, T denotes maturity and r_0 corresponds to the constant risk-free rate.

In [6], it is claimed that these assumptions are insufficient to price contingent claims, because we have not made an assumption that gives the price of "volatility risk". By no arbitrage arguments (see [1] or [6]), the value of any claim must satisfy:

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v S\frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + r_0 S\frac{\partial U}{\partial S} + (k(\theta - v) - \lambda(S,v,t))\frac{\partial U}{\partial v} - r_0 U + \frac{\partial U}{\partial t} = 0, \quad (2)$$

where $\bar{S}(t) = S,\ \bar{v}(t) = v$ and $\lambda(S,v,t)$ represents the price of volatility risk.

Since we are working with Heston's model (see [6]), we will assume that any risk premia is of the form $\lambda(S,v,t) = \lambda v$. It should be remarked that, once the components of the market are fixed, the risk premia is independent of the claim, i.e. the same risk premia is used to price all the claims (see [1]).

As Heston points in [6], this choice of risk premia is not arbitrary (see [2] and [4]).

Thus, the price of the European call option $U(S,v,t)$ satisfies the PDE:

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v S\frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + r_0 S\frac{\partial U}{\partial S} + (k(\theta - v) - \lambda v)\frac{\partial U}{\partial v} - r_0 U + \frac{\partial U}{\partial t} = 0, \quad (3)$$

subject to the following conditions:

$$\begin{aligned} &U(S,v,T) = \max(0, S-K), \\ &U(0,v,t) = 0, && r_0 S\frac{\partial U}{\partial S} + k\theta\frac{\partial U}{\partial v} - r_0 U + U_t\Big|_{(S,0,t)} = 0, \\ &\frac{\partial U}{\partial S}(\infty, v, t) = 1, && U(S,\infty,t) = S. \end{aligned} \quad (4)$$

Heston conjectures a solution similar to the Black-Scholes model for the price of an European call option:

$$U(S,v,t,T,K) = SR_1 - KB(t,T)R_2, \quad (5)$$

where K corresponds to the strike of the option and T denotes the maturity of the option.

The following semi-explicit formula for the price of the European call option is obtained

$$U(x,v,\tau,\ln(K)) = xR_1(x,v,\tau;\ln(K)) - \ln(K)B(t,T)R_2(x,v,\tau;\ln(K)), \quad (6)$$

where $x = \ln(S)$, $\tau = T - t$ and function $R_j,\ j \in \{1,2\}$ are given by

$$R_j(x,v,\tau;\ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln(K)}f_j(x,v,\tau,\phi)}{i\phi}\right]d\phi, \quad (7)$$

where

$$\begin{aligned} f_j(x,v,\tau;\phi) &= e^{C(\tau;\phi)+D(\tau;\phi)+i\phi x}, \\ C(\tau;\phi) &= r_0\phi i\tau + \frac{a}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d)\tau - 2\ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}, \\ D(\tau;\phi) &= \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right], \\ g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \quad d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2\zeta_j\phi i - \phi^2)}, \\ \zeta_1 &= \frac{1}{2}, \quad \zeta_2 = -\frac{1}{2}, \quad a = k\theta, \quad b_1 = k + \lambda - \rho\sigma, \quad b_2 = k + \lambda. \end{aligned} \quad (8)$$

3 The extended model

We propose (see [6]) the following market dynamics in the physical measure:

$$\begin{cases} d\bar{S}(t) = \mu_s \bar{S}(t)dt + \sigma_s(t)\sqrt{\bar{v}(t)}\bar{S}(t)d\bar{z}_1(t), \\ d\bar{v}(t) = k[\theta - \bar{v}(t)]dt + \sigma\sqrt{\bar{v}(t)}d\bar{z}_2(t), \\ d\bar{B}(t,T) = \mu_b \bar{B}(t,T)dt + \sigma_b(t)\sqrt{\bar{v}(t)}\bar{B}(t,T)d\bar{z}_3(t), \end{cases} \tag{9}$$

where $\bar{B}(t,T)$ corresponds to the price of the risk-free bond with maturity T when stochastic interest rates are considered.

We also denote

$$\rho_{sv}dt = \mathrm{Cov}(d\bar{z}_1, d\bar{z}_2), \quad \rho_{sb}dt = \mathrm{Cov}(d\bar{z}_1, d\bar{z}_3), \quad \rho_{vb}dt = \mathrm{Cov}(d\bar{z}_2, d\bar{z}_3). \tag{10}$$

Let $\bar{\mathbf{X}}(t) = (\bar{S}(t), \bar{v}(t), \bar{B}(t,T))$. Let us assume that the short rate of interest is a deterministic function of the state factors, i.e. $\bar{r} = \bar{r}(\bar{\mathbf{X}}(t))$, (short rates are stochastic but, at any fixed time t, they can be computed from the state of the market). Assuming as in [6] that the risk premia is of the form λv, any claim satisfies the PDE (see [1], pg 218):

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}\sigma_s^2 v S^2 \frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\sigma_b^2 v B^2 \frac{\partial^2 U}{\partial B^2} + \rho_{sv}\sigma_s\sigma S v \frac{\partial^2 U}{\partial s \partial v} + \rho_{sb}\sigma_s\sigma_b v S B \frac{\partial^2 U}{\partial S \partial B} \\ &+ \rho_{vb}\sigma_b\sigma B v \frac{\partial^2 U}{\partial v \partial B} + rS\frac{\partial U}{\partial S} + [k(\theta - v) - \lambda v]\frac{\partial U}{\partial v} - rU + rB\frac{\partial U}{\partial B} = 0, \end{aligned} \tag{11}$$

where, with a small abuse of notation, the value of today's state factors are $\mathbf{X} = (S, v, B)$, $r = r(\mathbf{X})$ and subject to the terminal condition of the claim (European call option), proper boundary data (see (4)) and $B(T,T) = 1$.

There also exists a risk-neutral measure π. The value of any T-claim $U(t, \mathbf{X})$ is given by the conditional expectation:

$$U(t, \mathbf{X}) = E^{\pi}\left[e^{-\int_t^T \bar{r}(\bar{\mathbf{X}}(s))ds} U(\bar{\mathbf{X}}(T)) \middle| \bar{\mathbf{X}}(t) = \mathbf{X} \right], \tag{12}$$

and the market dynamics in the risk neutral measure is given by

$$\begin{cases} d\bar{S}(t) = r\bar{S}(t)dt + \sigma_s(t)\sqrt{\bar{v}(t)}\bar{S}(t)d\bar{z}_1(t), \\ d\bar{v}(t) = [k\theta - k\bar{v}(t) - \lambda\bar{v}(t)]dt + \sigma\sqrt{\bar{v}(t)}d\bar{z}_2(t), \\ d\bar{B}(t,T) = r\bar{B}(t,T)dt + \sigma_b(t)\sqrt{\bar{v}(t)}\bar{B}(t,T)d\bar{z}_3(t). \end{cases} \tag{13}$$

The change of variable $x = \ln\left(\frac{S}{B(t,T)}\right)$ implies that the PDE in the new variable is:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \left(\frac{1}{2}\sigma_s^2 v + \frac{1}{2}\sigma_b^2 v - \rho_{sb}\sigma_s\sigma_b v\right)\frac{\partial^2 U}{\partial x^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\sigma_b^2 v B^2 \frac{\partial^2 U}{\partial B^2} \\ &+ \left(-\sigma_b^2 v P + \rho_{sb}\sigma_s\sigma_b v B\right)\frac{\partial^2 U}{\partial x \partial B} + \left(\rho_{sv}\sigma_s\sigma v - \rho_{vb}\sigma_b\sigma v\right)\frac{\partial^2 U}{\partial x \partial v} + \left(\rho_{vb}\sigma_b\sigma v B\right)\frac{\partial^2 U}{\partial v \partial B} \\ &+ \left(-\frac{1}{2}\sigma_s^2 v + \frac{1}{2}\sigma_b^2 v\right)\frac{\partial U}{\partial x} + [k(\theta - v) - \lambda v]\frac{\partial U}{\partial v} + rB\frac{\partial U}{\partial B} - rU = 0. \end{aligned} \tag{14}$$

Similar to the simple SV model, Heston (see [6]) conjectures a solution of the form:

$$U(t, x, P, v) = e^x B(t,T)R_1(t,x,v) - KB(t,T)R_2(t,x,v). \tag{15}$$

Substituting (15) into equation (14), we obtain that $R_j(t, x, v)$ must satisfy, for $j = 1, 2$:

$$\frac{1}{2}\sigma_x^2 v \frac{\partial^2 R_j}{\partial x^2} + \rho_{xv}\sigma_x \sigma v \frac{\partial^2 R_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 R_j}{\partial v^2} + \zeta_j v \frac{\partial R_j}{\partial x} + (a - b_j v)\frac{\partial R_j}{\partial v} + \frac{\partial R_j}{\partial t} = 0, \tag{16}$$

where

$$\begin{aligned} &\frac{1}{2}\sigma_x^2 = \frac{1}{2}\sigma_s^2 - \rho_{sb}\sigma_s\sigma_b + \frac{1}{2}\sigma_b^2, \quad \rho_{xv} = \frac{\rho_{sv}\sigma_s\sigma - \rho_{bv}\sigma_b\sigma}{\sigma_x\sigma}, \\ &\zeta_1 = \frac{1}{2}\sigma_x^2, \quad \zeta_2 = -\frac{1}{2}\sigma_x^2, \quad a = k\theta, \\ &b_1 = k + \lambda - \rho_{sv}\sigma_s\sigma, \quad b_2 = k + \lambda - \rho_{bv}\sigma_b\sigma, \end{aligned} \tag{17}$$

subject to the condition at maturity corresponding to the European call option:

$$R_j(T, x, v; \ln(K)) = I_{\{x \geq \ln(K)\}},$$

where I denotes the indicator function.

In order to apply Heston's results, we need to find an stochastic bond model such that the risk premia is of the form $\lambda(S, B, v, t) = \lambda v$ and where short rates can be computed from the state of the market.

The bond model is developed in the next Section and the proof that the requirements are fulfilled is in Section 5.

4 The stochastic bond

We are looking for a bond formula which can be nested in (9). Longstaff and Schwartz develop in [8] a model for interest rates that we are partly going to use.

Without loss of generality, we can assume that the bond is offered to the market by an entity (the US government, for example), whose only purpose is to trade the bond. This bond is constructed, by no arbitrage arguments, upon a certain asset $\bar{Q}$ with dynamics:

$$\begin{cases} d\bar{Q} = (\mu + \delta\bar{v})\bar{Q}dt + \sigma_{\bar{Q}}\sqrt{\bar{v}}\bar{Q}d\bar{Z}, \\ d\bar{v} = [\kappa(\theta - \bar{v})]dt + \sigma\sqrt{\bar{v}}d\bar{z}_2, \end{cases} \tag{18}$$

where $\bar{v}(t)$ is the same volatility process of (9).

We assume that asset $\bar{Q}$, although dependent of the state of the market, is only accessible to the entity which offers the bond. Therefore, any other investor who invests in the market described by (9) can only negotiate upon the traded stock $\bar{S}$ and the bond.

Following the development in [8], we assume that individuals have time-additive preferences of the form

$$E_t\left[\int_t^\infty \exp(-\rho s)\log(\bar{C}_s)ds\right], \tag{19}$$

where $E[\cdot]$ is the conditional expectation operator, ρ is the utility discount factor and $\bar{C}_s$ represents consumption at time s.

The representative investor's decision problem is equivalent to maximizing (19) subject to the budget constraint

$$d\bar{W} = \bar{W}\frac{d\bar{Q}}{\bar{Q}} - \bar{C}dt, \tag{20}$$

where $\bar{W}$ denotes wealth.

Standard maximization arguments employed in [8] lead to the following equation for the wealth dynamics

$$d\bar{W} = (\mu + \delta\bar{v}(t) - \rho)\bar{W}dt + \sigma_{\bar{Q}}\bar{W}\sqrt{\bar{v}(t)}d\bar{Z}. \tag{21}$$

Applying Theorem 3 in [3], the value of a contingent claim $B(t,v)$ must satisfy the PDE

$$-\frac{\partial B}{\partial t}=\frac{\sigma^2 v}{2}\frac{\partial^2 B}{\partial v^2}+(k\theta-kv-\lambda v)\frac{\partial B}{\partial v}-rB, \tag{22}$$

where $\bar{v}(t)=v$, the market price of risk is λv and $\bar{r}(t)=r$ is the instantaneous riskless rate.

To obtain the equilibrium interest rate $\bar{r}$, Theorem 1 of [3] is applied. This theorem relates the riskless rate to the expected rate of change in marginal utility. The result obtained is that

$$\bar{r}(t)=\mu+(\delta-\sigma_Q^2)\bar{v}(t)=\mu+\beta\bar{v}(t) \tag{23}$$

The price of a riskless unit discount bond $B(\tau,v)$, where $\tau=T-t$ is obtained solving equation (22) subject to the maturity condition $B(0,v)=1$.

For the rest of the paper, we assume that $\beta>0$. We will see that when parameter $\beta\to 0^+$, the function $B(\tau,v)$ approaches to the bond price when the risk-free rate is considered constant ($B(\tau,v)=e^{-\mu\tau}$).

Now, we proceed to give the main result of this Section.

Theorem 4.1. *The riskless unit discount bond $B(\tau,v)$, where $\tau=T-t$ denotes the time until maturity, $\bar{v}(\tau)=v$ and $\bar{r}(t)=r=\mu+\beta v$, is given by the formula*

$$B(\tau,v)=F(\tau)e^{G(\tau)v}, \tag{24}$$

where

$$\begin{aligned}F(\tau)&=\exp\left(-\left(\mu+\frac{k\theta}{b}\right)\tau+k\theta\left(\frac{b+c}{bc}\right)\ln\left(b+ce^{d\tau}\right)-k\theta\left(\frac{b+c}{bc}\right)\ln(b+c)\right),\\ G(\tau)&=\frac{e^{d\tau}-1}{b+ce^{d\tau}},\end{aligned} \tag{25}$$

and

$$d=-\sqrt{(k+\lambda)^2+2\beta\sigma^2},\quad b=\frac{(k+\lambda)-d}{2\beta},\quad c=\frac{-(k+\lambda)-d}{2\beta}. \tag{26}$$

Proof. For simplicity, along the proof, we will employ the notation:

$$\eta=k\theta,\quad \alpha=k+\lambda.$$

The claim satisfies the partial differential equation (22) subject to the maturity condition $B(0,v)=1$. With the notation that we have just introduced, we have to solve:

$$\begin{cases}\dfrac{\partial B}{\partial\tau}=\dfrac{\sigma^2}{2}v\dfrac{\partial^2 B}{\partial v^2}+(\eta-\alpha v)\dfrac{\partial B}{\partial v}-(\mu+\beta v)B,\\ B(0,v)=1.\end{cases} \tag{27}$$

We conjecture a solution of the form $B(\tau,v)=F(\tau)e^{G(\tau)v}$, thus, B_v, B_{vv} and B_τ are explicitly computable. Condition $B(0,v)=1$ imposes that $F(0)=1$ and $G(0)=0$.

Substituting the conjectured solution in (27) we obtain:

$$\frac{\sigma^2}{2}vF(\tau)G^2(\tau)+(\eta-\alpha v)F(\tau)G(\tau)-(\mu+\beta v)F(\tau)=F'(\tau)+F(\tau)G'(\tau)v. \tag{28}$$

As the previous equation is an identity in v, we have two equations:

$$\begin{cases}\dfrac{\sigma^2}{2}F(\tau)G^2(\tau)-\alpha F(\tau)G(\tau)-\beta F(\tau)=F(\tau)G'(\tau),\\ \eta F(\tau)G(\tau)-\mu F(\tau)=F'(\tau).\end{cases} \tag{29}$$

For $G(\tau)$, as candidate for solution we take:

$$G(\tau) = \frac{a + e^{d\tau}}{b + ce^{d\tau}} = \frac{e^{d\tau} - 1}{b + ce^{d\tau}},$$

as $G(0) = 0$ implies $a = -1$ and $b \neq -c$.

Thus, obtaining $G^2(\tau)$, $G'(\tau)$ and substituting in the first equation in (29), we obtain a second degree equation given in function of $\exp(2d\tau), \exp(d\tau), 1$, which implies that:

$$\begin{aligned}
\sigma^2 - 2\alpha c - 2\beta c^2 &= 0, \\
-2\sigma^2 - 2\alpha(b - c) - 4\beta bc &= 2(bd + cd), \\
\sigma^2 + 2\alpha b - 2\beta b^2 &= 0.
\end{aligned}$$

Solved for b and c, we obtain:

$$c = \frac{-\alpha \pm \sqrt{\alpha^2 + 2\beta\sigma^2}}{2\beta}, \quad b = \frac{\alpha \pm \sqrt{\alpha^2 + 2\beta\sigma^2}}{2\beta}.$$

As $b \neq -c$, two solutions are eliminated. Another one is rejected when solving the second ODE in (29), as it appears $\ln(b + c)$, which must be positive. The solution is then:

$$c = \frac{-\alpha + \sqrt{\alpha^2 + 2\beta\sigma^2}}{2\beta}, \quad b = \frac{\alpha + \sqrt{\alpha^2 + 2\beta\sigma^2}}{2\beta}, \quad d = -\sqrt{\alpha^2 + 2\beta\sigma^2}.$$

For the second equation in (29), we have to solve:

$$\begin{cases} \eta F(\tau)G(\tau) - \mu F(\tau) = F'(\tau), \\ F(0) = 1. \end{cases} \tag{30}$$

After substituting the value of $G(\tau)$ in (30), the solution is:

$$F(\tau) = \exp\left(-\left(\mu + \frac{\eta}{b}\right)\tau + \eta\frac{b + c}{bc}\ln(b + ce^{d\tau}) - \eta\frac{b + c}{bc}\ln(b + c)\right),$$

which completes the proof.

$\square$

For the rest of the paper, we denote $\bar{B}(\tau, \bar{v}) = B(\tau, \bar{v})$.

To finish the Section, we give some auxiliary results which will be needed in Section 5. They will be employed to prove that, when we incorporate the bond to the pricing model of the option, we can employ Heston's results to price options in the extended model (see [6]).

The proofs of the following results are not included since they are quite straightforward to prove.

Proposition 4.1 *The bond dynamics in the physical measure is given by*

$$\begin{aligned}
d\bar{B}(\tau, \bar{v}) &= [\mu + \beta\bar{v} + \lambda\bar{v}]\,\bar{B}(\tau, \bar{v})dt + G(\tau)\sigma\sqrt{\bar{v}}\bar{B}(\tau, \bar{v})d\bar{z}_2 \\
&= (\bar{r}(t) + \lambda\bar{v})\bar{B}(\tau, \bar{v})dt + G(\tau)\sigma\sqrt{\bar{v}}\bar{B}(\tau, \bar{v})d\bar{z}_2,
\end{aligned} \tag{31}$$

where $\bar{r}(t)$ denotes the instantaneous riskless rate and $\bar{z}_2$ is the same Wiener process as in equation (9).

The following result states that, when parameter β approaches to 0^+, then function $B(\tau, v)$ converges to the price of the bond when constant risk-free rates are employed, i.e., the bond employed in the simple SV model with constant interest rates (see Section 2).

Proposition 4.2. *Consider the functions $F(\tau)$ and $G(\tau)$ given by (25)-(26). If $\beta \to 0^+$, then we have that $F(\tau) \to \exp(-\mu\tau)$ and $G(\tau) \to 0$.*

As $\bar{B}(\tau, \bar{v})$ is the stochastic process of a bond price, the stochastic component $G(\tau)\sigma\sqrt{\bar{v}}$ of equation (31) must vanish at maturity so the bond reaches par at maturity with probability one. This is also satisfied due to following lemma.

Lemma 4.1. *Let $G(\tau)$ be given by (25). Then it holds:*

$$\begin{cases} G(\tau) = \dfrac{e^{d\tau} - 1}{b + ce^{d\tau}} \underset{\tau \to 0}{\longrightarrow} 0, \\ G(\tau) \neq 0, \quad \tau > 0, \\ G(0) = 0. \end{cases}$$

5 Valuation Formula

Suppose that the market is formed by a stock given by (physical measure)

$$\begin{cases} d\bar{S}(t) = \mu_s \bar{S}(t)dt + \sigma_s(t)\sqrt{\bar{v}(t)}\bar{S}(t)d\bar{z}_1(t), \\ d\bar{v}(t) = k[\theta - \bar{v}(t)]dt + \sigma\sqrt{\bar{v}(t)}d\bar{z}_2(t), \end{cases}$$

and by a bond

$$\bar{B}(t, T; \bar{v}) = \bar{B}(\tau; \bar{v}) = F(\tau)e^{G(\tau)\bar{v}},$$

where $\tau = T - t$ and $F(\tau),\ G(\tau)$ are explicitly given by formulas (25)-(26).

If we compute the bond dynamics, Proposition 4.1 enforces that, in order to be consistent with model (9),

$$\begin{cases} \sigma_b(\tau) = \sigma G(\tau), \\ \quad \rho_{bv} = 1, \\ \quad \rho_{bs} = \rho_{vs}, \end{cases} \tag{32}$$

and for simplicity reasons (other specifications could be considered), we have taken $\sigma_s(t) \equiv 1$.

The sign and magnitude of the correlation between the bond and the stock seems to be difficult to estimate from market data (see [7]). Condition $\rho_{bs} = \rho_{vs}$, although restrictive, does not violate market empirical observations in the sense of the sign (positive/negative).

Proposition 5.1. *The short interest rate is given by $r = \mu + \beta v$ and the risk premia $\lambda(S, v, B, t) = \lambda v$ where λ is the constant employed in the bond formula (24).*

Proof. Let us assume that it exists a deterministic function $\bar{r} = \bar{r}\left(\bar{\mathbf{X}}(t)\right)$ where $\bar{\mathbf{X}}(t) = (\bar{S}(t), \bar{v}(t), \bar{B}(t, T))$ for the short interest rate. Using the results in [1], pg 218, any contingent claim must satisfy

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma_s^2 vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\sigma_b^2 vB^2\frac{\partial^2 U}{\partial B^2} + \rho_{sv}\sigma_s\sigma Sv\frac{\partial^2 U}{\partial s\partial v} + \rho_{sb}\sigma_s\sigma_b vSB\frac{\partial^2 U}{\partial S\partial B} +$$
$$+ \rho_{vb}\sigma_b\sigma Bv\frac{\partial^2 U}{\partial v\partial B} + rS\frac{\partial U}{\partial S} + [k(\theta - v) - \lambda(S, v, B, t)]\frac{\partial U}{\partial v} - rU + rB\frac{\partial U}{\partial B} = 0,$$

where $\bar{\mathbf{X}}(t) = \mathbf{X} = (S, v, B)$.

Suppose that, fixed a maturity T ($\tau = T - t$), we want to price the contingent claim which values 1 at maturity. In order to avoid any arbitrage opportunity, this claim has to be the bond,

$$U(S, v, B, \tau) = F(\tau)e^{G(\tau)v},$$

thus, it must hold that

$$
\begin{aligned}
&-\left(F'(\tau)e^{G(\tau)v}+F(\tau)G'(\tau)ve^{G(\tau)v}\right)+\frac{1}{2}\sigma^2 vF(\tau)G^2(\tau)e^{G(\tau)v}+\\
&+[k(\theta-v)-\lambda(S,v,B,t)]F(\tau)G(\tau)e^{G(\tau)v}-rF(\tau)e^{G(\tau)v}=0.
\end{aligned}
$$

On the other hand, by construction of the bond, we know that

$$
\begin{aligned}
&-\left(F'(\tau)e^{G(\tau)v}+F(\tau)G'(\tau)ve^{G(\tau)v}\right)+\frac{1}{2}\sigma^2 vF(\tau)G^2(\tau)e^{G(\tau)v}+\\
&+[k(\theta-v)-\lambda v]F(\tau)G(\tau)e^{G(\tau)v}-(\mu+\beta v)F(\tau)e^{G(\tau)v}=0.
\end{aligned}
$$

We subtract both expressions and divide by $F(\tau)e^{G(\tau)v}$ to get to

$$(-\lambda(S,v,B,t)+\lambda v)\,G(\tau)+(-r+(\mu+\beta v))=0.$$

The previous expression must hold for all v,τ. From Proposition 4.1 we know that $G(\tau)\neq 0,\ \tau\neq 0$ and that $G(0)=0$. Standard arguments yield the desired result.

□

In the riskless measure, the dynamics is:

$$
\begin{cases}
d\bar{S}(t)=r\bar{S}(t)dt+\sqrt{\bar{v}(t)}\bar{S}(t)d\bar{z}_1(t),\\
d\bar{v}(t)=[k\theta-k\bar{v}(t)-\lambda\bar{v}(t)]dt+\sigma\sqrt{\bar{v}(t)}d\bar{z}_2(t),\\
d\bar{B}(t,T)=r\bar{B}(t,T)dt+\sigma G(\tau)\sqrt{\bar{v}(t)}\bar{B}(t,T)d\bar{z}_2(t),
\end{cases}
\tag{33}
$$

where the riskless rate is $\bar{r}(t)=\mu+\beta\bar{v}(t)$.

If we compare it with the original SV model of Heston (see Section 2), note that just one new parameter has appeared, β, which models the stochastic component of the bond.

Proposition 4.2 states that, as β approaches to 0^+, the function which gives the bond price $B(\tau,v)$ converges, for any fixed v, to $e^{-\mu\tau}$, which is the price of a bond when constant risk free rates are employed. Therefore, the original SV model can be considered a particular case of this one and we allow $\beta\geq 0$ where $\beta=0$ denotes the the original SV model.

Now we are going to develop a semi-explicit formula. We point that Heston conjectured in [6] a solution for the extended model:

$$U(t,x,P,v)=e^x B(t,T)R_1(t,x,v)-KB(t,T)R_2(t,x,v),$$

where $R_j,\ j\in\{1,2\}$ satisfies (16)-(17).

Substituting the parameter restrictions (32) into (16)-(17), we obtain

$$\frac{1}{2}\sigma_x^2 v\frac{\partial^2 R_j}{\partial x^2}+\rho_{xv}\sigma_x\sigma v\frac{\partial^2 R_j}{\partial x\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 R_j}{\partial v^2}+\zeta_j v\frac{\partial R_j}{\partial x}+(a-b_j v)\frac{\partial R_j}{\partial v}+\frac{\partial R_j}{\partial t}=0, \tag{34}$$

where

$$
\begin{aligned}
&\frac{1}{2}\sigma_x^2=\frac{1}{2}-\rho_{sv}\sigma G(\tau)+\frac{1}{2}\sigma^2 G^2(\tau),\quad \rho_{xv}=\frac{\rho_{sv}-\sigma G(\tau)}{\sigma_x},\\
&\zeta_1=\frac{1}{2}\sigma_x^2,\quad \zeta_2=-\frac{1}{2}\sigma_x^2,\quad a=k\theta,\\
&b_1=k+\lambda-\rho_{sv}\sigma,\quad b_2=k+\lambda-\sigma^2 G(\tau).
\end{aligned}
\tag{35}
$$

The following result is proved in Appendix in [6].

Lemma 5.1. *Let $\tau = T - t$. The solution of equation*

$$\frac{1}{2}\sigma_x^2 v \frac{\partial^2 f_j}{\partial x^2} + \rho_{xv}\sigma_x \sigma v \frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f_j}{\partial v^2} + \zeta_j v \frac{\partial f_j}{\partial x} + (a - b_j v)\frac{\partial f_j}{\partial v} - \frac{\partial f_j}{\partial \tau} = 0, \tag{36}$$

subject to $f_j(x, v, 0; \phi) = e^{i\phi x}$, $j \in \{1, 2\}$ is the characteristic function of R_j.

In order to obtain the solution of (36), the characteristic function is conjectured to be

$$f_j(x, v, \tau, \phi) = e^{C_j(\tau,\phi) + D_j(\tau,\phi)v + i\phi x}.$$

Thus it holds that:

$$\begin{aligned}
&\frac{\partial f}{\partial t} = f\left(\frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}v\right) = f\left(-\frac{\partial C}{\partial \tau} - \frac{\partial D}{\partial \tau}v\right)\\
&\frac{\partial f}{\partial x} = fi\phi, \quad \frac{\partial f}{\partial v} = fD,\\
&\frac{\partial^2 f}{\partial x^2} = -f\phi^2, \quad \frac{\partial^2 f}{\partial v^2} = fD^2, \quad \frac{\partial^2 f}{\partial v \partial x} = i\phi D f.
\end{aligned}$$

Substituting in the PDE (36), we come to:

$$-\frac{1}{2}\sigma_x^2 v f \phi^2 + \rho_{xv}\sigma_x \sigma v i\phi D f + \frac{1}{2}\sigma^2 v f D^2 + u_j v f i\phi + (a - b_j v) f D + f\left(-\frac{\partial C}{\partial \tau} - \frac{\partial D}{\partial \tau}v\right) = 0.$$

As the previous expression is a second degree polynomial identity in v, we obtain the next two equations:

$$\begin{cases}
-\frac{1}{2}\sigma_x^2 \phi^2 + \rho_{xv}\sigma_x \sigma i\phi D + \frac{1}{2}\sigma^2 D^2 + u_j i\phi - b_j D - \frac{\partial D}{\partial \tau} = 0,\\
aD - \frac{\partial C}{\partial \tau} = 0,
\end{cases}$$

plus the condition $C(0) = D(0) = 0$.

The first equation is a Ricatti equation, but as $\sigma_x(t)$ depends on time and not being constant, a direct solution has not been found and it has to be solved numerically, for example, by means of the routine of Matlab ode45.

Finally, the price of the European call option is computed with (15), where

$$R_j(x, v, \tau, \ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi \ln(K)} f_j(x, v, \tau, \phi)}{i\phi}\right] d\phi,$$

and $f_j(x, v, \tau, \phi) = e^{C_j(\tau,\phi) + D_j(\tau,\phi)v + i\phi x}$.

6 Numerical Experiments

In this section we are going to study the effect of incorporating stochastic interest rates to the pricing model. As in [6], we perform the change of variables

$$k^* = k + \lambda, \quad \theta^* = \frac{k\theta}{k + \lambda},$$

since, in the risk-neutralized process, the variance drifts toward a long-run mean θ^*, with mean-reversion speed k^*.

The model parameter values are:

$$k^* = 2, \quad \theta^* = 0.01, \quad v(0) = 0.01, \quad \rho_{vs} = -0.5, \quad \sigma = 0.1, \quad \mu = 0.04, \quad \beta = 0.5.$$

First, we are going to study the stochastic bond. For a maturity $T \in [0,1]$, we are going to compute the price of the riskless unit discount bond $\bar{B}(0,T;\bar{v})$.

In order to study how interest rates evolve, we have also computed the equivalent constant interest rate r^c_T, i.e. for the same maturities, we have computed the values r^c_T such that:

$$e^{-r^c_T T} = \bar{B}(0,T;\bar{v})$$

The results are plotted in Figure 1, where in the left side we have plotted the bond price and in the right side the equivalent constant interest rate r^c_T.

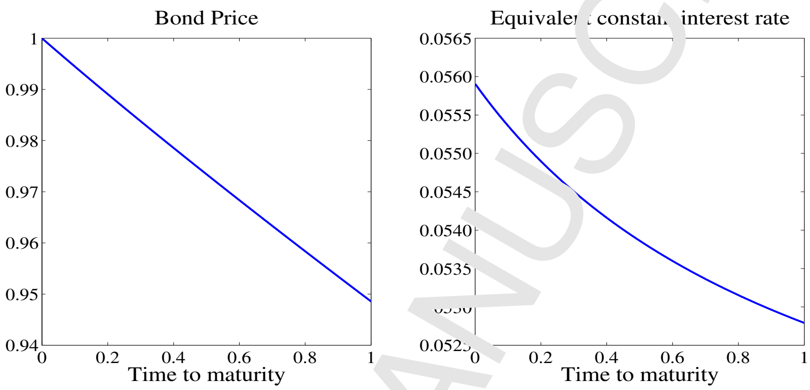

Figure 1: Price of the riskless unit discount bond $\bar{B}(0,T;\bar{v})$ (left) and equivalent constant interest rate (right).

The value r^c_T decreases as maturity grows but, as the variance is a mean-reversion process, r^c_T converges and, as $T \to \infty$, the value of $r^c_T \to 0.0491$.

Now we are going to study the effects of the stochastic interest rates in option prices. For a stock price $S(0) \in [70,130]$, a strike $K = 100$, a maturity $T = 0.5$ years and the same parameter values of the previous experiment, we have computed the price of the European call option with the SVSIR model (see Section 5).

In order to compare it with Heston's model, we have computed the price of the European call option with the original SV model (see Section 2) for $r_A = 0.056$ and $r_B = 0.053$, which, for a maturity of $T = 0.5$, are respectively above and below the values of r^c_T (see Figure 1).

As we can see in Figure 2, the option prices of the SVSIR model are consistent with option prices of the SV model with nearby interest rates.

7 Conclusions

We have presented an extension of Heston's SV model which, just increasing in one the number of parameters, allows to incorporate SIR to the pricing model of the European call option. We have also derived a semi-explicit formula which is easy to implement, for example, with Matlab routines.

Although the option price can be computed fast with a small numerical error, one of the main concerns in option pricing is that we usually need to compute a large amount of contracts with different strikes, maturities and even different parameter values. This problem can be radically reduced employing the reduced basis method presented in [5], where the model presented in this work can be employed to construct the polynomial basis which interpolate the value function for several different parameter values at the same time.

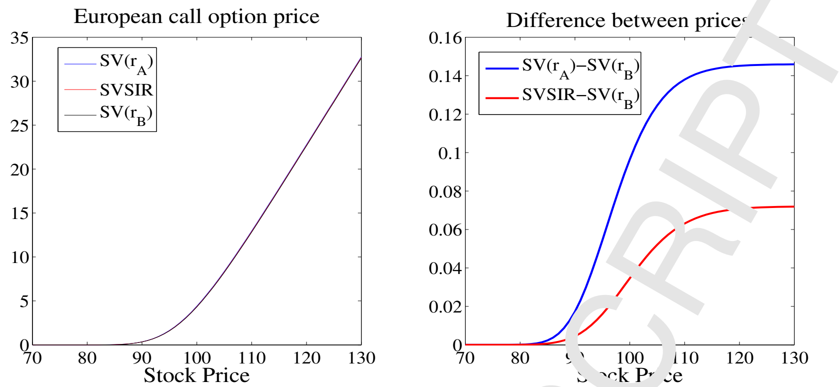

Figure 2: Price (left) and difference between prices (right) of European call options. In the left, the colour code is blue for SV(r_A),red for SVSIR and black for SV(r_B) models. In the right side, the colour code is blue for SV(r_A)-SV(r_B) and red for SVSIR-SV(r_B).

Further work may include the analysis of this model with real market data. Since we have an explicit formula for the bond price, the estimation of market parameter values can be done not only adjusting to option prices, but also enforcing the price of, for example, the US-Treasure bond.

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