Journal of Computational and Applied Mathematics 33 (1990) 181-198
North-Holland

Disequilibrium and variational inequalities

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Received 1 March 1990
Revised 25 June 1990

Abstract: In this paper we introduce a new market disequilibrium model in a spatial economic setting, which generalizes a recent spatial disequilibrium model to the asymmetric case. We derive two alternative variational inequality formulations of the market conditions, in the case of price rigidities and/or controls, and discuss existence and uniqueness properties. We then propose a decomposition algorithm which resolves the variational inequality problem into three distinct and simpler variational inequality subproblems with special structure, which are then solved in sequential fashion. Any appropriate algorithm can then be used to solve the individual subproblems. The first variational inequality subproblem, however, is identical to the one governing the well-known spatial price equilibrium problem and, hence, a plethora of algorithms are available for its solution. We conclude with computational experience with the decomposition algorithm on large-scale market examples.

This work bridges the study of disequilibrium and equilibrium problems via the theory of variational inequalities.

Keywords: Disequilibrium, spatial equilibrium, economics, variational inequalities.

1. Introduction

In this paper we show that the theory of variational inequalities can be utilized for the study of economic market problems in *disequilibrium*. The analysis is conducted for a new economic model in a spatial setting.

Heretofore, the methodology of variational inequalities has been used exclusively in the formulation and solution of a spectrum of *equilibrium* problems, in which the markets clear. Examples of such applications in economics and operations research include: the traffic network equilibrium problem (see, e.g., [1,3,5-7,18,33]), spatial price equilibrium problems [8,11,17,19,24,28-30], oligopolistic market equilibrium problems [13,20,21,26], and general economic equilibrium problems [4,8,37]. For an overview of the theory and applications, see [9] and [25].

Recently, Thore [36] introduced the concept of spatial disequilibrium and showed that in the special case of separable supply and demand price functions, and fixed transportation cost

* This author's research was supported by NSF Grant RII-8800361 under the sponsorship of the NSF VPW program while the author was visiting MIT.
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functions, the well-known [32] optimization reformulation of the spatial price equilibrium problem could be extended to handle the case of disequilibrium caused by rigid prices and/or price controls. However, in the case of asymmetries in the governing functions, such an approach could no longer be used. The concept of disequilibrium had been studied in general, rather than partial economic systems, by, amongst others, Dreze [15], Malinvaud [23], and Artus, Laroque and Michel [2]. Thompson and Thore [35] present both optimization and complementarity formulations for models of economic disequilibrium.

Our goal, hence, is to bridge the study of equilibrium and disequilibrium problems through the unifying framework of variational inequalities.

In Section 2, we introduce an economic market model in which there are several producers and several consumers which can, in general, be spatially separated. The supply price at a supply market may depend upon the total supplies of the commodity at every supply market. Similarly, the demand price at a demand market may depend upon the total demands of the commodity at every demand market. The transaction cost between a pair of supply and demand markets is surcharged with a unit transaction cost, which also includes the transportation cost. The transaction cost may depend upon the commodity shipments between every pair of supply and demand markets.

In this model the supply price at a supply market can be regulated by a fixed minimum price level, whereas the demand price at a demand market can be regulated by a fixed maximum demand price level. Such regulatory instruments of price floors and ceilings are used by, for example, governments in the case of agricultural commodities and energy resources. We then state the market conditions governing disequilibrium and give alternative variational inequality (VI) formulations of the problem.

In Section 3, we discuss the qualitative properties of the market disequilibrium model and give conditions for existence and uniqueness.

In Section 4, we present a decomposition algorithm and establish conditions for convergence. The decomposition algorithm resolves the original VI problem into three simpler and distinct VI subproblems which are then solved in sequential fashion. This decomposition algorithm, therefore, allows one the flexibility of selecting any appropriate algorithm for the computation of the individual variational inequality subproblems. The first VI subproblem, however, has a structure identical to the one characterizing the spatial price equilibrium problem and, hence, is particularly amenable to solution by any of the existing algorithms developed especially for this problem with special structure (cf. [14,24,25,30]). The second and third VI subproblems reflect, respectively, the excess supply and excess demand side conditions and possess a simple special structure.

In Section 5, we provide computational results for the decomposition procedure for large-scale market problems. We conclude with a summary and discussion in Section 6.

2. The market model

In this section we introduce a generalized version of the Thore [36] model of spatial disequilibrium which also generalizes the market model of Dafermos and Nagurney [12] (see, also, [24]) to the case of disequilibrium.

We assume that a certain commodity is produced at m supply markets and is consumed at n demand markets. We denote a typical supply market by i and a typical demand market by j. Let s_i denote the total supply associated with supply market i, d_j the total demand associated with demand market j, and let Q_{ij} denote the nonnegative commodity shipment between the pair of supply and demand markets (i, j). We group the supplies s_i into a column vector s in $\mathbb{R}^m$, the demands d_j into a column vector d in $\mathbb{R}^n$, and the commodity shipments Q_{ij} into a column vector Q in $\mathbb{R}^{mn}$. We let u_i denote the nonnegative possible excess supply at supply market i and v_j the nonnegative possible excess demand at demand market j. We then group the excess supplies into a column vector u in $\mathbb{R}^m$ and the excess demands into a column vector v in $\mathbb{R}^n$.

The following equations must hold:

$$s_i = \sum_j Q_{ij} + u_i, \quad i = 1,\dots, m, \tag{1}$$

and

$$d_j = \sum_i Q_{ij} + v_j, \quad j = 1,\dots, n. \tag{2}$$

The feasible set $K^1 = (s, d, Q, u, v)$ is then defined such that (1) and (2) hold.

We further associate with each supply market i a supply price π_i and with each demand market j a demand price ρ_j. We also assume that there is a fixed minimum supply price $\underline{\pi}_i$ for each supply market i and a fixed maximum demand price $\bar{\rho}_j$ at each demand market j. Thus $\underline{\pi}_i$ represents the price floor imposed upon the producers at supply market i, whereas $\bar{\rho}_j$ represents the price ceiling imposed at the demand market j. We group the supply prices and demand prices into respective row vectors π in $\mathbb{R}^m$ and ρ in $\mathbb{R}^n$. Similarly, we group the supply price floors into a row vector $\underline{\pi}$ in $\mathbb{R}^m$ and the demand price ceilings into a row vector $\bar{\rho}$ in $\mathbb{R}^n$. We also define the vector $\tilde{\pi}$ in $\mathbb{R}^{mn}$ consisting of m vectors, where the ith vector, $\{\tilde{\pi}_i\}$, consists of n components $\{\pi_i\}$. Similarly, we define the vector $\tilde{\rho}$ in $\mathbb{R}^{mn}$ consisting of m vectors $\{\tilde{\rho}_j\}$ in $\mathbb{R}^n$ with components $\{\rho_1, \rho_2, \dots, \rho_n\}$.

The unit transaction cost, which includes the tranportation cost, associated with the market pair (i, j) is denoted by c_{ij}. The costs are then grouped into a row vector c in $\mathbb{R}^{mn}$.

The economic market conditions for the above model, assuming perfect competition, take, cf. Thore [36], the following form: For all pairs of supply and demand markets (i, j), $i = 1,\dots, m$, $j = 1,\dots, n$:

$$\pi_i + c_{ij} \begin{cases} = \rho_j, & \text{if } Q_{ij} > 0, \\ \geqslant \rho_j, & \text{if } Q_{ij} = 0, \end{cases} \tag{3}$$

$$\pi_i \begin{cases} = \underline{\pi}_i, & \text{if } u_i > 0, \\ \geqslant \underline{\pi}_i, & \text{if } u_i = 0, \end{cases} \tag{4}$$

$$\rho_j \begin{cases} = \bar{\rho}_j, & \text{if } v_j > 0, \\ \leqslant \bar{\rho}_j, & \text{if } v_j = 0. \end{cases} \tag{5}$$

Conditions (3) are the well-known Samuelson [32], Takayama and Judge [34] equilibrium conditions. Conditions (4) state that the supply price at each supply market i must be greater than or equal to the imposed supply price floor at i. If there is an excess supply at i, then the supply price must be equal to the supply price floor at i. Conditions (5) state that the demand price at each demand market j cannot exceed the demand price ceiling at j. In the case of excess demand at j, then the demand price must be equal to the demand price ceiling at j.

In the absence of price floors and price ceilings, where (4) and (5) are absent, the above model collapses to the model of Dafermos and Nagurney [12] which had been solved in [24].

We now discuss the supply price, demand price, and transaction cost structure. We assume that the supply price associated with any supply market may depend upon the total supply of the commodity at every supply market, that is,

$$\pi = \pi(s), \tag{6}$$

where π is a known smooth function and, similarly, the demand price associated with any demand market may depend upon the total demand for the commodity at every demand market, that is,

$$\rho = \rho(d), \tag{7}$$

where ρ is a known smooth function.

The transaction cost between a pair of supply and demand markets may depend, in general, upon the shipments of the commodity between every pair of markets, that is,

$$c = c(Q), \tag{8}$$

where c is a known smooth function.

Note that the level of generality of the governing functions is identical to that in general spatial price equilibrium problems (cf. [11,12,17]).

In the special case where the number of supply markets m is equal to the number of demand markets n, the supply price and the demand price functions π and ρ are assumed to be separable, and the transaction cost c are assumed to be fixed and equal to the transportation cost, the above model collapses to the one introduced in [36]. In this symmetric case, as was shown therein, there is an equivalent optimization formulation of conditions (3), (4) and (5).

We further define the vectors $\hat{\pi} = \pi \in \mathbb{R}^m$, and $\hat{\rho} = \rho \in \mathbb{R}^n$. In view of the feasibility conditions (1) and (2), we can express $\hat{\pi}$ and $\hat{\rho}$ in the following manner:

$$\hat{\pi} = \hat{\pi}(Q, u) \quad \text{and} \quad \hat{\rho} = \hat{\rho}(Q, v). \tag{9}$$

We also define the vector $\tilde{\hat{\pi}} \in \mathbb{R}^{mn}$ consisting of m vectors, where the ith vector, $\{\tilde{\hat{\pi}}_i\}$, consists of n components $\{\hat{\pi}_i\}$ and the vector $\tilde{\hat{\rho}} \in \mathbb{R}^{mn}$ consisting of m vectors $\{\hat{\rho}_j\} \in \mathbb{R}^n$ with components $\{\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_n\}$.

As mentioned in the Introduction, a spectrum of equilibrium problems has been formulated and studied as variational inequality problems. We now show that the above system (3), (4), and (5) of equilibrium/disequilibrium conditions can also be formulated as a variational inequality problem.

Theorem 1. *A pattern of total supplies, total demands, commodity shipments, excess supplies and excess demands* $(s, d, Q, u, v) \in K^1$ *satisfies inequalities* (3), (4), *and* (5) *governing the disequilibrium market problem if and only if it satisfies the variational inequality*

$$(\tilde{\pi}(s) + c(Q) - \tilde{\rho}(d)) \cdot (Q' - Q) + (\pi(s) - \underline{\pi}) \cdot (u' - u) + (\bar{\rho} - \rho(d)) \cdot (v' - v) \geqslant 0$$
$$\textit{for all } (s', d', Q', u', v') \in K^1, \tag{10}$$

or, equivalently, the variational inequality

$$\left(\tilde{\hat{\pi}}(Q, u) + c(Q) - \tilde{\hat{\rho}}(Q, v)\right) \cdot (Q' - Q) + \left(\hat{\pi}(Q, u) - \underline{\pi}\right) \cdot (u' - u) + (\bar{\rho} - \hat{\rho}(Q, v)) \cdot (v' - v) \geqslant 0 \quad \textit{for all } (Q', u', v') \in K^2 \equiv \mathbb{R}_+^{mm} \times \mathbb{R}_+^m \times \mathbb{R}_+^m. \tag{11}$$

Proof. Assume that a $(s, d, Q, u, v) \in K^1$ satisfies (3), (4), and (5). Then for each pair (i, j), and any $Q'_{ij} \geqslant 0$:

$$\left(\pi_i(s) + c_{ij}(Q) - \rho_j(d)\right) \cdot \left(Q'_{ij} - Q_{ij}\right) \geqslant 0. \tag{12}$$

Summing over all pairs (i, j) we have that

$$\left(\tilde{\pi}(s) + c(Q) - \tilde{\rho}(d)\right) \cdot (Q' - Q) \geqslant 0. \tag{13}$$

Using similar arguments, we obtain

$$\left(\pi(s) - \underline{\pi}\right) \cdot (u' - u) \geqslant 0 \quad \text{and} \quad (\bar{\rho} - \rho(d)) \cdot (v' - v) \geqslant 0. \tag{14}$$

Summing then the inequalities (13) and (14),

$$\left(\tilde{\pi}(s) + c(Q) - \tilde{\rho}(d)\right) \cdot (Q' - Q) + \left(\pi(s) - \underline{\pi}\right) \cdot (u' - u) + (\bar{\rho} - \rho(d)) \cdot (v' - v) \geqslant 0. \tag{15}$$

Analogously, by definition of $\hat{\pi}$ and $\hat{\rho}$, we obtain that if $(Q, u, v) \in K^2$ satisfies (3), (4), (5), then

$$\left(\tilde{\hat{\pi}}(Q, u) + c(Q) - \tilde{\hat{\rho}}(Q, v)\right) \cdot (Q' - Q) + \left(\hat{\pi}(Q, u) - \underline{\pi}\right) \cdot (u' - u) + (\bar{\rho} - \hat{\rho}(Q, v)) \cdot (v' - v) \geqslant 0. \tag{16}$$

Assume now that VI (10) holds. Let $u' = u$ and $v' = v$. Then

$$\left(\tilde{\pi}(s) + c(Q) - \tilde{\rho}(d)\right) \cdot (Q' - Q) \geqslant 0, \tag{17}$$

which, in turn, implies that (3) holds. Similar arguments demonstrate that (4) and (5) also then hold.

By definition, we can establish the same inequalities, when we utilize the functions $\hat{\pi}(Q, u)$ and $\hat{\rho}(Q, v)$. □

3. Properties of the disequilibrium solution

We have shown in Section 2 that the spatial market disequilibrium problem can be cast into a variational inequality problem (11) over the unbounded Cartesian product set $K^2 = K_1 \times K_2 \times K_3$, where $K_1 = \mathbb{R}_+^{mn}$, $K_2 = \mathbb{R}_+^m$, and $K_3 = \mathbb{R}_+^n$. This variational inequality is usually an asymmetric variational inequality. In this section we will study several properties of the disequilibrium solution (Q, u, v), in particular, existence and uniqueness.

We first establish the existence conditions.

Denote the row vector $F(Q, u, v)$ by

$$F(Q, u, v) \equiv \left(\tilde{\hat{\pi}}(Q, u) + c(Q) - \tilde{\hat{\rho}}(Q, v),\ \hat{\pi}(Q, u) - \underline{\pi},\ \bar{\rho} - \hat{\rho}(Q, v)\right). \tag{18}$$

It is well known (see [22]) that variational inequality (11) admits at least one solution provided that the function $F(Q, u, v)$ is coercive. More precisely, we have the following theorem.

Theorem 2. *Assume that the function $F(Q, u, v)$ is coercive, i.e., there exists a point $(Q^0, u^0, v^0) \in K^2$, such that*

$$\lim_{\|(Q,u,v)\| \to \infty} \frac{\left(F(Q, u, v) - F(Q^0, u^0, v^0)\right)\begin{pmatrix} Q - Q^0 \\ u - u^0 \\ v - v^0 \end{pmatrix}}{\|(Q - Q^0, u - u^0, v - v^0)\|} = \infty$$

$$\text{for all } (Q, u, v) \in K^2. \tag{19}$$

Then variational inequality (11) *admits at least one solution or, equivalently, a disequilibrium solution exists.*

One of the sufficient conditions ensuring (19) in Theorem 2 is that the function $F(Q, u, v)$ is strongly monotone, that is, the following inequality holds:

$$\left[F(Q^1, u^1, v^1) - F(Q^2, u^2, v^2)\right]\left[\begin{pmatrix} Q^1 \\ u^1 \\ v^1 \end{pmatrix} - \begin{pmatrix} Q^2 \\ u^2 \\ v^2 \end{pmatrix}\right] \geqslant \alpha \left\|\begin{pmatrix} Q^1 - Q^2 \\ u^1 - u^2 \\ v^1 - v^2 \end{pmatrix}\right\|^2$$

$$\text{for all } (Q^1, u^1, v^1), (Q^2, u^2, v^2) \in K^2, \tag{20}$$

where α is a positive constant.

Under the same condition (20) uniqueness of the solution pattern (Q, u, v) is guaranteed.

We are able to show through the subsequent lemmas, that strong monotonicity of $F(Q, u, v)$ is equivalent to the strong monotonicity of the transaction cost $c(Q)$, the supply price $\pi(s)$, and the demand price $\rho(d)$ functions, which is a commonly imposed condition in the study of the spatial price equilibrium problem (see, e.g., [6,8,11,25,30]).

Lemma 3. *Let (Q, s, d) be a vector associated with $(Q, u, v) \in K^2$ via* (1) *and* (2). *There exist positive constants m_1 and m_2 such that:*

$$\|(Q, u, v)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}} \leqslant m_1 \|(Q, s, d)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}} \tag{21}$$

and

$$\|(Q, s, d)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}} \leqslant m_2 \|(Q, u, v)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}}, \tag{22}$$

where $\|\cdot\|_{\mathbf{R}^k}$ denotes the norm in the space $\mathbf{R}^k$.

Proof. For any $(Q, u, v)^{\mathrm{T}} \in K^2$ we have:

$$\|(Q, u, v)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}} = \|Q\|^2_{\mathbf{R}^{mn}} + \|u\|^2_{\mathbf{R}^m} + \|v\|^2_{\mathbf{R}^n}. \tag{23}$$

Substituting (1) and (2) into (23) yields:

$$\|(Q, u, v)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}} = \|Q\|^2_{\mathbf{R}^{mn}} + \left\|\begin{pmatrix} s_1 - \Sigma_j Q_{1j} \\ \vdots \\ s_m - \Sigma_j Q_{mj} \end{pmatrix}\right\|^2_{\mathbf{R}^m} + \left\|\begin{pmatrix} d_1 - \Sigma_i Q_{i1} \\ \vdots \\ d_n - \Sigma_i Q_{in} \end{pmatrix}\right\|^2_{\mathbf{R}^n}$$

$$\leqslant \|Q\|^2_{\mathbf{R}^{mn}} + \|s\|^2_{\mathbf{R}^m} + \|d\|^2_{\mathbf{R}^n} + \left\|\left(\sum_j Q_{1j}, \ldots, \sum_j Q_{mj}\right)^{\mathrm{T}}\right\|^2_{\mathbf{R}^m}$$

$$+ \left\|\left(\sum_i Q_{i1}, \ldots, \sum_i Q_{in}\right)^{\mathrm{T}}\right\|^2_{\mathbf{R}^n}. \tag{24}$$

There exists an $m_1 > 1$ such that:

$$\left\|\left(\sum_j Q_{1j}, \ldots, \sum_j Q_{mj}\right)^{\mathrm{T}}\right\|^2_{\mathbf{R}^m} \leqslant \frac{m_1 - 1}{2} \|Q\|^2_{\mathbf{R}^m} \tag{25}$$

and

$$\left\|\left(\sum_i Q_{i1}, \ldots, \sum_i Q_{in}\right)^{\mathrm{T}}\right\|^2_{\mathbf{R}^n} \leqslant \frac{m_1 - 1}{2} \|Q\|^2_{\mathbf{R}^n}. \tag{26}$$

A combination of (24), (25), and (26) yields:

$$\|(Q, u, v)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}} \leqslant m_1 \|Q\|^2_{\mathbf{R}^{mn}} + \|s\|^2_{\mathbf{R}^m} + \|d\|^2_{\mathbf{R}^n}$$
$$\leqslant m_1\left[\|Q\|^2_{\mathbf{R}^{mn}} + \|s\|^2_{\mathbf{R}^m} + \|d\|^2_{\mathbf{R}^n}\right] = m_1 \|(Q, s, d)^{\mathrm{T}}\|^2_{\mathbf{R}^{mn+m+n}}. \tag{27}$$

Similarly, we can prove that (22) holds for large enough m_2. The proof is complete. □

We now state the following lemma.

Lemma 4. *$F(Q, u, v)$ is a strongly monotone function of (Q, u, v), if and only if $\pi(s)$, $c(Q)$, and $-\rho(d)$ are strongly monotone functions of s, Q, and d, respectively.*

Proof. We always have the following relationships:

$$\left[F(Q^1, u^1, v^1) - F(Q^2, u^2, v^2)\right]\left[\begin{pmatrix} Q^1 \\ u^1 \\ v^1 \end{pmatrix} - \begin{pmatrix} Q^2 \\ u^2 \\ v^2 \end{pmatrix}\right] \tag{28}$$

$$= \left[\tilde{\pi}(Q^1, u^1) + c(Q^1) - \tilde{\rho}(Q^1, v^1) - \tilde{\pi}(Q^2, u^2) - c(Q^2) + \tilde{\rho}(Q^2, v^2)\right](Q^1 - Q^2)$$
$$+ \left[\hat{\pi}(Q^1, u^1) - \hat{\pi}(Q^2, u^2)\right](u^1 - u^2) + \left[\hat{\rho}(Q^2, v^2) - \hat{\rho}(Q^1, v^1)\right](v^1 - v^2)$$

$$= \left[\hat{\pi}(Q^1, u^1) - \hat{\pi}(Q^2, u^2)\right](s^1 - u^1 - s^2 + u^2) + \left[c(Q^1) - c(Q^2)\right](Q^1 - Q^2)$$
$$+ \left[\hat{\rho}(Q^2, v^2) - \hat{\rho}(Q^1, v^1)\right](d^1 - v^1 - d^2 + v^2)$$
$$+ \left[\hat{\pi}(Q^1, u^1) - \hat{\pi}(Q^2, u^2)\right](u^1 - u^2) + \left[\hat{\rho}(Q^2, v^2) - \hat{\rho}(Q^1, v^1)\right](v^1 - v^2)$$
$$= \left[\pi(s^1) - \pi(s^2)\right](s^1 - s^2) + \left[c(Q^1) - c(Q^2)\right](Q^1 - Q^2)$$
$$- \left[\rho(d^1) - \rho(d^2)\right](d^1 - d^2). \tag{29}$$

If $\pi(s)$, $c(Q)$, and $-\rho(d)$ are strongly monotone functions of s, Q, and d, respectively, we have that

$$\left(\pi(s^1) - \pi(s^2)\right)(s^1 - s^2) + \left(c(Q^1) - c(Q^2)\right)(Q^1 - Q^2) + \left(\rho(d^2) - \rho(d^1)\right)(d^1 - d^2)$$
$$\geqslant \alpha_1 \left\| (Q^1 - Q^2, u^1 - u^2, v^1 - v^2)^{\mathrm{T}} \right\|, \tag{30}$$

where $\alpha_1 > 0$. Recalling expressions (21) and (29), we obtain:

$$\left[F(Q^1, u^1, v^1) - F(Q^2, u^2, v^2)\right]\left[\begin{pmatrix} Q^1 \\ u^1 \\ v^1 \end{pmatrix} - \begin{pmatrix} Q^2 \\ u^2 \\ v^2 \end{pmatrix}\right] \geqslant \frac{\alpha_1}{m_1}\left\|\begin{pmatrix} Q^1 - Q^2 \\ u^1 - u^2 \\ v^1 - v^2 \end{pmatrix}\right\|^2 \tag{31}$$

which implies that $F(Q, u, v)$ is a strongly monotone function of (Q, u, v).

Conversely, if $F(Q, u, v)$ is a strongly monotone function of (Q, u, v), we have that

$$\left[F(Q^1, u^1, v^1) - F(Q^2, u^2, v^2)\right]\left[\begin{pmatrix} Q^1 - Q^2 \\ u^1 - u^2 \\ v^1 - v^2 \end{pmatrix}\right] \geqslant \alpha_2\left\|\begin{pmatrix} Q^1 - Q^2 \\ u^1 - u^2 \\ v^1 - v^2 \end{pmatrix}\right\|^2, \tag{32}$$

where $\alpha_2 > 0$. Substituting now (22) into (32) and recalling (29), we obtain

$$\left[\pi(s^1) - \pi(s^2)\right](s^1 - s^2) + \left[c(Q^1) - c(Q^2)\right](Q^1 - Q^2) + \left[\rho(d^2) - \rho(d^1)\right](d^1 - d^2)$$
$$\geqslant \frac{\alpha_2}{m_2}\left\|\begin{pmatrix} Q^1 - Q^2 \\ s^1 - s^2 \\ d^1 - d^2 \end{pmatrix}\right\|^2, \tag{33}$$

which implies that $\pi(s)$, $c(Q)$, and $-\rho(d)$ are strongly monotone functions of s, Q, and d, respectively. The proof is complete. □

At this point, we arrive at the following proposition.

Proposition 5. *Assume that $\pi(s)$, $c(Q)$, and $-\rho(d)$ are strongly monotone functions of s, Q, and d, respectively. Then there exists precisely one disequilibrium point $(Q, u, v) \in K^2$.*

Furthermore, it is well known (cf. [22]) that a variational inequality admits at most a single solution if the function entering the variational inequality is strictly monotone. Using formula (29) we, hence, obtain the following lemma.

Lemma 6. *$F(Q, u, v)$ is strictly monotone if and only if $\pi(s)$, $c(Q)$, and $-\rho(d)$ are strictly monotone functions of s, Q, and d, respectively.*

It is now clear that the following statement is true.

Theorem 7. *Assume that $\pi(s)$, $c(Q)$, and $-\rho(d)$ are strictly monotone in s, Q, and d, respectively. Then the disequilibrium solution $(Q, u, v) \in K^2$ is unique, if one exists.*

By further observation, we can see that if $\pi(s)$ and $-\rho(d)$ are monotone, then the disequilibrium commodity shipment Q is unique, provided that $c(Q)$ is a strictly monotone function of Q.

Existence and uniqueness of a disequilibrium solution (Q, u, v), therefore, crucially depend on the strong (strict) monotonicity of the functions $c(Q)$, $\pi(s)$, and $-\rho(d)$. If the Jacobian matrix of the transaction cost function $c(Q)$ is positive definite (strongly positive definite), i.e.,

$$x^{\mathrm{T}} \nabla c(Q) x > 0 \quad \text{for all } x \in \mathbb{R}^{mn}, \quad Q \in K_1, \quad x \neq 0, \tag{34}$$

$$x^{\mathrm{T}} \nabla c(Q) x \geqslant \alpha \| x \|^2, \ \alpha > 0, \quad \text{for all } x \in \mathbb{R}^{mn}, \quad Q \in K_1, \tag{35}$$

then the function $c(Q)$ is strictly (strongly) monotone. Monotonicity of $c(Q)$ is not economically unreasonable, since the transaction cost c_{ij} from supply market i to demand market j can be expected to mainly depend upon the shipment Q_{ij} which implies that the Jacobian matrix $\nabla c(Q)$ is diagonally dominant; hence, $\nabla c(Q)$ is positive definite.

Next we explore the economic meaning of monotonicity of the supply price function $\pi(s)$ and the demand price function $\rho(d)$.

Lemma 8. *Suppose that $f: D \to V$ is continuously differentiable on set D. Let $f^{-1}: V \to D$ be the inverse function of f, where D and V are subsets of $\mathbb{R}^k$. $\nabla f(x)$ is positive definite for all $x \in D$ if and only if $\nabla(f^{-1}(y))$ is positive definite for all $y \in V$.*

Proof. Since $\nabla f(x)$ is positive definite, we have that

$$w^{\mathrm{T}} \nabla f(x) w > 0, \quad \text{for all } w \in \mathbb{R}^k, \quad x \in D, \quad w \neq 0. \tag{36}$$

It is well known that

$$\nabla(f^{-1}) = (\nabla f)^{-1}. \tag{37}$$

(36) can be written as:

$$w^{\mathrm{T}} (\nabla f)^{\mathrm{T}} (\nabla f)^{-1} (\nabla f) w > 0, \quad \text{for all } w \in \mathbb{R}^k, \quad x \in D, \quad w \neq 0. \tag{38}$$

Letting $z = \nabla f \cdot w$ in (38) and using (37), we obtain

$$z^{\mathrm{T}} \nabla(f^{-1}(y)) z > 0, \quad \text{for all } z \in \mathbb{R}^k, \quad z \neq 0, \quad y \in V. \tag{39}$$

Thus, $\nabla(f^{-1}(y))$ is positive definite. Observing that each step of the proof is convertible, we can easily prove the converse part of the lemma. □

Denote the inverse of the supply price function $\pi(s)$ by π^{-1} and the inverse of the demand price function $\rho(d)$ by ρ^{-1}. Then

$$s = \pi^{-1}(\pi), \qquad d = \rho^{-1}(\rho). \tag{40}$$

By virtue of Lemma 8, $\pi(s)$ is a strictly (strongly) monotone function of s, provided that $\nabla_\pi s(\pi)$ is positive definite (strongly positive definite) for all $\pi \in \mathbf{R}^m_+$. Similarly, $-\rho(d)$ is a strictly (strongly) monotone function of d provided that $-\nabla_\rho d(\rho)$ is positive definite (strongly positive definite) for all $\rho \in \mathbf{R}^n_+$. In reality, the supply s_i is mainly affected by the supply price π_i, for each supply market i, $i=1,\ldots,n$, and the demand d_j is mainly affected by the demand price ρ_j for each demand market j, $j=1,\ldots,n$. Thus, in most cases, we can expect the matrices $\nabla_\pi s(\pi)$ and $-\nabla_\rho d(\rho)$ to be positive definite (strongly positive definite).

4. The decomposition algorithm

In this section we describe the decomposition algorithm for the computation of the disequilibrium solution and give conditions for convergence.

Making the observation the variational inequality governing the market disequilibrium problem is defined on the Cartesian product set $K^2 = K_1 \times K_2 \times K_3$, we decompose variational inequality (11) into three simpler variational inequality problems in lower dimensions.

The algorithm computes a sequence $(Q^0, u^0, v^0), (Q^1, u^1, v^1), \ldots,$ by solving three variational inequalities sequentially, and converges to the solution of (11).

The statement of the decomposition algorithm is as follows.

Step 0. Start with any $(u^0, v^0) \in K_2 \times K_3$.

Step 1. $(t=0, 1, 2,\ldots)$ Solve the following variational inequality

$$\left[\tilde{\pi}(Q, u^t) + c(Q) - \tilde{\rho}(Q, v^t)\right](Q' - Q) \geqslant 0, \quad \text{for all } Q' \in K_1. \tag{41}$$

The solution to (41) is termed Q^t.

Step 2. $(t=0, 1, 2,\ldots)$ Solve the variational inequality

$$\left[\hat{\pi}(Q^t, u) - \underline{\pi}\right](u' - u) \geqslant 0, \quad \text{for all } u' \in K_2. \tag{42}$$

The solution to (42) is u^{t+1}.

Step 3. $(t=0, 1, 2,\ldots)$ Solve the variational inequality

$$\left[\bar{\rho} - \hat{\rho}(Q^t, v)\right](v' - v) \geqslant 0, \quad \text{for all } v' \in K_3. \tag{43}$$

The solution to (43) is v^{t+1}. Let $t = t+1$, and go to Step 1.

Variational inequalities (41), (42), and (43) will admit a solution Q^t, u^{t+1}, and v^{t+1}, respectively, provided that the functions $c(Q)$, $\pi(s)$, and $-\rho(d)$ are strongly monotone. Thus, this sequence $\{(Q^t, u^t, v^t)\}$, $t=1, 2,\ldots,$ is well defined and can be obtained by applying any appropriate algorithm for the computation of the individual variational inequalities (41), (42), and (43). In particular, VI (41) is identical to the variational inequality governing the much-studied spatial price equilibrium problem and, hence, a plethora of algorithms are available for its solution (see, e.g., [14,16,17,24,25]).

In order to simplify the notation, for purposes of establishing convergence, let us denote:

$$\begin{aligned} A &= \frac{\partial\left[\tilde{\pi}(Q, u) + c(Q) - \tilde{\rho}(Q, v)\right]}{\partial Q} \\ &= \left[\frac{\partial \pi_i}{\partial s_k} + \frac{\partial c_{ij}}{\partial Q_{kl}} - \frac{\partial \rho_j}{\partial d_l},\ i, k = 1,\ldots,m,\ j, l = 1,\ldots,n\right], \end{aligned} \tag{44}$$

$$B=\frac{\partial \pi}{\partial u}=\left[\frac{\partial \pi_i}{\partial s_j},\ i,\ j=1,2,\ldots,m\right], \tag{45}$$

$$C=-\frac{\partial \rho}{\partial v}=\left[-\frac{\partial \rho_j}{\partial d_i},\ i,\ j=1,2,\ldots,n\right], \tag{46}$$

$$R=\frac{\partial\left[\tilde{\pi}(Q,u)+c(Q)-\tilde{\rho}(Q,v)\right]}{\partial u}=\begin{bmatrix} \frac{\partial \pi_1}{\partial s_1} & \cdots & \frac{\partial \pi_1}{\partial s_m} \\ \vdots & & \vdots \\ \frac{\partial \pi_1}{\partial s_1} & \cdots & \frac{\partial \pi_1}{\partial s_m} \\ \vdots & & \vdots \\ \frac{\partial \pi_m}{\partial s_1} & \cdots & \frac{\partial \pi_m}{\partial s_m} \\ \vdots & & \vdots \\ \frac{\partial \pi_m}{\partial s_1} & \cdots & \frac{\partial \pi_m}{\partial s_m} \end{bmatrix}_{mn\times m}, \tag{47}$$

$$S=\frac{\partial\left[\tilde{\pi}(Q,u)+c(Q)-\tilde{\rho}(Q,v)\right]}{\partial v}=\begin{bmatrix} -\frac{\partial \rho_1}{\partial d_1} & \cdots & -\frac{\partial \rho_1}{\partial d_n} \\ \vdots & & \vdots \\ -\frac{\partial \rho_n}{\partial d_1} & \cdots & -\frac{\partial \rho_n}{\partial d_n} \\ \vdots & & \vdots \\ -\frac{\partial \rho_1}{\partial d_1} & \cdots & -\frac{\partial \rho_1}{\partial d_n} \\ \vdots & & \vdots \\ -\frac{\partial \rho_n}{\partial d_1} & \cdots & -\frac{\partial \rho_n}{\partial d_n} \end{bmatrix}_{mn\times n}, \tag{48}$$

$$T=\frac{\partial \hat{\pi}(Q,u)}{\partial Q}=\begin{bmatrix} \frac{\partial \pi_1}{\partial s_1} & \cdots & \frac{\partial \pi_1}{\partial s_1} & \cdots & \frac{\partial \pi_1}{\partial s_m} & \cdots & \frac{\partial \pi_1}{\partial s_m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \frac{\partial \pi_m}{\partial s_1} & \cdots & \frac{\partial \pi_m}{\partial s_1} & \cdots & \frac{\partial \pi_m}{\partial s_m} & \cdots & \frac{\partial \pi_m}{\partial s_m} \end{bmatrix}_{m\times mn}, \tag{49}$$

$$U=-\frac{\partial \hat{\rho}(Q,u)}{\partial Q}=\begin{bmatrix} -\frac{\partial \rho_1}{\partial d_1} & \cdots & -\frac{\partial \rho_1}{\partial d_n} & \cdots & -\frac{\partial \rho_1}{\partial d_1} & \cdots & -\frac{\partial \rho_1}{\partial d_n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ -\frac{\partial \rho_n}{\partial d_1} & \cdots & -\frac{\partial \rho_n}{\partial d_n} & \cdots & -\frac{\partial \rho_n}{\partial d_1} & \cdots & -\frac{\partial \rho_n}{\partial d_n} \end{bmatrix}_{n\times mn} \tag{50}$$

Then

$$\nabla_{(Q,u,v)}F(Q,u,v)=\begin{bmatrix} A & R & S \\ T & B & 0 \\ U & 0 & C \end{bmatrix}_{(mn+m+n)\times(mn+m+n)} \tag{51}$$

We now define the $(mn+m+n)\times(mn+m+n)$ matrix

$$\Omega=\begin{bmatrix} 0 & \tilde{A}^{-1/2}R\tilde{B}^{-1/2} & \tilde{A}^{-1/2}S\tilde{C}^{-1/2} \\ \tilde{B}^{-1/2}T\tilde{A}^{-1/2} & 0 & 0 \\ \tilde{C}^{-1/2}U\tilde{A}^{-1/2} & 0 & 0 \end{bmatrix}, \tag{52}$$

where

$$\tilde{A}=\tfrac{1}{2}(A+A^{\mathrm{T}}), \tag{53}$$

$$\tilde{B}=\tfrac{1}{2}(B+B^{\mathrm{T}}), \tag{54}$$

$$\tilde{C}=\tfrac{1}{2}(C+C^{\mathrm{T}}). \tag{55}$$

For any $x\in\mathbf{R}^{mn+m+n}$, we further define the block ∞-norm as:

$$\|x\|^{\infty}=\max\{\|x_1\|_{\mathbf{R}^{mn}},\ \|x_2\|_{\mathbf{R}^{m}},\ \|x_3\|_{\mathbf{R}^{n}}\}, \tag{56}$$

where $x_1\in\mathbf{R}^{mn}$, $x\in\mathbf{R}^{n}$, and $x_3\in\mathbf{R}^{n}$ are partitions of the vector x and $\|\cdot\|_{\mathbf{R}^k}$ is the standard 2-norm in the space $\mathbf{R}^k$, $k=mn,\ m,\ n$.

Associated with the so-defined block norm, we define the block ∞-norm on the matrix M as:

$$\|M\|^{\infty}=\max\Big\{\max_{\|x\|^{\infty}=1}\|M_1x\|_{\mathbf{R}^{mn}},\ \max_{\|x\|^{\infty}=1}\|M_2x\|_{\mathbf{R}^{m}},\ \max_{\|x\|^{\infty}=1}\|M_3x\|_{\mathbf{R}^{n}}\Big\}, \tag{57}$$

where M_1, M_2, and M_3 are partitions of the matrix M, i.e.,

$$M=\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix}. \tag{58}$$

We are now ready to state our convergence results (see, e.g., [31]).

Theorem 9 (Local Convergence Theorem). *Suppose that (Q,u,v) is a solution to the disequilibrium problem* (11).

Assume that the matrices A, B, C, defined in (44), (45), *and* (46) *are positive definite and that $\|\Omega\|^{\infty}<1$ at point (Q,u,v). Then there exists a neighborhood of (Q,u,v) such that whenever the initial iterate (Q^0,u^0,v^0) is chosen in it, the sequence $\{(Q^t,u^t,v^t)\}$ generated by the algorithm will converge to the locally unique solution (Q,u,v).*

Proposition 10. *Assume that the matrices A, B, and C are positive definite at the point (Q,u,v). Assume also that the following inequalities hold at the disequilibrium solution (Q,u,v):*

$$\begin{aligned} &\|\tilde{A}^{-1/2}\|\cdot\|\tilde{B}^{-1/2}\|\cdot\|R\|<\frac{1}{\sqrt{2}}, \qquad \|\tilde{A}^{-1/2}\|\cdot\|\tilde{B}^{-1/2}\|\cdot\|T\|<1,\\ &\|\tilde{A}^{-1/2}\|\cdot\|\tilde{C}^{-1/2}\|\cdot\|S\|<\frac{1}{\sqrt{2}}, \qquad \|\tilde{A}^{-1/2}\|\cdot\|\tilde{C}^{-1/2}\|\cdot\|U\|<1, \end{aligned} \tag{59}$$

where $\|\cdot\|$ denote the norm associated with the standard 2-norm on a matrix.

The sequence $\{(Q^t, u^t, v^t)\}$ *generated by the algorithm then converges locally to the locally unique solution* (Q, u, v).

Proof. Denote by

$$M_1 = (0, \tilde{A}^{-1/2} R \tilde{B}^{-1/2}, \tilde{A}^{-1/2} S \tilde{C}^{-1/2})_{mn \times (mn+m+n)}, \tag{60}$$

$$M_2 = (\tilde{B}^{-1/2} T \tilde{A}^{-1/2} 00)_{m \times (mn+m+n)}, \tag{61}$$

$$M_3 = (\tilde{C}^{-1/2} U \tilde{A}^{-1/2} 00)_{n \times (mn+m+n)}. \tag{62}$$

By virtue of (57) and the definition of matrix Ω, we have:

$$\| \Omega \|^{\infty} = \max\Big\{ \max_{\|x\|^{\infty}=1} \| M_1 x \|_{\mathbf{R}^{mn}}, \max_{\|x\|^{\infty}=1} \| M_2 x \|_{\mathbf{R}^{m}}, \max_{\|x\|^{\infty}=1} \| M_3 x \|_{\mathbf{R}^{n}} \Big\}. \tag{63}$$

Observing that

$$\max_{\|x\|^{\infty}=1} \| M_2 x \|_{\mathbf{R}^m} \leqslant \| \tilde{B}^{-1/2} T \tilde{A}^{-1/2} \|, \tag{64}$$

$$\max_{\|x\|^{\infty}=1} \| M_3 x \|_{\mathbf{R}^n} \leqslant \| \tilde{C}^{-1/2} U \tilde{A}^{-1/2} \| \tag{65}$$

and

$$\begin{aligned} \max_{\|x\|^{\infty}=1} \| M_1 x \|^2_{\mathbf{R}^{mn}} &= \max_{\|x\|^{\infty}=1} \Big[\| \tilde{A}^{-1/2} R \tilde{B}^{-1/2} x_1 \|^2_{\mathbf{R}^m} + \| \tilde{A}^{-1/2} S \tilde{C}^{-1/2} x_2 \|^2_{\mathbf{R}^n} \Big] \\ &\leqslant \| \tilde{A}^{-1/2} R \tilde{B}^{-1/2} \|^2 + \| \tilde{A}^{-1/2} S \tilde{C}^{-1/2} \|^2 \\ &\leqslant \max\{ 2 \| \tilde{A}^{-1/2} R \tilde{B}^{-1/2} \|^2, 2 \| \tilde{A}^{-1/2} S \tilde{C}^{-1/2} \|^2 \}, \end{aligned} \tag{66}$$

we obtain

$$\begin{aligned} \| \Omega \|^{\infty} \leqslant \max\{ & \sqrt{2} \| \tilde{A}^{-1/2} R \tilde{B}^{-1/2} \|, \sqrt{2} \| \tilde{A}^{-1/2} S \tilde{C}^{-1/2} \|, \| \tilde{B}^{-1/2} T A^{-1/2} \|, \\ & \| \tilde{C}^{-1/2} U \tilde{A}^{-1/2} \| \}. \end{aligned} \tag{67}$$

From assumption (58), we see that $\| \Omega \|^{\infty} < 1$. Therefore, all of the conditions in Theorem 9 are satisfied. Hence, the algorithm converges. The proof is complete. □

Theorem 11 (Global Convergence Theorem). *Assume that the matrices* A, B, *and* C *defined in* (44), (45), *and* (46) *are positive definite for all* $(Q', u', v') \in K^2$. *If there exists a scalar* b, *where* $0 < b < 1$, *such that*

$$\| \Omega \|^{\infty} \leqslant b \quad \textit{for all } (Q', u', v') \in K^2, \tag{68}$$

then for any initial vector $(Q^0, u^0, v^0) \in K^2$, *the sequence* $\{(Q^t, u^t, v^t)\}$ *generated by the algorithm will converge to the unique disequilibrium solution* (Q, u, v).

In a manner similar to the one followed in the proof of Proposition 10, we can prove the following.

Proposition 12. *Assume that the matrices A, B, and C are positive definite for all $(Q', u', v') \in K^2$. Assume also that the following inequalities hold for all $(Q', u', v') \in K^2$:*

$$\begin{aligned} &\|\tilde{A}^{-1/2}\| \cdot \|\tilde{B}^{-1/2}\| \cdot \|R\| \leqslant b\frac{1}{\sqrt{2}}, \qquad \|\tilde{A}^{-1/2}\| \cdot \|\tilde{B}^{-1/2}\| \cdot \|T\| \leqslant b, \\ &\|\tilde{A}^{-1/2}\| \cdot \|\tilde{C}^{-1/2}\| \cdot \|S\| \leqslant b\frac{1}{\sqrt{2}}, \qquad \|\tilde{A}^{-1/2}\| \cdot \|\tilde{C}^{-1/2}\| \cdot \|V\| \leqslant b, \end{aligned} \tag{69}$$

where b is a constant with $0 < b < 1$.

Then the sequence $\{(Q^t, u^t, v^t)\}$ generated by the algorithm converges to the unique disequilibrium solution (Q, u, v).

5. Computational results

In this section we consider the market model outlined in Section 2 and we provide computational experience with the decomposition algorithm proposed in Section 4.

Recall that the decomposition algorithm resolves the solution of variational inequality (11) into the solution of three simpler and distinct variational inequality problems, which are then solved in sequential fashion. Hence, this decomposition algorithm allows one the flexibility of selecting any appropriate algorithm for the computation of the individual variational inequality problems, which have special structure.

Note, however, that the first VI problem (41) to be solved is identical to the VI governing the well-known spatial price equilibrium problem (cf. [24]), whereas the second and third VI problems (42) and (43) govern, respectively, the possible excess supply and excess demand side conditions.

Hence, for the computation of VI (41) we utilize the Gauss–Seidel type serial decomposition algorithm by demand markets introduced in [24] which has performed well in practice (see, also [28,29]). For the solution of the embedded quadratic programming problem encountered at each step we utilize the demand market equilibration algorithm proposed in [14] (see, also, [27]), which solves each restricted demand market equilibrium subproblem exactly, rather than iteratively. For a theoretical analysis of such progessive equilibration algorithms, see [16]. We emphasize, however, that a plethora of algorithms can be applied to solve problem (41). For alternative algorithms and a list of references, see [25].

Variational inequalities (42) and (43) also have a special structure. Observe that the Gauss–Seidel serial linearization algorithm (akin to the decomposition algorithms over Cartesian products of sets in [24]) can be easily adapted to solve each of these problems. The resulting subproblem is a one-variable minimization problem in either an excess supply or excess demand variable which can be solved explicitly in closed form.

For all of the numerical examples, we, hence, utilized the above described algorithms for the solution of the individual variational inequality problems.

As an illustration of how the decomposition algorithm performs computationally, we consider market problems with linear asymmetric functions. Here we assume that the supply price functions are given by

$$\pi_i = \pi_i(s) = \sum_j r_{ij} s_j + t_i = \hat{\pi}_i(Q, u) = \sum_j r_{ij}\Big(\sum_k Q_{jk} + u_j\Big) + t_i, \tag{70}$$

Table 1
Computational experience on large-scale market problems

Number of cross terms = 5; CPU-time in seconds (*, **)			
(m, n)	$\underline{\pi}=0$, $\bar{\rho}=1000$	$\underline{\pi}=150$, $\bar{\rho}=250$	$\underline{\pi}=175$, $\bar{\rho}=200$
(45, 45)	2.59(0, 0)	9.09(0, 18)	15.33(20, 23)
(60, 60)	7.53(0, 0)	18.91(0, 28)	37.64(19, 33)
(75, 75)	15.13(0, 0)	27.89(0, 31)	60.14(11, 37)
(90, 90)	29.52(0, 0)	38.36(0, 35)	93.43(1, 49)

* Number of supply markets with excess supply.
** Number of demand markets with excess demand.

the demand price functions are given by

$$\rho_j=\rho_j(d)=-\sum_k m_{jk}d_k+q_j=\hat{\rho}_j(Q, v)=-\sum_k m_{jk}\left(\sum_i Q_{ik}+v_k\right)+q_j \tag{71}$$

and the transaction cost functions are given by

$$c_{ij}=c_{ij}(Q)=\sum_{kl} g_{ijkl}Q_{kl}+h_{ij}, \tag{72}$$

where the not necessarily symmetric Jacobians of the supply price and transaction cost functions are positive definite, whereas the Jacobian of the demand price functions is negative definite.

In this section we considered randomly generated market problems in which the supply price (70), demand price (71), and transaction cost functions (72) were generated uniformly in the same manner as described in [24]. In particular, the function term ranges were as follows: $r_{ii}\in[3, 10]$, $t_i\in[10, 25]$, $-m_{jj}\in[-1, -5]$, $q_j\in[150, 650]$, $g_{ijij}\in[1, 15]$, $h_{ij}\in[10, 25]$, $i=1,\dots, m$, $j=1,\dots, n$. The remaining r_{ij}, $-m_{jk}$, and g_{ijkl} terms were generated to ensure that the Jacobian matrices were strictly diagonal dominant and, hence, positive definite. We set the number of supply markets m equal to the number of demand markets n and varied the problem sizes from 45 supply markets and 45 demand markets (90 markets total) to 90 supply markets and 90 demand markets (180 markets total) in increments of 15 markets. These problems are larger than those considered in [24].

In Table 1 we fixed the number of cross terms in the functions (70), (71), and (72) to 5, whereas in Table 2, we fixed the number of cross terms to 10. The termination criterion utilized was $|\pi_i+c_{ij}-\rho_j|\leqslant\varepsilon=5$, if $Q_{ij}>0$, and $\pi_i+c_{ij}-\rho_j\geqslant-\varepsilon$ if $Q_{ij}=0$, and $\pi_i\geqslant\underline{\pi}_i$, $\rho_j\leqslant\bar{\rho}_j$, and

Table 2
Computational experience on large-scale market problems

Number of cross terms = 10; CPU-time in seconds (*, **)			
(m, n)	$\underline{\pi}=0$, $\bar{\rho}=1000$	$\underline{\pi}=150$, $\bar{\rho}=250$	$\underline{\pi}=175$, $\bar{\rho}=200$
(45, 45)	5.32(0, 0)	16.26(0, 19)	29.92(12, 20)
(60, 60)	17.12(0, 0)	32.66(0, 24)	64.17(16, 28)
(75, 75)	27.78(0, 0)	49.41(0, 30)	104.64(8, 38)
(90, 90)	22.60(0, 0)	65.78(0, 32)	147.79(5, 45)

* Number of supply markets with excess supply.
** Number of demand markets with excess demand.

$(\pi_i - \underline{\pi}_i)u_i \leqslant 5$, $(\bar{\rho}_j - \rho_j)v_j \leqslant 5$. Since verification of convergence can in itself be computationally time-consuming, especially in large-scale examples, we verified convergence for VI (41) after every other iteration. Overall verification was implemented after every other solution of the three VI problems.

The algorithm was coded in FORTRAN and compiled using the FORTVS compiler, optimization level 3 on the IBM 4381-14 mainframe at the Cornell National Supercomputer Facility. The CPU-times reported in Tables 1 and 2 are exclusive of input and output. The initial pattern was set at $Q_{ij} = 0$ for all i and j, $u_i = \max(0, (\underline{\pi}_i - t_i)/r_{ii})$, for all i, and $v_j = \max(0, (\bar{\rho}_j - q_j)/-m_{jj})$ for all j.

In each of the first column examples in Tables 1 and 2, we set $\underline{\pi}_i = 0$ for all i and $\bar{\rho}_j = 1000$ for all j. (In view of the generation of functions such price floors and ceilings would generate equilibrium solutions in the sense that the excess supplies and excess demand would be equal to zero). Hence, the reported CPU-times in these columns reflect the computational time for the decomposition algorithm to solve the market model in which all supply and demand markets clear. Note that there is overhead present which would not exist in a pure equilibrium code, since verification of convergence for the two additional VI problems is also made.

To the same problems, we then tightened the bounds in column 2 of each table where now $\underline{\pi} = 150$ and $\bar{\rho} = 250$. We also report the number of supply and demand markets in disequilibrium or, equivalently, those with excess supply and/or demand. In column 3 of each table we further tightened the bounds to $\underline{\pi} = 175$ and $\bar{\rho} = 200$ and report the number of supply and demand markets in disequilibrium.

As can be seen from Tables 1 and 2, the decomposition algorithm was robust, converging for all the examples. Although the CPU-time for the computation of the disequilibrium examples exceeded that for the equilibrium examples, this is not unexpected. Finally, the algorithm can solve a greater spectrum of problems than heretofore was possible, thus expanding the potential scope of applications for policy analyses.

6. Summary and conclusions

In this paper we introduced a market model in a spatial setting which can also handle price rigidities and/or controls imposed on the supply price and demand price functions. This model generalizes the model of Thore [36] to the asymmetric case and brings the same level of generality to the study of disequilibrium problems, as had been realized by equilibrium problems. We then showed that the market conditions governing the disequilibrium state—in which the markets need no longer clear—can be formulated as a variational inequality problem over a Cartesian product of sets. We established existence and uniqueness of the disequilibrium solution under certain monotonicity conditions. A decomposition algorithm which resolves the variational inequality problem into three simpler and distinct variational inequality subproblems was proposed, and convergence conditions established. Finally, computational results were provided on large-scale examples.

This contribution bridges the study of disequilibrium and equilibrium through the unifying framework of variational inequalities.

Acknowledgements

This research was carried out while the first author was visiting MIT. The hospitality and cordiality of the Sloan School, the Operations Research Center and the Center for Transportation Studies are warmly appreciated.

This research was conducted on the Cornell National Supercomputer Facility, a resource of the Center for Theory and Simulation in Science and Engineering at Cornell University, which is funded in part by the National Science Foundation, New York State, and IBM Corporation.

The authors are indebted to the referees for their careful reading of the paper.

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