JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 82 (1997) 59–78

Asymptotics of extreme zeros of the Meixner–Pollaczek polynomials

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Received 19 August 1996

Abstract

We study the asymptotics of the smallest and largest zeros of the symmetric and asymmetric Meixner–Pollaczek polynomials using two techniques. One is a Coulomb fluid technique developed earlier where the primary input is the weight function. The second uses the method of chain sequences which supplies inequalities for the largest zeros from the knowledge of the recurrence coefficients. An upper bound for the largest zero of Meixner polynomials is also given.

Keywords: Zeros; Asymptotics; Chain sequences; Inequalities; Coulomb fluid method; Meixner polynomials

AMS classification: Primary 35D45; Secondary 30E05

1. Introduction

This note is part of a continuing program of establishing the 'edge' asymptotic behavior of special classes of orthogonal polynomials. In order to compute such asymptotics, a rather precise knowledge is required on the smallest and largest zeros of the associated orthogonal polynomials.

It is known from the important work of Ullman, Lubinsky, Mhaskar, Nevia, Rakhmanov, Saff, Van Assche, and others, see [14, 15] for references and details, that the distribution function of the zeros, i.e. the equilibrium distribution, denoted as $\sigma(x)$ is the solution to the minimization problem:

$$\min_{\sigma} F[\sigma] \text{ subject to } \int_J \sigma(x)\,\mathrm{d}x = N, \tag{1.1}$$

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[1]Research partially supported by NSF grant DMS 9625459 and a visiting fellowship from the Leverhulme Foundation.

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PII S0377-0427(97)00093-9

and the functional F is

$$F[\sigma] = \int_J u(x)\sigma(x)\,\mathrm{d}x - \int_J\int_J \sigma(x)\ln|x-y|\,\sigma(y)\,\mathrm{d}y\,\mathrm{d}x. \tag{1.2}$$

Here $\exp[-u(x)] = w(x)$ is the weight function and N is the degree of the polynomial orthonormal with respect to weight $w(x)$; i.e., the p's satisfy

$$\int_K p_M(x)p_N(x)w(x)\,\mathrm{d}x = \delta_{M,N}, \tag{1.3}$$

where K is the support of w and $K \subset (-\infty, \infty)$. It assumed that $J \subset K$ and determining J is part of the minimization problem.

In this work we shall first focus our attention on the symmetric Meixner–Pollaczek polynomials $\{P_n^{(\lambda)}(x)\}$ whose weight function is [5, 20]

$$w(x) = \Gamma(\lambda + \mathrm{i}x)\Gamma(\lambda - \mathrm{i}x), \quad x \in K = (-\infty, \infty),\ \lambda > 0, \tag{1.4}$$

and later we also compute the smallest and largest zeros of the nonsymmetric generalization of the Meixner–Pollaczek polynomials. In this case $J = (-b, b)$.

In general when $w(x)$ is even and K is symmetric about the origin then $J = (-b, b)$ and b is an approximation to the largest zeros of $p_N(x)$. Furthermore, the Euler–Lagrange equations of the variational problem (1.1)–(1.2) imply that the minimizing function σ satisfies a singular integral equation,

$$u'(x) = 2(\mathrm{P})\int_{-b}^{b} \frac{\sigma(y)}{x-y}\,\mathrm{d}y, \tag{1.5}$$

where (P) denotes principal value. The edge parameter b is determined by the normalization condition $\int_J \sigma(x)\,\mathrm{d}x = N$. The solution of the integral equation (1.5) is

$$\sigma(x) = \frac{1}{2\pi^2}\sqrt{\frac{b-x}{x+b}}\,(\mathrm{P})\int_{-b}^{b} \frac{u'(y)}{y-x}\sqrt{\frac{y+b}{b-y}}\,\mathrm{d}y, \quad x \in (-b, b). \tag{1.6}$$

The function $\sigma(x)$ given by (1.6) is indeed the potential theoretic approximation of

$$\sigma_N(x) := w(x)\sum_{n=0}^{N-1}[p_n(x)]^2,$$

expected to be valid for sufficiently large N. This technique, the Coulomb fluid method, was first developed by Dyson on certain random matrix ensembles in the 1960s [6] and has recently seen application to other matrix ensembles [1–3]. Some of the details of this approach are in [1, 2]. Once the problem is reduced to finding nonnegative solutions σ of the Euler–Lagrange equations of the variational problem (1.1)–(1.2), one can then take advantage of the extensive theory of singular integral equations and boundary value problems as is available, e.g., in Gakhov's excellent book [9].

In Section 2 we carry out the Coulomb fluid method for the weight function (1.4) and find an asymptotic estimate for b. In Section 3 we give an estimate for the largest zero of the symmetric

Meixner–Pollaczek polynomials based on our earlier estimate for b. Section 4 contains derivations for the bounds of the largest zero of symmetric Meixner–Pollaczek polynomials using chain sequences. We also give bounds for the largest zeros of the Meixner polynomials. The Meixner polynomials are

$$m_n(x; \beta, c) := (\beta)_n {}_2F_1(-n, -x; \beta; 1 - c^{-1}), \tag{1.7}$$

[7, Section 10.24]. These polynomials are orthogonal with respect to a purely discrete measure supported on the non negative integers, so their kth smallest zero converges to $k - 1$, $k = 1, 2, \ldots$. Goh [10] established the Plancherel Rotach asymptotics for the Charlier polynomials from which one can get the asymptotics of the largest zeros. Section 5 deals with the extreme zeros of the general Meixner–Pollaczek polynomials whose weight function $w(x, \alpha)$, see (5.1), is not even. In this case J is no longer a symmetric interval and the problem of estimating the largest and smallest zeros becomes more complicated.

In this work we shall use $x \sim y$ to mean y is the first part of an asymptotic series and the error is 0 (the first term neglected). On the other hand, we use $x \approx y$ to mean y is an approximation to x.

An interesting theorem due to Maté, Nevai and Totik [17] may be stated as follows.

Theorem 1.1. *Let $Q_n(z)$ be a sequence of monic polynomials generated by*

$$Q_0(x) := 1, \qquad Q_1(x) := x - \alpha_0, \qquad Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \quad n > 0, \tag{1.8}$$

with α_n real and β_n positive. Assume $\alpha_n = 0$ and

$$\beta_n = c^2 n^{2\delta}[1 + o(n^{-2/3})], \tag{1.9}$$

as $n \to \infty$ and $\delta > 0$. Let

$$X_{N,1} > X_{N,2} > \cdots > X_{N,N} \tag{1.10}$$

be the zeros of $Q_N(z)$ then

$$X_{N,k} = 2cN^{\delta}[1 - \delta^{2/3}6^{-1/3}i_k N^{-2/3} + o(N^{-2/3}], \tag{1.11}$$

where $i_1 < i_2 < \cdots$ are positive zeros of the Airy function.

The special case $k = 1$ of Theorem 1.1 was established earlier by the same authors in [16]. In Section 6 we give an equivalent formulation of Theorem 1.1 which covers orthogonal polynomials that arise from birth and death processes. Such polynomials have their zeros in $(0, \infty)$ so the α_n's in (1.8) are positive. Our reformulation of Theorem 1.1 yields results about the very important Wilson polynomials. In fact all classical polynomials, either are birth and death process polynomials or limiting cases of such polynomials This will be explored in Section 6.

Let $\{p_n(z)\}$ be the orthonormal polynomials associated with the Q_n's of (1.8), so that

$$p_n(z) := Q_n(z) \Big/ \prod_{k=1}^{n} \beta_k^{1/2}. \tag{1.12}$$

Assume that $Q_n^*(x)$ satisfies the recursion in (1.8) and is given initially by

$$Q_0^*(x) := 0, \qquad Q_1^*(x) = 1. \tag{1.13}$$

It is known that the polynomials $B_n(z)$ and $D_n(z)$ [19],

$$B_{n+1}(z) := [Q_{n+1}(z)Q_n^*(0) - Q_{n+1}^*(0)Q_n(z)](\beta_1, \beta_2 \dots \beta_n)^{-1}, \tag{1.14}$$

$$D_{n+1}(z) := [Q_{n+1}(z)Q_n(0) - Q_{n+1}(0)Q_n(z)](\beta_1, \beta_2 \dots \beta_n)^{-1}, \tag{1.15}$$

converge uniformly to entire functions $B(z)$ and $D(z)$, respectively, on compact subsets of the complex plane if the moment problem is indeterminate, [19].

In [3] Chen and Ismail conjectured that if the moment problem is indeterminate and

$$\alpha_n = 0, \qquad \beta_n = \beta n^\gamma [1 + o(1)] \tag{1.16}$$

then there is an $\eta > 0$ such that

$$\sqrt{X_{N,1}\psi'(x)}\, p_N(x) = -\sqrt{\frac{2}{\pi}} \cos(\Theta + N\pi/2)[1 + o(1)], \tag{1.17}$$

where

$$\Theta = \Theta(x) := \arg[B(x) + i\eta D(x)], \tag{1.18}$$

and

$$\psi' = \frac{b/\pi}{B^2(x) + D^2(x)} \tag{1.19}$$

and $X_{N,1}$ is the largest zero of $p_N(z)$.

Typical cases covered by this conjecture are the Freud weights $\exp(-|x|^\alpha)$ when $1/\alpha$ is a positive integer. In Section 6 of this paper we confirm a special case of this conjecture. Our result is as follows

Theorem 1.2. *Let*

$$\beta_n = \beta\lambda_n \frac{\prod_{j=1}^k (n + a_j)}{\prod_{j=1}^l (n + b_j)}, \quad n > 0, \ 0 \leqslant l < k - 4. \tag{1.20}$$

with

$$\beta > 0, \quad \lambda_n > 0, \quad n > 0 \quad \text{and} \quad \lambda_n = 1 + O(n^{-1-\varepsilon}), \tag{1.21}$$

as $n \to \infty$, for a fixed positive ε. Then (1.17) *holds.*

It is tacitly assumed in Theorems 1.1 and 1.2 that $\beta_n > 0$ for $n > 0$.

2. The symmetric Meixner–Pollaczek polynomials

In this section we illustrate the Coulomb fluid approach by applying to the symmetric Meixner–Pollaczek polynomials. Although the main result of this section follows from the Maté–Nevai–Totik theorem, Theorem 1.1, we nevertheless wish to include our derivation because the proof does not suffer from the technical complications of the nonsymmetric Meixner–Pollaczek polynomials treated in Section 5. The nonsymmetric Meixner–Pollaczek polynomials are not covered by Maté–Nevai–Totik theorem.

For the Meixner–Pollaczek polynomials (1.4) gives

$$\begin{aligned} u'(x) &= \mathrm{i}\left[\frac{\Gamma'(\lambda - \mathrm{i}x)}{\Gamma(\lambda - \mathrm{i}x)} - \frac{\Gamma'(\lambda + \mathrm{i}x)}{\Gamma(\lambda + \mathrm{i}x)}\right] \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{x - \mathrm{i}(\lambda + \mathrm{n})} + \frac{1}{x + \mathrm{i}(\lambda + n)}\right]. \end{aligned}$$

A simple calculation using (1.6) and the above representation gives,

$$\sigma(x) = \frac{\sqrt{b^2 - x^2}}{\pi} \sum_{n=0}^{\infty} \frac{\lambda + n}{\sqrt{b^2 + (\lambda + n)^2}} \frac{1}{x^2 + (\lambda + n)^2}, \tag{2.1}$$

where we used

$$\int_{-b}^{b} (b^2 - t^2)^{-1/2} \frac{\mathrm{d}t}{z - t} = \frac{\pi}{\sqrt{z^2 - b^2}}, \quad z \notin [-b, b]. \tag{2.2}$$

The normalization condition now becomes

$$N = \int_{-b}^{b} \sigma(x)\mathrm{d}x = \sum_{n=0}^{\infty}\left(1 - \frac{\lambda + n}{\sqrt{b^2 + (\lambda + n)^2}}\right). \tag{2.3}$$

The next step is to solve the transcendental equation (2.3) and find b as a function of N. This is impossible to do directly so we approximate the sum in (2.3). Clearly, $1 - x/\sqrt{b^2 + x^2}$ is decreasing and convex on $(0, \infty)$, hence

$$\begin{aligned} \int_{1}^{\infty}\left(1 - \frac{\lambda + x}{\sqrt{b^2 + (\lambda + x)^2}}\right)\mathrm{d}x &< \sum_{n=1}^{\infty}\left(1 - \frac{\lambda + n}{\sqrt{b^2 + (\lambda + n)^2}}\right) \\ &< \int_{0}^{\infty}\left(1 - \frac{\lambda + x}{\sqrt{b^2 + (\lambda + x)^2}}\right)\mathrm{d}x. \end{aligned} \tag{2.4}$$

The error in (2.4) is at most

$$\int_{0}^{1}\left(1 - \frac{\lambda + x}{\sqrt{b^2 + (\lambda + x)^2}}\right)\mathrm{d}x < 1.$$

Therefore, by adding the term $n = 0$ in the sum in (2.4) and comparing the result with (2.3) we get

$$\sqrt{b^2+(\lambda+1)^2} - \lambda - \frac{\lambda}{\sqrt{b^2+\lambda^2}} < N < \sqrt{b^2+\lambda^2} - \lambda + 1 - \frac{\lambda}{\sqrt{b^2+\lambda^2}},$$

which implies

$$b \sim \sqrt{N(N+2\lambda)}. \tag{2.5}$$

Since in (2.1) $\sigma(\pm b) = 0$, the parameter b defines the edges beyond which the density will vanish. This b is approximately the largest zero. A further correction to the largest zero can be obtained from

$$1 = \int_a^b \sigma(x)\,\mathrm{d}x, \tag{2.6}$$

for sufficiently large N and b. Thus formula (2.1) supplies the asymptotics of the largest zero. To see how this works, we first determine the behavior of $\sigma(x)$ for $x \to b^-$. This is

$$\sigma(x) \sim G(b)\sqrt{b-x}, \quad \text{as } x \to b, \tag{2.7}$$

hence

$$1 \approx \tfrac{2}{3} G(b)(b-a)^{3/2}$$

and we get

$$a \approx b - \left(\frac{3}{2G(b)}\right)^{2/3}. \tag{2.8}$$

Therefore in the case under consideration and as $b \to \infty$ we get

$$\begin{aligned} G(b) &= \frac{\sqrt{2b}}{\pi} \sum_{n=0}^{\infty} \frac{\lambda+n}{[b^2+(\lambda+n)^2]^{3/2}} = \frac{\sqrt{2b}}{\pi} \int_0^{\infty} \frac{(\lambda+x)\,\mathrm{d}x}{[b^2+(\lambda+x)^2]^{3/2}} + \mathrm{o}(1) \\ &= \frac{\sqrt{2b}}{\pi} \frac{1}{\sqrt{b^2+\lambda^2}}. \end{aligned} \tag{2.9}$$

Thus we have established the following approximation to a, the largest zero,

$$a \approx b - \left(\frac{3\pi}{2}\right)^{2/3} \left(\frac{b^2+\lambda^2}{2b}\right)^{1/3} = \sqrt{N(N+2\lambda)} - \frac{(3\pi)^{2/3}}{2} \frac{(N+\lambda)^{2/3}}{[N(N+2\lambda)]^{1/6}}. \tag{2.10}$$

The Coulomb fluid approximation gives the correct powers of N but the coefficient in the second term is given only as an approximation. In this way we have proved the following theorem.

Theorem 2.3. *The larger zero $X_{N,1}$ of the Meixner–Pollaczek polynomial $P_N^{(\lambda)}(x)$ satisfies*

$$X_{N,1} \sim \sqrt{N(N+2\lambda)} - c_1 \frac{(N+\lambda)^{2/3}}{[N(N+2\lambda)]^{1/6}}, \tag{2.11}$$

where

$$c_1 \approx \tfrac{1}{2}3\pi^{2/3}. \tag{2.12}$$

Let i_1 be the smallest positive zero of the Airy function. In [4] we advocated the view that the Coulomb fluid method overestimates the second coefficient. In fast Coulomb fluid approximation seems to read $6^{-1/3}i_1$ as $(3\pi)^{2/3}/2$. Indeed the correct value of c_1 is

$$c_1 = 6^{-1/3}i_1. \tag{2.13}$$

as implied by the Maté–Nevai–Totik theorem, with $c = \frac{1}{2}$ and $\delta = 1$.

3. A further estimate of the largest zero

In this section instead of extracting the behavior of $\sigma(x)$ for $x \approx b$, we will show using a different approach that the largest zero given by (2.11) is asymptotically exact. First we compute $\sigma(x)$ from (2.1), replace the sum by an integral then evaluate the integral explicitly using Mathematica this gives

$$\sigma(x) = \frac{1}{2\pi}\ln\left[\frac{2b^2+\lambda^2-x^2+2\sqrt{(b^2+\lambda^2)(b^2-x^2)}}{x^2+\lambda^2}\right], \quad x \in (-b, b). \tag{3.1}$$

Now (2.6) becomes

$$\begin{aligned} 1 &= \int_a^b \sigma(x)\,\mathrm{d}x \\ &= -\frac{\lambda}{2} + \sqrt{b^2+\lambda^2}\left[\frac{1}{2} - \frac{1}{\pi}\tan^{-1}\left(\frac{a}{\sqrt{b^2-a^2}}\right)\right] + \frac{\lambda}{\pi}\tan^{-1}\left(\frac{a}{\lambda}\sqrt{\frac{b^2+\lambda^2}{b^2-a^2}}\right) \\ &\quad + \frac{a}{2\pi}\ln(a^2+\lambda^2) + \frac{a}{2\pi}\ln\left[2b^2+\lambda^2-a^2+s\sqrt{(b^2-a^2)(b^2+\lambda^2)}\right]. \end{aligned} \tag{3.2}$$

This is an equation for a. In the limit under consideration, put $a = b - \varepsilon$, and expand the right-hand side of (3.2) in a series in $\varepsilon^{1/2}$. After some computations using Mathematica, we find

$$a_1\varepsilon^{3/2} + a_2\varepsilon^{5/2} + \mathrm{O}(\varepsilon^{7/2}) = 1, \tag{3.3}$$

where

$$a_1 := \frac{2}{3\pi}\sqrt{\frac{2b}{b^2+\lambda^2}}, \qquad a_2 := \frac{1}{15\pi\sqrt{2b}}\frac{5b^2-3\lambda^2}{(b^2+\lambda^2)^{3/2}}.$$

Thus substituting $\varepsilon = a_1^{-2/3} + \delta$, with $|\delta| \ll |a_1^{-2/3}|$ for sufficiently large b, we find to first order in δ,

$$\delta = -\frac{2a_2}{3a_1^{7/3} + 5a_2a_1^{2/3}} = \mathrm{O}(b^{-1/3}). \tag{3.4}$$

Therefore

$$\varepsilon = \left(\frac{3\pi}{2}\right)^{2/3}\left(\frac{b^2+\lambda^2}{2b}\right)^{1/3} + \mathrm{O}(b^{-1/3}). \tag{3.5}$$

Here again, as per the discussion at the end of Section 2, it is very likely that

$$\varepsilon = 6^{-1/3} i_1 \left(\frac{b^2+\lambda^2}{b}\right)^{1/3} + \mathrm{O}(b^{-1/3}). \tag{3.6}$$

4. Chain sequences

A sequence $\{a_n : 0 < n < N\}$ is a called chain sequence if there exists a parameter sequence g_n, such that

$$a_n = g_n(1-g_{n-1}), \quad 0 \leqslant g_0 < 1, \ 0 < g_n < 1, \ 0 < n < N. \tag{4.1}$$

For detailed information, see [5]. Here N may be finite or infinite.

Theorem 4.1. *Assume that $p_n(z)$ is a sequence of monic polynomials generated by*

$$p_0(x) := 1, \quad p_1(x) := x - \alpha_0, \qquad p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad n > 0, \tag{4.2}$$

with α_n real and β_n positive and let

$$B := \max\{x_n : 0 < n < N\}, \qquad A := \min\{y_n : 0 < n < N\}, \tag{4.3}$$

where x_n and y_n, $x_n \geqslant y_n$, are the roots of

$$(x-\alpha_n)(x-\alpha_{n-1})a_n = \beta_n, \tag{4.4}$$

and a_n is any chain sequence. Then all the zeros of $p_N(z)$ lie in (A, B).

This is Theorem 2 in [11] and is a variation on the Wall–Wetzel theorem on continued fractions [21]. The case $N = \infty$ is essentially in Chihara [5], who did not attach any names to the theorem. It is also known that if the zeros of $p_N(x)$ are less (greater) than A (respectively B), then $\{\beta_n/[(\alpha_n - A)(\alpha_{n-1} - A)]: \ 0 < n < N\}$ (respectively $\{\beta_n/[(B-\alpha_n)(B-\alpha_{n-1})]: \ 0 < n < N\}$) is a chain sequence, [11, 12]. For the monic Meixner–Pollaczek polynomials

$$\alpha_n = 0 \quad \text{and} \quad \beta_n = \tfrac{1}{4}n(n+2\lambda-1)/4. \tag{4.5}$$

With the chain sequence $a_n = \frac{1}{4}$, as was done in [11], we get

$$-A = B = \sqrt{(N-1)(N+2\lambda-2)}.$$

Therefore, $X_{N,1}$, the largest zero of $P_N^{(\lambda)}(x)$, satisfies the inequality

$$X_{N,1} < \sqrt{(N-1)(N+2\lambda-2)}. \tag{4.6}$$

Although the inequality (4.6) is not as sharp as the estimate (2.11), it is nevertheless sharper than the first estimate of the Coulomb fluid method, (2.5). One can establish better bounds by using chain sequences tailored to the case at hand. Let $h_{N,1}$ be the largest zeros of the Hermite polynomial $H_N(x)$. For the Hermite polynomials $\alpha_n = 0$ and $\beta_n = n/2$ hence $n/[2(h_{n,1}^2 + \varepsilon)]$ is a chain sequence and the Wall–Wetzel theorem yields the inequality

$$X_{N,1}^2 \leqslant \sqrt{\tfrac{1}{2}(N-2+2\lambda)}\, h_{N,1}.$$

On the other hand, it is known that

$$h_{N,1} < \sqrt{2N+1} - \frac{6^{-1/3} i_1}{(2N+1)^{1/6}}, \tag{4.7}$$

where i_1, as indicated earlier, is the smallest positive zero of the Airy function, see, e.g. [20, (6.32.3)]. Thus we have proved the following theorem.

Theorem 4.2. *The largest zero of $P_N^{(\lambda)}(x)$ satisfies*

$$X_{N,1} < \sqrt{(N-2+2\lambda)/2}\left(\sqrt{2N+1} - \frac{6^{-1/3} i_1}{(2N+1)^{1/6}}\right). \tag{4.8}$$

It is worth pointing out that the first two terms in the asymptotic expansion of the right-hand side of (4.8) are in agreement with the corresponding terms in (1.9) with $c = \frac{1}{2}$ and $\delta = 1$. This shows that the inequality (4.8) is quite sharp because it gives correctly the first two terms in the asymptotic expansion for large N.

In [8] a formula of the Plancherel-Rotach type was derived for the symmetric Meixner–Pollaczek polynomials by implicitly assuming the largest zero is of order $N+1$. The bound (4.8) and the asymptotic formula (2.11) strongly indicate that the order of the largest zero of the symmetric Meixner–Pollaczek polynomials depends on λ and is likely to be $N+\lambda$. One can easily carry out the analysis in [8] with $N+1$ replaced by $N+\lambda$ and will similarly obtain a more accurate Plancherel–Rotach asymptotic formula. We leave this exercise to the interested reader who will be well-advised to first read [8].

We now discuss an inequality satisfied by the largest zero of the Meixner polynomials. Let

$$m_{N,1}(\beta, c) > m_{N,2}(\beta, c) > \cdots > m_{N,N}(\beta, c) \quad \text{and} \quad l_{N,1}(\alpha) > l_{N,2}(\alpha) > \cdots > l_{N,N}(\alpha) \tag{4.9}$$

be the zeros of the Meixner and Laguerre polynomials $m_N(x; \beta, c)$ and $L_N^{(\alpha)}(x)$, respectively. It is easy to see from (1.7) that

$$\lim_{c \to 1^-} m_N((1-c)x/\sqrt{c}; \beta, c) = n!\, L_N^{(\beta-1)}(x), \tag{4.10}$$

holds uniformly on compact subsets of x and β, $\beta > 0$ [7, Section 10.24]. Therefore one would expect the zeros of $L_N^{(\beta-1)}(x)$ to approximate the zeros of $m_N((1-c)c^{-1/2}x; \beta, c)$. Ismail and Muldoon [12] studied the rate of approximation and showed that

$$\begin{aligned} l_{N,j}(\beta-1) - \beta(1-\sqrt{c}) &< \frac{1-c}{\sqrt{c}}\, m_{N,j}(\beta, c) \\ &< l_{N,j}(\beta-1) - \beta(1-\sqrt{c}) + \frac{n-1}{\sqrt{c}}(1-\sqrt{c})^2. \end{aligned} \tag{4.11}$$

This provides a sharp estimate for $m_{N,j}(\beta, c)$ for fixed N and c close to 1. We now find estimate from $m_{N,1}(\beta, c)$ for large N and fixed c using the Wall–Wetzel theorem, Theorem 4.1. In the case under consideration, the recursion coefficients of (4.2) are [5].

$$\alpha_n = \sqrt{c}\beta + n(1+c)/\sqrt{c} \quad \text{and} \quad \beta_n = n(n+\beta-1). \tag{4.12}$$

Recall that

$$\sqrt{l_{N,j}}(\alpha) < (4N+2\alpha+2)^{1/2} - 6^{-1/3}(4N+2\alpha+2)^{-1/6} i_j, \tag{4.13}$$

holds for $\alpha > -1$ and $|\alpha| \geqslant \frac{1}{4}$ [20, Theorem 6.32]. Thus by Theorem 4.1 we may choose

$$a_n = \frac{n(n+\alpha)}{[\varepsilon + l_{N,1}(\alpha) - (2n+\alpha+1)][\varepsilon + l_{N,1}(\alpha) - (2n+\alpha-1)]}, \quad 0 < n < N, \tag{4.14}$$

for any $\varepsilon > 0$ and $\alpha = \beta - 1$. It is easy that, with a_n as in (4.14), the largest root of (4.4) increases with n, hence $(1-c)m_{N,1}(\beta, c)$ is less than the largest root of (4.4) with $n = N-1$. Since this holds for any $\varepsilon > 0$ we can let $\varepsilon \to 0$ and obtain the upper bound

$$\begin{aligned}\frac{(1-c)}{\sqrt{c}} m_{N,1}(\beta, c) &\leqslant \sqrt{c}\beta + \frac{1+c}{\sqrt{c}}(N - \tfrac{3}{2}) \\ &+ \sqrt{\left\{\frac{(1+c)^2}{4c} + [l_{N,1}(\beta-1) - 2N - \beta + 2][l_{N,1}(\beta-1) - 2N - \beta + 4]\right\}}\end{aligned} \tag{4.15}$$

The quantity under the square root is

$$[l_{N,1}(\beta-1) - 2N - \beta + 3]^2 + \frac{(1-c)^2}{4c}.$$

Since $\sqrt{A^2+B^2} < |A| + |B|$ for $AB \neq 0$ we find from (4.15) that

$$\frac{(1-c)}{\sqrt{c}} m_{N,1}(\beta, c) < \sqrt{c}\beta + \frac{1+c}{\sqrt{c}}(N - \tfrac{3}{2}) + \frac{1-c}{2\sqrt{c}} + l_{N,1}(\beta-1) - 2N - \beta + 3, \tag{4.16}$$

for $\beta > 0$. Observe that (4.16) is sharp in its dependence on c since both sides converge to $l_{N,1}(\beta-1)$ as $c \to 1^-$. Substitute for $l_{N,1}(\beta-1)$ from (4.13) into (4.16) to arrive at the inequality

$$\begin{aligned}\frac{(1-c)}{\sqrt{c}} m_{N,1}(\beta, c) &< (1+\sqrt{c})\beta + \frac{(1-\sqrt{c})^2}{\sqrt{c}} N \\ &+ \frac{(1-\sqrt{c})(2\sqrt{c}-1)}{\sqrt{c}} - 2\left(\frac{2N+\beta}{3}\right)^{1/3} i_1,\end{aligned} \tag{4.17}$$

for $\beta > 0$ and $|\beta - 1| \geqslant \frac{1}{4}$, that is $\beta > 0$ and $\beta \notin (\frac{3}{4}, \frac{5}{4})$.

It is expected that (4.17) is sharp for large N in the sense that the coefficients of N and $N^{1/3}$ on the right-hand side of (4.17) agree with the corresponding coefficients in the large N asymptotic development of the left-hand side of (4.17).

It is important to note that uniform asymptotic expansions of Meixner polynomials were established in [10] and later considerably extended by Jin and Wong in their work [13]. Building on this important work of Jin and Wong may lead to asymptotic developments of all zeros of Meixner polynomials of fixed rank, i.e., $m_{N,1}(\beta, c)$ for fixed k. Ismail and Li [11] gave lower bounds for $m_{N,N}(\beta, c)$.

5. The zeros of general Meixner–Pollaczek polynomials

With the introduction of an extra parameter, α, where $0 < \alpha < \pi$, the potential of the asymmetric Meixner–Pollaczek polynomials is

$$\begin{aligned} u(x,\alpha) &:= -\ln w(x,\alpha) \\ &= (\pi - 2\alpha)x - \ln[\Gamma(\lambda + \mathrm{i}x)\Gamma(\lambda - \mathrm{i}x)], \quad -\infty < x < \infty,\ \lambda > 0,\ 0 < \alpha < \pi. \end{aligned} \tag{5.1}$$

Note that $u(x, \pi/2)$ reduces to the potential for the symmetric Meixner–Pollaczek polynomials considered in the previous sections. It is clear that since the potential is no longer an even function of x, then Theorem 1.1 is not applicable and the density function, $\sigma(x)$, will not be supported in an interval which is symmetric about the origin. In this case the integral equation for $\sigma(\cdot)$ reads,

$$u'(x,\alpha) := \frac{\mathrm{d}u(x,\alpha)}{\mathrm{d}x} = 2(\mathrm{P}) \int_a^b \frac{\sigma(x)}{x-y}\,\mathrm{d}y, \quad a < x < b. \tag{5.2}$$

The solution of (5.2) is again required to satisfy the normalization condition,

$$N = \int_a^b \sigma(x)\mathrm{d}x. \tag{5.3}$$

In this case there are two quantities, a and b, to be determined in terms of the degree N and the parameters α and λ. Recall that the integral equation arose from a minimization problem. It is appropriate here to assume

$$\sigma(a) = \sigma(b) = 0. \tag{5.4}$$

The general solution of (5.2) involves $C(b^2 - x^2)^{-1/2}$ and (5.4) will imply $C = 0$. Technically, one can carry the term $C(b^2 - x^2)^{-1/2}$ through and at the end from requiring $\sigma(x) \geqslant 0$ on $(-b, b)$ and minimizing $F[\sigma]$ we can show that $C = 0$. This point will be discussed in the future paper using arguments based on computing the free energy of the system of N particles under external field. With condition (5.4) the minimization problem at hand is equivalent to a scalar Riemann–Hilbert problem with index $\chi = -1$. From the standard theory described in Gakhov [9], the unique solution subject to the boundary condition is given by

$$\sigma(x) = \frac{R(x)}{2\pi^2}(P)\int_a^b \frac{u'(y,\alpha)}{(y-x)R(y)}\,\mathrm{d}y \tag{5.5}$$

with the supplementary condition

$$\int_a^b \frac{u'(y,\alpha)}{R(y)}\,\mathrm{d}y = 0, \tag{5.6}$$

where

$$R(x) := \sqrt{(b-x)(x-a)}, \quad x \in (a,b). \tag{5.7}$$

The branch of $R(x)$ is chosen in such a way that $R(x) > 0$, for $a < x < b$. Using (5.3) and (5.6), we can determine the parameters a and b, to be seen later. Thus in a crude estimate a and b will be the smallest and largest zeros, respectively. We first deal with (5.3). Substituting

$$u'(y,\alpha) = (\pi - 2\alpha) + \sum_{n=0}^{\infty}\left(\frac{1}{y + \mathrm{i}c_n} + \text{c.c.}\right),$$

into (5.6) gives,

$$(2\alpha - \pi) = \sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{(a + c_n)(b + \mathrm{i}c^n)}} + \text{c.c.}\right), \tag{5.8}$$

where we have used c.c. to denote complex conjugate and used the evaluation

$$\int_a^b \frac{\mathrm{d}y}{R(y)(y \pm \mathrm{i}c_n)} = \frac{\pi}{\sqrt{(a \pm \mathrm{i}c_n)(b \pm \mathrm{i}c_n)}}, \tag{5.9}$$

to arrive at (5.8). Now approximating the sum over n by an integral gives

$$\mathrm{i}(2\alpha - \pi) = \ln\left[\frac{2\lambda + \mathrm{i}(a+b) + \sqrt{(\lambda + \mathrm{i}a)(\lambda + \mathrm{i}b)}}{2\lambda - \mathrm{i}(a+b) + \sqrt{(\lambda - \mathrm{i}a)(\lambda - \mathrm{i}b)}}\right] + o(1),$$

or equivalently

$$\mathrm{e}^{i(\alpha - \pi/2)} = \frac{\sqrt{\lambda + \mathrm{i}a} + \sqrt{\lambda + \mathrm{i}b}}{\sqrt{\lambda - \mathrm{i}a} + \sqrt{\lambda - \mathrm{i}b}}. \tag{5.10}$$

With the introduction of $2\beta := \alpha - \pi/2$, formula (5.10) can be recast in a useful form

$$\mathrm{e}^{\mathrm{i}\beta}(\sqrt{\lambda - \mathrm{i}a} + \sqrt{\lambda - \mathrm{i}b}) = \mathrm{e}^{-\mathrm{i}\beta}(\sqrt{\lambda + \mathrm{i}a} + \sqrt{\lambda + \mathrm{i}b}),$$

which upon squaring leads to

$$a\sin 2\beta - \sqrt{\lambda^2 + a^2} = b\sin 2\beta - \sqrt{\lambda^2 + b^2} =: -\Xi, \tag{5.11}$$

say, where now a and b are parameterized by Ξ. Therefore (5.11) shows that a and b, are the roots of the equation

$$X^2\cos^2 2\beta - 2\Xi X\sin 2\beta + \lambda^2 - \Xi^2 = 0. \tag{5.12}$$

This implies

$$a + b = 2\Xi\tan 2\beta\sec 2\beta, \; ab = (\lambda^2 - \Xi^2)\sec^2 2\beta. \tag{5.13}$$

We now determine Ξ from the normalization condition; thus

$$N = \int_a^b \mathrm{d}y\frac{u'(y,\alpha)}{R(y)}\left(\frac{1}{2\pi^2}(P)\int_a^b \mathrm{d}x\frac{R(x)}{y - x}\right). \tag{5.14}$$

From the partial fraction decomposition,

$$\frac{(b-x)(x-a)}{y-x}=(x-a)+(y-b)+\frac{(y-b)(a-y)}{y-x}, \tag{5.15}$$

we find

$$(P)\int_a^b \frac{R(x)\,\mathrm{d}x}{y-x}=(P)\int_a^b\left[\frac{x-a}{R(x)}+\frac{y-b}{R(x)}\right]\mathrm{d}x$$

$$=\frac{\pi}{2}(b-a)+\pi(y-b),\quad y\in(a,b), \tag{5.16}$$

where we have used,

$$(P)\int_a^b \frac{\mathrm{d}x}{(y-x)R(x)}=0,\quad y\in(a,b),$$

to arrive at (5.16). Using the supplementary condition (5.6), the normalization condition (5.3) becomes,

$$2\pi N=\int_a^b \frac{yu'(y,\alpha)}{R(y)}\,\mathrm{d}y, \tag{5.17}$$

i.e.,

$$2\pi N=\int_a^b \frac{\mathrm{d}y}{R(y)}\left[y(\pi-2\alpha)+\sum_{n=0}^{\infty}\left(2-\frac{\mathrm{i}c_n}{y+\mathrm{i}c_n}+\frac{\mathrm{i}c_n}{y-\mathrm{i}c_n}\right)\right]. \tag{5.18}$$

One referee pointed out that (5.6) and (5.17) are special cases of general results in [18]. In the case under consideration, they just follow from the theory described in the much older book by Gakhov [9]. Approximating the sum in (5.18) by an integral gives

$$2N=(\pi/2-\alpha)(a+b)$$

$$+\int_\lambda^\infty\left(2-\frac{t}{\sqrt{(t-\mathrm{i}a)(t-\mathrm{i}b)}}-\frac{t}{\sqrt{(t+\mathrm{i}a)(t+\mathrm{i}b)}}\right)\mathrm{d}t+\mathrm{o}(1). \tag{5.19}$$

Evaluating the t integral using Mathematica, we find,

$$2N=(\pi/2-\alpha)(a+b)$$

$$-2\lambda+\sqrt{(\lambda-\mathrm{i}a))(\lambda-\mathrm{i}b)}+\sqrt{(\lambda+\mathrm{i}a)(\lambda+\mathrm{i}b)}$$

$$+\mathrm{i}(a+b)\ln\left[\frac{\sqrt{\lambda-\mathrm{i}a}+\sqrt{\lambda-\mathrm{i}b}}{\sqrt{\lambda+\mathrm{i}a}+\sqrt{\lambda+\mathrm{i}b}}\right]+\mathrm{o}(1). \tag{5.20}$$

Note that from (5.10), the sum of the first and last term in (5.20) is zero. We have, finally,

$$2N = -2\lambda + \sqrt{(\lambda + \mathrm{i}a)(\lambda + \mathrm{i}b)} + \sqrt{(\lambda - \mathrm{i}a)(\lambda - \mathrm{i}b)}. \tag{5.21}$$

Clearly,

$$\begin{aligned}(\lambda + \mathrm{i}a)(\lambda + \mathrm{i}b) &= \lambda^2 + \mathrm{i}\lambda(a+b) - ab \\ &= (\Xi + \mathrm{i}\lambda \sin 2\beta)^2 \sec^2 2\beta,\end{aligned}$$

where we have used (5.13). Thus (5.21) yields

$$\Xi = (N + \lambda)\cos 2\beta. \tag{5.22}$$

Furthermore,

$$a = (N + \lambda)\tan\alpha - \sqrt{N(N + 2\lambda)}\sec\alpha < 0, \tag{5.23}$$

$$b = (N + \lambda)\tan\alpha - \sqrt{N(N + 2\lambda)}\sec\alpha > 0. \tag{5.24}$$

To determine the smallest and the largest zeros, we simply adapt the procedure described in Section 1. Therefore,

$$\sigma(x) \sim H\sqrt{x - a}, \quad \text{as } x \to a, \tag{5.25}$$

$$\sigma(x) \sim G\sqrt{b - a}, \quad \text{as } x \to b, \tag{5.26}$$

where

$$\begin{aligned} H :&= \frac{\sqrt{b-a}}{2\pi}\sum_{n=0}^{\infty}\left[\frac{1}{(c_n + \mathrm{i}a)^{3/2}}\frac{1}{\sqrt{c_n + \mathrm{i}b}} + \text{c.c.}\right] \\ &= \frac{\sqrt{b-a}}{2\pi}\left[\int_{\lambda}^{\infty}\frac{\mathrm{d}t}{(t + \mathrm{i}a)^{3/2}}\frac{1}{\sqrt{t + \mathrm{i}b}} + \text{c.c.}\right] + \mathrm{o}(1) \\ &= \frac{1}{\pi\mathrm{i}\sqrt{b-a}}\left(\sqrt{\frac{\lambda - \mathrm{i}b}{\lambda - \mathrm{i}a}} - \sqrt{\frac{\lambda + \mathrm{i}a}{\lambda + \mathrm{i}b}}\right) + \mathrm{o}(1), \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} G :&= \frac{\sqrt{b-a}}{2\pi}\sum_{n=0}^{\infty}\left[\frac{1}{(c_n + \mathrm{i}b)^{3/2}}\frac{1}{\sqrt{c_n + \mathrm{i}a}} + \text{c.c.}\right] \\ &= \frac{\sqrt{b-a}}{2\pi}\left[\int_{\lambda}^{\infty}\frac{\mathrm{d}t}{(t + \mathrm{i}b)^{3/2}}\frac{1}{\sqrt{t + \mathrm{i}a}} + \text{c.c.}\right] + \mathrm{o}(1) \\ &= \frac{1}{\pi\mathrm{i}\sqrt{b-a}}\left(\sqrt{\frac{\lambda - \mathrm{i}a}{\lambda - \mathrm{i}b}} - \sqrt{\frac{\lambda + \mathrm{i}a}{\lambda + \mathrm{i}b}}\right) + \mathrm{o}(1). \end{aligned} \tag{5.28}$$

Therefore following Section 1, the smallest and the largest zeros of the asymmetric Meixner–Polaczek polynomials are, respectively,

$$d \approx a + \left(\frac{3}{2H}\right)^{2/3} \tag{5.29}$$

$$c \approx b - \left(\frac{3}{2G}\right)^{2/3}, \tag{5.30}$$

where c.c denotes the complex conjugate of the quantity on the same line. The following convenient form for H and G can be obtained by squaring and extracting a square root,

$$H = \frac{\sqrt{2}}{\pi\sqrt{b-a}}\left[\left(\frac{\lambda^2+b^2}{\lambda^2+a^2}\right)^{1/2} - \frac{\lambda^2+ab}{\lambda^2+a^2}\right]^{1/2}, \tag{5.31}$$

$$G = \frac{\sqrt{2}}{\pi\sqrt{b-a}}\left[\left(\frac{\lambda^2+a^2}{\lambda^2+b^2}\right)^{1/2} - \frac{\lambda^2+ab}{\lambda^2+a^2}\right]^{1/2}. \tag{5.32}$$

Thus (5.29) and (5.30) become

$$d \sim a + c_1(b-a)^{1/3}\left[\left(\frac{\lambda^2+b^2}{\lambda^2+a^2}\right)^{1/2} - \frac{\lambda^2+ab}{\lambda^2+a^2}\right]^{-1/3}, \tag{5.33}$$

and

$$c \sim b - c_1(b-a)^{1/3}\left[\left(\frac{\lambda^2+a^2}{\lambda^2+b^2}\right)^{1/2} - \frac{\lambda^2+ab}{\lambda^2+b^2}\right]^{-1/3}. \tag{5.34}$$

where

$$c_1 \approx \tfrac{1}{2}(3\pi)^{2/3}. \tag{5.35}$$

Here again we believe the correct value of c_1 is $6^{-1/3}i_1$. We next summarize the above findings in the form of a theorem, as suggested by a referee.

Theorem 5.1. *The largest and smallest zeros $X_{N,1}$ and $X_{N,N}$ of a Meixner–Pollaczek polynomial of order N are asymtotically given by*

$$X_{N,1} \sim b - c_1(b-a)^{1/3}\left[\left(\frac{\lambda^2+a^2}{\lambda^2+b^2}\right)^{1/2} - \frac{\lambda^2+ab}{\lambda^2+b^2}\right]^{-1/3}. \tag{5.36}$$

and

$$X_{N,N} \sim a + c_1(b-a)^{1/3}\left[\left(\frac{\lambda^2+b^2}{\lambda^2+a^2}\right)^{1/2} - \frac{\lambda^2+ab}{\lambda^2+a^2}\right]^{-1/3}, \tag{5.37}$$

for some constant c_1, where a and b are given by (5.23) *and* (5.24), *respectively.*

Note for $u(x, \pi/2)$ reduces to the potential $-\ln(|\Gamma(\lambda + ix)|)^2$ and the previous results on the largest zero of the symmetric Meixner–Pollaczek polynomials are recovered.

6. On Theorems 1.1 and 1.2

We first indicate a proof of Theorem 1.2.

Proof of Theorem 1.2. Recall that $\{p_n(x)\}$ are the orthonormal polynomials associated with the Q_n's. From (1.8) and (1.13) it follows that

$$Q_{2n+1}(0) = Q^*_{2n}(0) = 0, \quad Q_{2n}(0) = (-1)^n \prod_{k=1}^{n} \beta_{2k-1}, \tag{6.1}$$

$$Q^*_{2n+1} = (-1)^n \prod_{k=1}^{n} \beta_{2k}. \tag{6.2}$$

Therefore (1.15) gives

$$D(z) = \lim_{n\to\infty} (-1)^n Q_{2n+1}(z) \prod_{k=1}^{n} \left(\frac{\beta_{2k-1}}{\beta_{2k}}\right).$$

In terms of the p_N's the above relationship is

$$\beta^{1/2}_{2N+1} p_{2N+1}(z) = (-1)^N D(z)\left(\prod_{j=1}^{N} (\beta^{1/2}_{2j}/\beta^{1/2}_{2j-1})\right)[1 + o(1)]. \tag{6.3}$$

Similarly (6.1), (6.2) and (1.14) imply

$$p_{2N}(z) = (-1)^{N+1} B(z)\left(\prod_{j=1}^{N} (\beta^{1/2}_{2j-1}/\beta^{1/2}_{2j})\right)[1 + o(1)]. \tag{6.4}$$

In the case when the β_n's are given by (1.20), we apply

$$\Gamma(a + z)/\Gamma(b + z) = n^{a-b}[1 + o(1)] \tag{6.5}$$

and find

$$\prod_{j=1}^{N} \frac{\beta_{2j-1}}{\beta_{2j}} = \frac{N^{(1-k)/2}}{M}\left(\prod_{j=1}^{N} \frac{\lambda_{2j-1}}{\lambda_{2j}}\right)[1 + o(1)], \tag{6.6}$$

where

$$M := \left[\prod_{j=1}^{k} \frac{\Gamma((1 + a_j)/2)}{\Gamma(1 + a_j/2)}\right]\left[\prod_{s=1}^{l} \frac{\Gamma(1 + b_s/2)}{\Gamma(1 + b_s/2)}\right]. \tag{6.7}$$

In view of the assumptions (1.21) it follows that the infinite product

$$\prod_{j=1}^{\infty} \frac{\lambda_{2j}}{\lambda_{2j-1}} \text{ converges to } \Lambda, \tag{6.8}$$

say. Since

$$X_{N,1} \sim 2\sqrt{\beta_N} \tag{6.9}$$

then it follows from (1.20), (6.3), (6.6) and (6.9) that

$$\sqrt{X_{2N+1,1}}\,p_{2N+1}(x) = \frac{\sqrt{2\Lambda M}(-1)^N}{\beta^{1/4}2^{(k-l)/4}} D(x)[1 + o(1)].$$

Set

$$w(x) := \frac{\eta/\pi}{B^2(x) + \eta^2 D^2(x)}. \tag{6.10}$$

Therefore

$$\sqrt{X_{2N+1,1} w(x)}\,p_{2N+1}(x) = \sqrt{\frac{2M\Lambda}{\pi\eta\sqrt{\beta}2^{(k-l)/2}}}\frac{(-1)^N \eta D(x)}{\sqrt{B^2(x) + \eta^2 D^2(x)}}[1 + o(1)]. \tag{6.11}$$

We now choose

$$\eta = 2^{(l-k)}\beta^{-1/2}\Lambda M \tag{6.12}$$

and reduce (6.11) to (6.17) for odd N. The case of even N follows in a similar fashion and the choice of η in (6.12) is consistent. This completes the proof of Theorem 1.2.

It is important to observe that the choice of η in Theorem 1.2 is unique as can be seen from the construction, so η must be given by (6.12) in order for (1.17) to hold.

We note that the density of the zeros σ, [3], is related to Θ through, [3]

$$\Theta'(x) = \pi\sigma(x). \tag{6.13}$$

Thus Θ in (1.17) may be replaced by $\pi\int_0^x \sigma(u)\,du$. This is how (1.18) is stated in [3].

We now come to discussing Theorem 1.1 in the context of birth and death processes polynomials. A birth and death process is a stationary Markov process with birth rates λ_n and death rates μ_n. It is assumed that $\lambda_n > 0$, $\mu_{n+1} > 0$ for $n \geqslant 0$ and $\mu_0 \geqslant 0$. Every such process leads to orthogonal polynomials $\{F_n(x)\}$ generated by

$$\begin{gathered} F_0(x) = 0, \qquad F_1(x) = (\lambda_0 + \mu_0 - x)/\lambda_1, \\ -xF_n(x) = \lambda_n F_{n+1}(x) - (\lambda_n + \mu_n)F_n(x) - \mu_n F_{n_1}(x). \end{gathered} \tag{6.14}$$

The zeros of such a sequence of polynomials are always positive. The F_n's generate a set of symmetric orthogonal polynomials $\mathscr{F}_n(x)$ through

$$\begin{gathered} \mathscr{F}_0(x) = 1. \qquad \mathscr{F}_1(x) = x, \\ x\mathscr{F}_n(x) = \mathscr{F}_{n+1}(x) - \beta_n \mathscr{F}_{n-1}(x), \end{gathered} \tag{6.15}$$

where

$$\beta_{2n+1} = \lambda_n, \qquad \beta_{2n} = \mu_n. \tag{6.16}$$

It is well known that

$$\mathscr{F}_{2n}(x) = (-1)^n \left\{ \prod_{k=1}^{n} \mu_k \right\} F_n(x^2), \tag{6.17}$$

see, e.g., [4].

A birth and death process is called asymptotically symmetric in $\lim_{n\to\infty} \mu_n/\lambda_n = 1$.

Theorem 6.1. *Let* $\{F_n\}$ *be a family of birth and death process polynomials satisfying* (6.14) *and let*

$$\lambda_n = a^2 n^{2\delta}[1 + o(n^{-2/3})], \tag{6.18}$$

and

$$\mu_n = a^2 n^{2\delta}[1 + o(n^{-2/3})], \tag{6.19}$$

as $n \to \infty$ *and assume* $\delta > 0$. *Let the zeros of* F_n *be arranged as*

$$x_{n,1} > x_{n,2} > \cdots > x_{n,n} > 0, \tag{6.20}$$

and assume $\mu_0 = 0$. *Then we have*

$$\sqrt{x_{n,k}} = 2an^{\delta}[1 - \tfrac{1}{2}\delta^{2/3} 3^{-1/3} i_k n^{-2/3} + o(n^{-2/3})], \tag{6.21}$$

or equivalently

$$\sqrt{x_{n,k}} = \sqrt{2(\lambda_n + \mu_n)}\left[1 - \frac{1}{2}\delta^{2/3} 3^{-1/3} i_k \left(\frac{\lambda_n + \mu_n}{2a^2}\right)^{-1/(3\delta)} + o(n^{-2/3})\right], \tag{6.22}$$

Proof. Use (6.16), (6.14) and (6.15) to see that

$$\beta_n = a^2 (n/2)^{\delta}[1 + o(n^{-2/3})].$$

Now apply Theorem 1.1 to $\mathscr{F}_n(x)$ to see that the zeros $X_{n,k}$ of the $\mathscr{F}_n$'s satisfy

$$X_{n,k} = 2a(n/2)^{\delta}[1 - \delta^{2/3} 6^{-1/3} i_k n^{-2/3} + o(n^{-2/3})].$$

The result (6.21) now follows since $x_{n,k} = X_{2n,k}^2$, as can be seen from (6.17). Finally, (6.21) and (6.22) are equivalent since their right-hand sides differ by terms that are of the same order as the error term.

Example. The Wilson polynomials have

$$\lambda_n = \frac{(a+b+n)(a+c+n)(a+d+n)(a+b+c+d+n-1)}{(a+b+c+d+2n-1)(a+b+c+d+2n)}, \quad n \geqslant 0, \tag{6.23}$$

$$\mu_n = \frac{n(b+c+n)(b+d+n)(c+d+n)}{(a+b+c+d+2n-1)(a+b+c+d+2n-2)}, \quad n \geqslant 0, \tag{6.24}$$

where a, b, c and d are positive parameters, Thus $a = \frac{1}{2}$, $\delta = 1$ in (6.18), (6.19), and for $x_{n,k}(W)$, the zeros of the Wilson polynomials, we get

$$x_{n,k}(W) = n^2[1 - 3^{-1/3}n^{-2/3}i_k + o(n^{-2/3})]. \tag{6.25}$$

Although formula (6.25) is certainly correct it does not show the dependence on the parameters a, b, c and d. On the other hand (6.22) gives the more informative formula

$$x_{n,k}(W) = (n + a + b + c + d)^2 - 3^{-1/3}(n + a + b + c + d)^{4/3}i_k + o(n^{4/3}). \tag{6.26}$$

Based on our earlier work using the Coulomb fluid asymptotics we conjecture that

$$\sqrt{x_{n,k}(W)} = (n + a + b + c + d) - \tfrac{1}{2}3^{-1/3}(n + a + b + c + d)^{1/3}\{i_k + \varepsilon_n\}, \tag{6.27}$$

where ε_n is positive for all n, $n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Of course (6.26) shows that $\varepsilon_n \to 0$ as $n \to \infty$. We think the presence of $a + b + c + d$ in (6.27) is important to ensure the positivity ε_n. Since the weight function of the Wilson polynomials is symmetric in the parameters a, b, c, d, the asymptotic formulas for the orthogonal polynomials and their zeros must be symmetric in the four parameters involved.

Example. The Laguerre polynomials have

$$\lambda_n = n + \alpha + 1, \quad \mu_n = n. \tag{6.28}$$

Hence $a = 1$, $\delta = \frac{1}{2}$ and (6.22) gives the classical result

$$\sqrt{l_{n,k}^{(\alpha)}} = (4n + 2\alpha + 2)^{1/2} - 6^{-1/3}(4n + 2\alpha + 2)^{-1/6}i_k + o(n^{-1/6}). \tag{6.29}$$

The Meixner polynomials arise from a birth and death process which is not asymptotically symmetric, so we do not see how to apply Theorem 6.1 to them.

Acknowledgements

This work was done while the second named author (M.I.) was visiting Imperial College. He gratefully acknowledges the hospitality and support from both Imperial College and the University of South Florida. The authors are grateful to the two referees for many useful comments and suggestions.

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