ELSEVIER

Journal of Computational and Applied Mathematics 103 (1999) 281–285

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

A note on the convergence of the MAOR method

Ljiljana Cvetković *

Faculty of Science, Institute of Mathematics, Trg. D. Obradovica 4, 21000 Novi Sad, Yugoslavia

Received 27 October 1997; received in revised form 12 October 1998

Abstract

The MAOR method as a generalization of the well-known MSOR method was introduced by Hadjidimos et al. (Appl. Numer. Math. 10 (1992) 115–127) and investigated in Y. Song (J. Comput. Appl. Math. 79 (1997) 299–317) where some convergence results for the case when matrix of the system is strictly diagonally dominant are obtained. In this paper we shall improve these results. © 1999 Elsevier Science B.V. All rights reserved.

AMS classification: 65F10

Keywords: Linear systems; Iterative methods; Convergence

1. Introduction

Let

$$Ax = b \tag{1}$$

be a system of linear equations with the $n \times n$ nonsingular matrix A of the following form:

$$A = \begin{bmatrix} D_1 & -H \\ -K & D_2 \end{bmatrix}, \tag{2}$$

where D_1 and D_2 are square nonsingular diagonal matrices.

For solving system (1) Young [4] proposed the modified SOR (MSOR) method, which was investigated by many authors (for detailed comments see [2, 4]).

* E-mail: lila@unsim.ns.ac.yu.

0377-0427/99/$ – see front matter © 1999 Elsevier Science B.V. All rights reserved.
PII: S0377-0427(98)00263-5

In [1] a generalization of the MSOR method, called MAOR method, was proposed. In [2] for matrix A having the form (2) and strictly diagonally dominance property, among others, some sufficient conditions for the convergence of the MAOR method were given. Here we shall improve this result.

Definition 1. Matrix $A \in \boldsymbol{C}^{n,n}$ is a strictly diagonally dominant (SDD) matrix if

$$|a_{ii}| > r_i(A) := \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n.$$

2. MAOR method

Let A be two-cyclic matrix of the form (2) and let

$$A = D - C_L - C_U$$

be the standard splitting of A into diagonal (D), strictly lower (C_L) and strictly upper (C_U) triangular matrices, respectively. D is suppose to be nonsingular. Obviously,

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \qquad C_L = \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}, \qquad C_U = \begin{bmatrix} 0 & H \\ 0 & 0 \end{bmatrix}.$$

Let us denote

$$L = D^{-1}C_L = \begin{bmatrix} 0 & 0 \\ D_2^{-1}K & 0 \end{bmatrix}, \qquad U = D^{-1}C_U = \begin{bmatrix} 0 & D_1^{-1}H \\ 0 & 0 \end{bmatrix}.$$

The modified AOR (MAOR) method is defined as follows:

$$\boldsymbol{x}^{k+1} = \mathscr{L}_{\omega_1,\omega_2,\gamma}\boldsymbol{x}^k + \Phi_{\omega_1,\omega_2,\gamma,b}, \quad k = 0, 1, \ldots, \tag{3}$$

where

$$\mathscr{L}_{\omega_1,\omega_2,\gamma} = (I - \gamma L)^{-1}(I - \Omega + (\omega_2 - \gamma)L + \omega_1 U),$$

$$\Phi_{\omega_1,\omega_2,\gamma,b} = (I - \gamma L)^{-1}D^{-1}\Omega b,$$

and $\Omega = \operatorname{diag}(\omega_1 I_1, \omega_2 I_2)$, $\omega_1\omega_2 \neq 0$, $\omega_1, \omega_2, \gamma \in \mathbb{R}$ and I_1, I_2 are the identity matrices of the same dimensions as D_1, D_2, respectively.

When the parameter γ equals ω_2 the MAOR method reduces to the MSOR method.

3. Convergence properties

It is well known that MAOR method will converge for any start vector if and only if $\rho(\mathscr{L}_{\omega_1,\omega_2,\gamma}) < 1$. The following Lemma gives us an upper bound for $\|\mathscr{L}_{\omega_1,\omega_2,\gamma}\|_\infty$.

Let n_1 and n_2 be the dimensions of the diagonal matrices D_1 and D_2, respectively and

$$N_1=\{1,2,\dots,n_1\}, \qquad N_2=\{n_1+1,n_1+2,\dots,n\}.$$

For all $i=1,2,\dots,n$ we shall denote

$$l_i=r_i(L) \quad \text{and} \quad u_i=r_i(U).$$

We also denote

$$\ell=\max_{i\in N_2} l_i \quad \text{and} \quad u=\max_{i\in N_1} u_i.$$

Lemma 2.

$$\|\mathscr{L}_{\omega_1,\omega_2,\gamma}\|_\infty \leqslant \max\{|1-\omega_1|+|\omega_1|u,\ |1-\omega_2|+(|\omega_2-\gamma\omega_1|+|\gamma\omega_1|u)\ell\}.$$

Proof. There exists an n-dimensional vector y such that $\|y\|_\infty=1$ and

$$\|\mathscr{L}_{\omega_1,\omega_2,\gamma}\|_\infty=\|\mathscr{L}_{\omega_1,\omega_2,\gamma}y\|_\infty.$$

Let us denote $\mathscr{L}_{\omega_1,\omega_2,\gamma}y=z$. Obviously,

$$(I-\gamma L)z=(I-\Omega+(\omega_2-\gamma)L+\omega_1U)y$$

or

$$z_i=(1-\omega_1)y_i+\omega_1\sum_{k\in N_2}(U)_{ik}y_k, \quad i\in N_1, \tag{4}$$

$$z_j=\gamma\sum_{k\in N_1}(L)_{jk}z_k+(1-\omega_2)y_j+(\omega_2-\gamma)\sum_{k\in N_1}(L)_{jk}y_k, \quad j\in N_2. \tag{5}$$

Here we use the notation $(A)_{ij}$ for the ijth element of the matrix A.

From the equality (4) it follows that

$$|z_i-(1-\omega_1)y_i|\leqslant|\omega_1|u_i, \quad i\in N_1$$

and

$$|z_i|\leqslant|1-\omega_1|+|\omega_1|u.$$

From the second equality (5) for all $j\in N_2$ we have

$$\begin{aligned}
|z_j-(1-\omega_2)y_j| &\leqslant \sum_{k\in N_1}|(L)_{jk}||\gamma z_k+(\omega_2-\gamma)y_k| \\
&= \sum_{k\in N_1}|(L)_{jk}||\gamma(z_k-(1-\omega_1)y_k)+(\omega_2-\gamma\omega_1)y_k| \\
&\leqslant \sum_{k\in N_1}|(L)_{jk}|(|\gamma||\omega_1|u_k+|\omega_2-\gamma\omega_1|) \\
&\leqslant (|\gamma||\omega_1|u+|\omega_2-\gamma\omega_1|)\ell.
\end{aligned}$$

Finally,

$$|z_j| \leqslant |1-\omega_2| + (|\gamma||\omega_1|u + |\omega_2-\gamma\omega_1|)\ell$$

and the lemma is proved. □

This Lemma allows us to obtain the convergence area for MAOR method when the matrix of our system is strictly diagonally dominant.

3.1. SDD matrices

From now on we shall use the following notations: for $x \in \mathbb{R}$, $x \geqslant 0$,

$$t(x) = \frac{2}{1+x}, \qquad f^-(x) = \frac{1-|1-x|-x\ell}{\ell(1+u)}, \qquad f^+(x) = \frac{1-|1-x|+x\ell}{\ell(1+u)},$$

$$g^-(x) = -f^-(x)\frac{1+u}{1-u}.$$

Theorem 3. *Let A be an SDD matrix. Let ℓ and u are as defined above and*

$$t_1 = t(u), \qquad t_2 = t(\ell), \qquad t_3 = t(\ell u).$$

Then the MAOR iterative method is a convergent one if the parameters ω_1, ω_2 and γ are chosen in the following way:

$$0 < \omega_1 < t_1, \quad 0 < \omega_2 < t_2, \quad -f^-(\omega_2) < \gamma\omega_1 < f^+(\omega_2) \tag{6}$$

or

$$0 < \omega_1 < t_1, \quad t_2 \leqslant \omega_2 < t_3, \quad g^-(\omega_2) < \gamma\omega_1 < f^+(\omega_2). \tag{7}$$

Proof. Since A is an SDD matrix, it follows that $\ell < 1$ and $u < 1$. Under this assumption it is easy to prove that for all above choices of ω_1, ω_2 and γ the inequalities

$$|1-\omega_1| + |\omega_1|u < 1$$

and

$$|1-\omega_2| + (|\omega_2-\gamma\omega_1| + |\gamma\omega_1|u)\ell < 1$$

hold. Now, the convergence follows from Lemma 2. □

Comments and remarks. We shall continue to use the notations ℓ and u which correspond to σ_2 and σ_1 from [2], respectively.

Let us compare our result with the Theorem 3.6 from [2]. The convergence area for parameters ω_1, ω_2 and γ in Theorem 3.6 from [2] is

$$0 < \omega_1 < t_1, \qquad 0 < \omega_2 < t_2, \qquad 0 \leqslant \gamma \leqslant \omega_2 \tag{8}$$

or

$$0<\omega_2\leqslant\gamma<t_2, \qquad 0<\gamma\omega_1<\omega_2 t_1. \tag{9}$$

If (8) is satisfied, then we have

$$0\leqslant\gamma\omega_1\leqslant\frac{2\omega_2}{1+u}=\omega_2 t_1.$$

Since

$$\omega_2 t_1<f^+(\omega_2)=\begin{cases}\dfrac{\omega_2+\omega_2\ell}{\ell(1+u)} & \text{for } 0<\omega_2\leqslant 1,\\[2ex] \dfrac{2-\omega_2+\omega_2\ell}{\ell(1+u)} & \text{for } 1<\omega_2<t_2,\end{cases}$$

we conclude that for such a choice of parameters (6) is satisfied.

In the second case, if (9) is satisfied, similarly we conclude that

$$0<\gamma\omega_1<f^+(\omega_2).$$

Since $0<\omega_2<t_2$ and

$$0<\omega_1<\frac{2}{1+u}\frac{\omega_2}{\gamma}\leqslant\frac{2}{1+u}=t_1,$$

we have again that (6) is satisfied. Hence, the convergence area from our Theorem 3 is always wider than the corresponding one in Theorem 3.6 from [2].

In the case of MSOR method, when $\gamma=\omega_2$ our Theorem 3 states that the MSOR method will converge if

$$0<\omega_1<t_1, \qquad 0<\omega_2<t_2$$

or

$$\frac{1+\ell-(2/\omega_2)}{\ell(1-u)}<\omega_1<\frac{(2/\omega_2)-1+\ell}{\ell(1+u)}, \quad t_2\leqslant\omega_2<t_3,$$

which is obviously wider than the corresponding one in Theorem 3.6 from [2].

References

[1] A. Hadjidimos, A. Psimarni, A.K. Yeyios, On the convergence of the modified accelerated overrelaxation (MAOR) method, Appl. Numer. Math. 10 (1992) 115–127.
[2] Y. Song, On the convergence of the MAOR method, J. Comput. Appl. Math. 79 (1997) 299–317.
[3] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
[4] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.