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Stability analysis of Cohen–Grossberg neural network with both time-varying and continuously distributed delays

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Abstract

In this paper, the Cohen–Grossberg neural network model with both time-varying and continuously distributed delays is considered. Without assuming both global Lipschitz conditions on these activation functions and the differentiability on these time-varying delays, applying the idea of vector Lyapunov function, M-matrix theory and inequality technique, several new sufficient conditions are obtained to ensure the existence, uniqueness, and global exponential stability of equilibrium point for Cohen–Grossberg neural network with both time-varying and continuously distributed delays. These results generalize and improve the earlier publications. Two numerical examples are given to show the effectiveness of the obtained results. It is believed that these results are significant and useful for the design and applications of the Cohen–Grossberg neural networks.

Keywords: Global exponential stability; Cohen–Grossberg neural network; Time-varying delays; Distributed delays

1. Introduction

The Cohen–Grossberg neural network models, first proposed and studied by Cohen and Grossberg [8], have been widely applied within various engineering and scientific fields such as neuro-biology, population biology, and computing technology. In such applications, it is of prime importance to ensure that the designed neural networks be stable. This neural network can be described by the following differential equations [8]:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}g_j(x_j(t)) + I_i\right] \tag{1}$$

for $i = 1, 2, \ldots, n$. In hardware implementation, however, time delays occur due to finite switching speed of the amplifiers and communication time [7]. On the other hand, it has also been shown that the process of moving images

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requires the introduction of delay in signal transmitted through the networks [20]. It is known that time delays may lead to oscillation, divergence, or instability which may be harmful to a system [16,2]. Hence, for the Cohen–Grossberg model (1), Ye et al. [27] also introduced delays by considering the following system of delayed differential equations:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} c_{ij}^{(k)} g_j(x_j(t-\tau_k))\right] \tag{2}$$

for $i = 1, 2, \ldots, n$. Further studies were taken by Wang and Zou [25,26], Lu and Chen [15], Chen and Rong [5], Rong [19], Liao et al. [13], Cao and Liang [3] about the following model:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij} g_j(x_j(t-\tau_{ij})) + I_i\right] \tag{3}$$

for $i = 1, 2, \ldots, n$. In [26,5], several sufficient conditions were obtained to ensure model (3) to be asymptotically stable. In [19,34], based on Lyapunov stability theory and linear matrix inequality (LMI), several sufficient conditions were obtained to ensure model (3) to be robustly stable. In [13], several sufficient conditions were obtained to ensure model (3) to be exponentially stable. A set of conditions ensuring global exponential stability of model (3) were derived in [25] when $c_{ij} = 0$ and $d_{ij} = 0$, respectively. And, by property of Lyapunov diagonal stable matrix, absolutely global stability was studied in [15] for model (3) when $d_{ij} = 0$.

Usually, constant fixed time delays in models of delayed feedback systems serve as good approximation in simple circuits having a small number of cells. In most situations, delays are time-varying. Therefore, the studies of neural networks with time-varying delays are more important and actual than those with constant delays. Recently, Hwang et al. [12], Cao and Liang [3], Arik and Orman [1] and Yuan and Cao [28] studied the Cohen–Grossberg neural networks with time-varying delays of form

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij} g_j(x_j(t-\tau_{ij}(t))) + I_i\right] \tag{4}$$

for $i = 1, 2, \ldots, n$. Several sufficient conditions were obtained to ensure global exponential stability for model (4). Meantime, Zhang et al. [31], Chen and Rong [6] have considered the following model:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij} f_j(x_j(t-\tau_{ij}(t))) + I_i\right] \tag{5}$$

for $i = 1, 2, \ldots, n$. Several sufficient conditions were given to ensure global exponential stability for model (5).

Since a neural network usually has a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, it is desired to model them by introducing continuously distributed delays over a certain duration of time such that the distant past has less influence compared to the recent behavior of the state [10]. Recently, it is noted that stability of Hopfield neural networks, cellular neural networks and bidirectional associative memory neural networks with the continuously distributed delays are discussed in [10,18,17,32,22,14,33,29,23,30]. Today, both time-varying delays and distributed delays have been widely accepted as important parameters associated with neural networks models. The recurrent neural networks models with both time-varying delays and distributed delays have been considered in [23,30]. To the best of our knowledge, few authors have considered Cohen–Grossberg neural network model with both time-varying delays and distributed delays. In this paper, without assuming both global Lipschitz conditions on these activation functions and the differentiability on these time-varying delays, as need in most other paper, we shall study global exponential stability of Cohen–Grossberg neural network with both time-varying delays and distributed delays.

2. Model description and preliminaries

In this paper, we consider the following model

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = &-a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}f_j(x_j(t-\tau_{ij}(t)))\right.\\ &\left.-\sum_{j=1}^{n} q_{ij}\int_{-\infty}^{t} K_{ij}(t-s)v_j(x_j(s))\,\mathrm{d}s + I_i\right]\end{aligned} \tag{6}$$

for $i=1,2,\ldots,n$, where n corresponds to the number of units in a neural network; $x_i(t)$ corresponds to the state of the ith unit at time t; $g_j(x_j(t))$, $f_j(x_j(t))$ and $v_j(x_j(t))$ denote the activation functions of the jth unit at time t; $\tau_{ij}(t)$ corresponds to the transmission delay along the axon of the jth unit from the ith unit and satisfies $0 \leqslant \tau_{ij}(t) \leqslant \tau_{ij}$ (τ_{ij} is a constant); $a_i(x_i(t))$ represents an amplification function at time t; $b_i(x_i(t))$ is an appropriately behaved function at time t such that the solutions of model (6) remain bounded; $C=(c_{ij})_{n\times n}$, $D=(d_{ij})_{n\times n}$ and $Q=(q_{ij})_{n\times n}$ are connection matrices, I_i is the constant input from outside of the network; the delay kernel K_{ij}: $[0,+\infty) \to [0,+\infty)$ is real valued nonnegative continuous function and satisfies [30]

$$\int_0^{+\infty} \mathrm{e}^{\beta s} K_{ij}(s)\,\mathrm{d}s = p_{ij}(\beta),$$

where $p_{ij}(\beta)$ is continuous function in $[0,\delta)$, $\delta > 0$, and $p_{ij}(0)=1$, $i,j=1,2,\ldots,n$.

The initial conditions of model (6) are of the form $x_i(s)=\varphi_i(s)$, $s \leqslant 0$, where φ_i is bounded and continuous on $(-\infty, 0]$.

Throughout this paper, we make the following assumptions:

(H1) Each function $a_i(u)$ is continuous and $0 < \underline{a}_i \leqslant a_i(u)$ for all $u \in R$, $i=1,2,\ldots,n$.

(H2) $b_i(u)$ is monotone increasing, i.e., there exists a positive diagonal matrix $B=\operatorname{diag}(b_1,b_2,\ldots,b_n)$ such that

$$\frac{b_i(u)-b_i(v)}{u-v} \geqslant b_i$$

for all $u, v \in R (u \neq v)$, $i=1,2,\ldots,n$.

(H3) For functions g_i, f_i and v_i, there exist three positive diagonal matrices $G=\operatorname{diag}(G_1,G_2,\ldots,G_n)$, $F=\operatorname{diag}(F_1, F_2,\ldots,F_n)$ and $V=\operatorname{diag}(V_1,V_2,\ldots,V_n)$ such that

$$G_i = \sup_{u_1\neq u_2}\left|\frac{g_i(u_1)-g_i(u_2)}{u_1-u_2}\right|, \quad F_i = \sup_{u_1\neq u_2}\left|\frac{f_i(u_1)-f_i(u_2)}{u_1-u_2}\right|, V_i = \sup_{u_1\neq u_2}\left|\frac{v_i(u_1)-v_i(u_2)}{u_1-u_2}\right|$$

for all $u_1 \neq u_2$, $i=1,2,\ldots,n$.

Remark 1. In [25–27,15,5,19,13,12,3,1], the amplification function was required to be bounded, positive and continuous. However, the upper bound of amplification function in this paper is not required. In addition, assumption (H2) in this paper is as same as that in [25,13,31], the condition of differentiability of behaved function in [27,26,15,5,19,12,3,1,6] is not required.

Remark 2. Assumption (H3) in this paper is weaker than the locally and partially Lipschitz condition which is mostly used in literature [25–27,15,5,19,13,12,3,1,31,6]. The activation functions such as sigmoid type and piecewise linear type are also the special case of the function satisfying assumption (H3).

For convenience, we introduce some notations. For matrix $A=(a_{ij})_{n\times n} \in R^{n\times n}$, $|A|$ denotes the absolute-value matrix given by $|A|=(|a_{ij}|)_{n\times n}$; $A^+=(a_{ij}^+)_{n\times n}$, where $a_{ii}^+=a_{ii}$ as $a_{ii}\geqslant 0$, and $a_{ii}^+=0$ as $a_{ii}<0$, $a_{ij}^+=|a_{ij}| (i \neq j)$; $\|A\|_2=(\lambda_{\max}(A^TA))^{1/2}$, where $\lambda_{\max}(A^{\mathrm{T}}A)$ represents the maximum eigenvalue of matrix $A^{\mathrm{T}}A$. For matrix $A=(a_{ij})_{m\times n} \in$

$R^{m\times n}$, $\|A\|$ denotes Euclid norm defined by $\|A\|=(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^2)^{1/2}$. For $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}\in R^{m\times n}$, $A>B$denotes $a_{ij}>b_{ij}$ for all $i=1,2,\ldots,m,\ j=1,2,\ldots,n$.

Definition 1. The equilibrium point $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^{\mathrm{T}}$ of model (6) is said to be globally exponentially stable, if there exist constants $\varepsilon>0$ and $K>0$ such that

$$\left[\sum_{i=1}^{n}|x_i(t)-x_i^*|^r\right]^{1/r}\leqslant K\|\phi-x^*\|_r\mathrm{e}^{-\varepsilon t}$$

for all $t>0$, where $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^{\mathrm{T}}$ is solution of (6) with initial value $x_i(s)=\varphi_i(s)$, $s\leqslant 0$, $i=1,2,\ldots,n$. $\phi(s)$: $(-\infty,0]\to R^n$ is a continuous function with $\phi(s)=(\varphi_1(s),\varphi_2(s),\ldots,\varphi_n(s))^{\mathrm{T}}$, and $\|\phi-x^*\|_r=\sup_{-\infty<s\leqslant 0}[\sum_{i=1}^{n}|\varphi_i(s)-x_i^*|^r]^{1/r}$, $r\geqslant 1$.

Definition 2 (*Zhang et al. [31]*). A real matrix $A=(a_{ij})_{n\times n}$ is said to be an M-matrix if $a_{ij}\leqslant 0\ (i,j=1,2,\ldots,n;\ i\neq j)$ and successive principle minors of A are positive.

Definition 3 (*Zhang et al. [30]*). A matrix A is said to belong to a class P_0 if A satisfies that all principal minors of A are nonnegative.

Definition 4 (*Cao and Wang [4]*). A map H: $R^n\to R^n$is a homeomorphism of R^n onto itself, if $H\in C^0$, H is one-to-one, H is onto and the inverse map $H^{-1}\in C^0$.

Lemma 1 (*Forti and Tesi [9]*). *If $H(x)\in C^0$ satisfies the following conditions*:

(i) *$H(x)$ is injective on R^n*,
(ii) *$\|H(x)\|\to+\infty$ as $\|x\|\to+\infty$*,

then $H(x)$ is homeomorphism of R^n onto itself.

Lemma 2 (*Cao and Liang [3]*). *Let $a\geqslant 0$, $b_k\geqslant 0$ $(k=1,2,\ldots,m)$, then*

$$a\prod_{k=1}^{m}b_k^{q_k}\leqslant\frac{1}{r}\left(a^r+\sum_{k=1}^{m}q_kb_k^r\right),$$

where $q_k>0$ $(k=1,2,\ldots,m)$ are some constants, $\sum_{k=1}^{m}q_k=r-1$, and $r\geqslant 1$.

3. Main results

Theorem 1. *Under assumptions* (H1), (H2) *and* (H3), *model* (6) *has a unique equilibrium point*, *which is globally exponentially stable if*

$$W=B-|C|G-|D|F-|Q|V \tag{7}$$

is an M-matrix. Where

$$B=\mathrm{diag}(b_1,b_2,\ldots,b_n),\quad C=(c_{ij})_{n\times n},\quad D=(d_{ij})_{n\times n},\quad Q=(q_{ij})_{n\times n}.$$

Proof. We shall prove this theorem in two steps.

Step 1: We will prove the existence and uniqueness of the equilibrium point.

Since the equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ of model (6) satisfy the following equation

$$-a_i(x_i^*)\left[b_i(x_i^*) - \sum_{j=1}^{n} c_{ij}g_j(x_j^*) - \sum_{j=1}^{n} d_{ij}f_j(x_j^*) - \sum_{j=1}^{n} q_{ij}v_j(x_j^*) + I_i\right] = 0 \tag{8}$$

for $i = 1, 2, \ldots, n$. From assumption (H1), we know that Eq. (8) is equivalent to the following equation:

$$-b_i(x_i^*) + \sum_{j=1}^{n} c_{ij}g_j(x_j^*) + \sum_{j=1}^{n} d_{ij}f_j(x_j^*) + \sum_{j=1}^{n} q_{ij}v_j(x_j^*) - I_i = 0 \tag{9}$$

for $i = 1, 2, \ldots, n$. Let $H(x) = (H_1(x), H_2(x), \ldots, H_n(x))^{\mathrm{T}}$, where

$$H_i(x) = -b_i(x_i) + \sum_{j=1}^{n} c_{ij}g_j(x_j) + \sum_{j=1}^{n} d_{ij}f_j(x_j) + \sum_{j=1}^{n} q_{ij}v_j(x_j) - I_i$$

for $i = 1, 2, \ldots, n$. In the following, we shall prove that $H(x)$ is a homeomorphism of R^n onto itself.

First, we prove that $H(x)$ is an injective map on R^n. In fact, if there exist $x=(x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y=(y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in R^n$ and $x \neq y$ such that $H(x) = H(y)$, then

$$\begin{aligned}&-(b_i(x_i) - b_i(y_i)) + \sum_{j=1}^{n} c_{ij}(g_j(x_j) - g_j(y_j)) + \sum_{j=1}^{n} d_{ij}(f_j(x_j) - f_j(y_j))\\ &\quad + \sum_{j=1}^{n} q_{ij}(v_j(x_j) - v_j(y_j)) = 0\end{aligned}$$

for $i = 1, 2, \ldots, n$.

From assumption (H2), we know that there exists matrix $\beta = \mathrm{diag}\{\beta_1, \ldots, \beta_n\}(\beta_i \geqslant b_i)$ such that

$$b_i(x_i) - b_i(y_i) = \beta_i(x_i - y_i)$$

for $i = 1, 2, \ldots, n$.

From assumption (H3), we know that there exist three matrices $K = \mathrm{diag}\{k_1, \ldots, k_n\} > 0(-G \leqslant K \leqslant G)$, $L = \mathrm{diag}\{l_1, \ldots, l_n\} > 0(-F \leqslant L \leqslant F)$ and $U = \mathrm{diag}\{u_1, \ldots, u_n\} > 0(-V \leqslant U \leqslant V)$ such that

$$g_i(x_i) - g_i(y_i) = k_i(x_i - y_i), \quad f_i(x_i) - f_i(y_i) = l_i(x_i - y_i), \quad v_i(x_i) - v_i(y_i) = u_i(x_i - y_i)$$

for $i = 1, 2, \ldots, n$. Hence, we have

$$-\beta_i(x_i - y_i) + \sum_{j=1}^{n} c_{ij}k_j(x_j - y_j) + \sum_{j=1}^{n} d_{ij}l_j(x_j - y_j) + \sum_{j=1}^{n} q_{ij}u_j(x_j - y_j) = 0$$

for $i = 1, 2, \ldots, n$. Furthermore, we get

$$-\beta_i|x_i - y_i| + \sum_{j=1}^{n} |c_{ij}||k_j||x_j - y_j| + \sum_{j=1}^{n} |d_{ij}||l_j||x_j - y_j| + \sum_{j=1}^{n} |q_{ij}||u_j||x_j - y_j| \geqslant 0$$

for $i = 1, 2, \ldots, n$. That is,

$$(\beta - |C||K| - |D||L| - |Q||U|)(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|)^{\mathrm{T}} \leqslant 0.$$

From $W = B - |C|G - |D|F - |Q|V$ is an M-matrix, we know that $\beta - |C||K| - |D||L| - |Q||U|$ is also an M-matrix, hence

$$x_i = y_i, \quad i = 1, 2, \ldots, n,$$

which is a contradiction. So $H(x)$ is an injective on R^n.

Second, we prove that $\|H(x)\| \to +\infty$ as $\|x\| \to +\infty$.

Since $W = B - |C|G - |D|F - |Q|V$ is an M-matrix, there exists a positive diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ such that $PW + W^{\mathrm{T}}P$ is a positive definite matrix. Let

$$\widetilde{H}(x) = (\widetilde{H}_1(x), \widetilde{H}_2(x), \ldots, \widetilde{H}_n(x))^{\mathrm{T}},$$

where

$$\begin{aligned}\widetilde{H}_i(x) = & -(b_i(x_i) - b_i(0)) + \sum_{j=1}^{n} c_{ij}(g_j(x_j) - g_j(0)) + \sum_{j=1}^{n} d_{ij}(f_j(x_j) - f_j(0)) \\ & + \sum_{j=1}^{n} q_{ij}(v_j(x_j) - v_j(0))\end{aligned}$$

for $i = 1, 2, \ldots, n$. Calculating

$$\begin{aligned} x^{\mathrm{T}}P\widetilde{H}(x) &= \sum_{i=1}^{n} x_i p_i \widetilde{H}_i(x) \\ &= \sum_{i=1}^{n}\Bigg[-p_i x_i(b_i(x_i) - b_i(0)) + \sum_{j=1}^{n} c_{ij} p_i x_i (g_j(x_j) - g_j(0)) \\ &\quad + \sum_{j=1}^{n} d_{ij} p_i x_i (f_j(x_j) - f_j(0)) + \sum_{j=1}^{n} q_{ij} p_i x_i (v_j(x_j) - v_j(0))\Bigg] \\ &= \sum_{i=1}^{n}\Bigg[-p_i x_i \beta_i x_i + \sum_{j=1}^{n} c_{ij} p_i x_i k_j x_j + \sum_{j=1}^{n} d_{ij} p_i x_i l_j x_j + \sum_{j=1}^{n} q_{ij} p_i x_i u_j x_j\Bigg] \\ &\leqslant \sum_{i=1}^{n}\Bigg[-p_i b_i x_i^2 + \sum_{j=1}^{n} |c_{ij}| p_i G_j |x_i| \cdot |x_j| + \sum_{j=1}^{n} |d_{ij}| p_i F_j |x_i| \cdot |x_j| \\ &\quad + \sum_{j=1}^{n} |q_{ij}| p_i V_j |x_i| \cdot |x_j|\Bigg] \\ &= -(|x_1|, |x_2|, \ldots, |x_n|)PW(|x_1|, |x_2|, \ldots, |x_n|)^{\mathrm{T}} \\ &= -\frac{1}{2}(|x_1|, |x_2|, \ldots, |x_n|)(PW + W^{\mathrm{T}}P)(|x_1|, |x_2|, \ldots, |x_n|)^{\mathrm{T}} \\ &\leqslant -\frac{1}{2}\lambda_{\min}(PW + W^{\mathrm{T}}P)\sum_{i=1}^{n} x_i^2 \\ &= -\frac{1}{2}\lambda_{\min}(PW + W^{\mathrm{T}}P)\|x\|^2. \end{aligned}$$

Using Schwartz inequality, we get

$$\|x\| \cdot \|P\| \cdot \|\widetilde{H}(x)\| \geqslant \tfrac{1}{2}\lambda_{\min}(PW + W^{\mathrm{T}}P)\|x\|^2.$$

When $\|x\| \neq 0$, we have $\|\widetilde{H}(x)\| \geqslant \frac{1}{2}\lambda_{\min}(PW + W^{\mathrm{T}}P)(\|x\|/\|P\|)$, therefore $\|\widetilde{H}(x)\| \to +\infty$ as $\|x\| \to +\infty$, hence $\|H(x)\| \to +\infty$ as $\|x\| \to +\infty$.

From Lemma 1, we know that $H(x)$ is a homeomorphism of R^n, thus, model (6) has a unique equilibrium point.

Step 2: We prove that the unique equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ of model (6) is globally exponentially stable.

Since $W = B - |C|G - |D|F - |Q|V$ is an M-matrix, there exists $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ such that

$$-\xi_i b_i + \sum_{j=1}^{n} |c_{ij}| G_j \xi_j + \sum_{j=1}^{n} |d_{ij}| F_j \xi_j + \sum_{j=1}^{n} |q_{ij}| V_j \xi_j < 0$$

for $i = 1, 2, \ldots, n$. Constructing the function

$$\Gamma_i(\theta) = \xi_i \left(-b_i + \frac{\theta}{a_i}\right) + \sum_{j=1}^{n} |c_{ij}| G_j \xi_j + \sum_{j=1}^{n} |d_{ij}| F_j \xi_j \mathrm{e}^{\tau\theta} + \sum_{j=1}^{n} |q_{ij}| V_j \xi_j p_{ij}(\theta)$$

for $i=1, 2, \ldots, n$, where $\tau = \max_{1 \leqslant i,j \leqslant n}\{\tau_{ij}\}$. Obviously, $\Gamma_i(0) < 0$, $\Gamma_i(\theta) \to +\infty$ as $\theta \to +\infty$. From the assumption of the delay kernels, we know that $\Gamma_i(\theta)$ are continuous, so there exist $\varepsilon > 0$ such that

$$\Gamma_i(\varepsilon) = \xi_i \left(-b_i + \frac{\varepsilon}{a_i}\right) + \sum_{j=1}^{n} |c_{ij}| G_j \xi_j + \sum_{j=1}^{n} |d_{ij}| F_j \xi_j \mathrm{e}^{\tau\varepsilon} + \sum_{j=1}^{n} |q_{ij}| V_j \xi_j p_{ij}(\varepsilon) < 0 \tag{10}$$

for $i = 1, 2, \ldots, n$. Let

$$y_i(t) = x_i(t) - x_i^*, \quad \widetilde{a}_i(y_i(t)) = a_i(y_i(t) + x_i^*), \quad \widetilde{b}_i(y_i(t)) = b_i(y_i(t) + x_i^*) - b_i(x_i^*),$$

$$\widetilde{g}_j(y_j(t)) = g_j(y_j(t) + x_j^*) - g_j(x_j^*), \quad \widetilde{f}_j(y_j(t)) = f_j(y_j(t) + x_j^*) - f_j(x_j^*),$$
$$\widetilde{v}_j(y_j(t)) = v_j(y_j(t) + x_j^*) - v_j(x_j^*),$$

then model (6) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = & -\widetilde{a}_i(y_i(t)) \left[\widetilde{b}_i(y_i(t)) - \sum_{j=1}^{n} c_{ij}\widetilde{g}_j(y_j(t)) - \sum_{j=1}^{n} d_{ij}\widetilde{f}_j(y_j(t - \tau_{ij}(t)))\right.\\ & \left. - \sum_{j=1}^{n} q_{ij} \int_{-\infty}^{t} K_{ij}(t-s)\widetilde{v}_j(y_j(s))\,\mathrm{d}s\right]\end{aligned} \tag{11}$$

for $i = 1, 2, \ldots, n$. Let

$$w_i(t) = \mathrm{e}^{\varepsilon t}|y_i(t)|$$

for $i=1,2,\ldots,n$. Calculating the upper right derivative $D^+w_i(t)$ of $w_i(t)$ along the solutions of (11), from assumption (H1), (H2) and (H3), we get

$$\begin{aligned}
D^+w_i(t) &= \mathrm{e}^{\varepsilon t}\,\mathrm{sgn}(y_i(t))\Bigg\{-\widetilde{a}_i(y_i(t))\Bigg[\widetilde{b}_i(y_i(t)) - \sum_{j=1}^n c_{ij}\widetilde{g}_j(y_j(t)) - \sum_{j=1}^n d_{ij}\widetilde{f}_j(y_j(t-\tau_{ij}(t))) \\
&\quad - \sum_{j=1}^n q_{ij}\int_{-\infty}^t K_{ij}(t-s)\widetilde{v}_j(y_j(s))\,\mathrm{d}s\Bigg]\Bigg\} + \varepsilon \mathrm{e}^{\varepsilon t}|y_i(t)| \\
&= \mathrm{e}^{\varepsilon t}\mathrm{sgn}(y_i(t))\Bigg\{-\widetilde{a}_i(y_i(t))\Bigg[\beta_i y_i(t) - \sum_{j=1}^n c_{ij}k_j y_j(t) - \sum_{j=1}^n d_{ij}l_j y_j(t-\tau_{ij}(t)) \\
&\quad - \sum_{j=1}^n q_{ij}\int_{-\infty}^t K_{ij}(t-s)u_j y_j(s)\,\mathrm{d}s\Bigg]\Bigg\} + \varepsilon \mathrm{e}^{\varepsilon t}|y_i(t)| \\
&\leqslant \mathrm{e}^{\varepsilon t}\widetilde{a}_i(y_i(t))\Bigg\{-\beta_i|y_i(t)| + \sum_{j=1}^n |c_{ij}||k_j||y_j(t)| + \sum_{j=1}^n |d_{ij}||l_j||y_j(t-\tau_{ij}(t))| \\
&\quad + \sum_{j=1}^n |q_{ij}|\int_{-\infty}^t K_{ij}(t-s)|u_j||y_j(s)|\,\mathrm{d}s\Bigg\} + \varepsilon \mathrm{e}^{\varepsilon t}|y_i(t)| \\
&\leqslant \mathrm{e}^{\varepsilon t}\widetilde{a}_i(y_i(t))\Bigg\{-b_i|y_i(t)| + \sum_{j=1}^n |c_{ij}|F_j|y_j(t)| + \sum_{j=1}^n |d_{ij}|G_j|y_j(t-\tau_{ij}(t))| \\
&\quad + \sum_{j=1}^n |q_{ij}|\int_{-\infty}^t K_{ij}(t-s)v_j|y_j(s)|\,\mathrm{d}s\Bigg\} + \varepsilon \mathrm{e}^{\varepsilon t}|y_i(t)| \\
&\leqslant \mathrm{e}^{\varepsilon t}\widetilde{a}_i(y_i(t))\Bigg\{\left(\frac{\varepsilon}{a_i} - b_i\right)|y_i(t)| + \sum_{j=1}^n |c_{ij}|G_j|y_j(t)| + \sum_{j=1}^n |d_{ij}|F_j|y_j(t-\tau_{ij}(t))| \\
&\quad + \sum_{j=1}^n |q_{ij}|\int_{-\infty}^t K_{ij}(t-s)V_j|y_j(s)|\,\mathrm{d}s\Bigg\} \\
&= \widetilde{a}_i(y_i(t))\Bigg\{\left(\frac{\varepsilon}{a_i} - b_i\right)w_i(t) + \sum_{j=1}^n |c_{ij}|G_j w_j(t) + \sum_{j=1}^n |d_{ij}|F_j \mathrm{e}^{\varepsilon\tau_{ij}(t)} w_j(t-\tau_{ij}(t)) \\
&\quad + \sum_{j=1}^n |q_{ij}|V_j\int_{-\infty}^t \mathrm{e}^{\varepsilon(t-s)}K_{ij}(t-s)w_j(s)\,\mathrm{d}s\Bigg\} \\
&\leqslant \widetilde{a}_i(y_i(t))\Bigg\{\left(\frac{\varepsilon}{a_i} - b_i\right)w_i(t) + \sum_{j=1}^n |c_{ij}|G_j w_j(t) + \mathrm{e}^{\varepsilon\tau}\sum_{j=1}^n |d_{ij}|F_j w_j(t-\tau_{ij}(t)) \\
&\quad + \sum_{j=1}^n |q_{ij}|V_j\int_{-\infty}^t \mathrm{e}^{\varepsilon(t-s)}K_{ij}(t-s)w_j(s)\,\mathrm{d}s\Bigg\} \qquad (12)
\end{aligned}$$

for $i=1,2,\ldots,n$.

Defining the curve $\gamma=\{z(l)=(\xi_1 l,\xi_2 l,\ldots,\xi_n l)|l>0\}$ and the set $\Omega(z)=\{u|0\leqslant u\leqslant z, z\in\gamma\}$. It is obvious that $\Omega(z(l))\supset\Omega(z(l'))$, when $l>l'$.

Let $l_0 = (1+\delta)\|\phi - x^*\|_r / \min_{1\leqslant i\leqslant n}\{\xi_i\}$ (δ is a positive constant), then

$$w_i(s) = \mathrm{e}^{\varepsilon s}|y_i(s)| \leqslant |y_i(s)| = |\varphi_i(s) - x_i^*| \leqslant \|\phi - x^*\|_r < \xi_i l_0, \quad -\infty < s \leqslant 0.$$

In the following, we will prove that

$$w_i(t) < \xi_i l_0 \tag{13}$$

for $t \geqslant 0$, $i = 1, 2, \ldots, n$. If (13) is not true, then there exist some i and t_1 such that

$$w_i(t_1) = \xi_i l_0, \quad D^+ w_i(t_1) \geqslant 0 \quad \text{and} \quad w_j(t) \leqslant \xi_j l_0$$

for $-\infty < t \leqslant t_1$, $j = 1, 2, \ldots, n$. However, from (10) and (12) we get

$$\begin{aligned} D^+ w_i(t_1) &\leqslant \widetilde{a}_i(y_i(t_1)) \left\{ \left(\frac{\varepsilon}{a_i} - b_i\right)\xi_i l_0 + \sum_{j=1}^n |c_{ij}| G_j \xi_j l_0 + \mathrm{e}^{\varepsilon\tau} \sum_{j=1}^n |d_{ij}| F_j \xi_j l_0 \right. \\ &\quad \left. + \sum_{j=1}^n |q_{ij}| V_j \int_{-\infty}^{t_1} \mathrm{e}^{\varepsilon(t_1 - s)} K_{ij}(t_1 - s)\xi_j l_0 \,\mathrm{d}s \right\} \\ &= \widetilde{a}_i(y_i(t_1)) \left\{ \left(\frac{\varepsilon}{a_i} - b_i\right)\xi_i + \sum_{j=1}^n |c_{ij}| G_j \xi_j + \mathrm{e}^{\varepsilon\tau} \sum_{j=1}^n |d_{ij}| F_j \xi_j \right. \\ &\quad \left. + \sum_{j=1}^n |q_{ij}| V_j p_{ij}(\varepsilon)\xi_j \right\} l_0 < 0, \end{aligned}$$

this is a contradiction, so

$$w_i(t) < \xi_i l_0$$

for $t \geqslant 0$, $i = 1, 2, \ldots, n$. That is

$$|x_i(t) - x_i^*| \leqslant \xi_i l_0 \mathrm{e}^{-\varepsilon t}$$

for $t \geqslant 0$, $i = 1, 2, \ldots, n$. Hence

$$\left[\sum_{i=1}^n |x_i(t) - x_i^*|^r\right]^{1/r} \leqslant K\|\phi - x^*\|_r \mathrm{e}^{-\varepsilon t}$$

for all $t \geqslant 0$, where

$$K = \frac{(1+\delta)(\sum_{i=1}^n \xi_i^r)^{1/r}}{\min_{1\leqslant i\leqslant n}\{\xi_i\}} > 1.$$

It means that the equilibrium point of model (6) is globally exponentially stable. The proof is completed. □

Remark 3. In Theorem 1 in this paper, the condition ensuring global exponential stability for model (6) is independent of amplification function and delays, which implies the strong self-regulation is dominant in the networks.

Corollary 1. *Under assumptions* (H1), (H2) *and* (H3), *model* (6) *has a unique equilibrium point, which is globally exponentially stable if there exist constants* $v_k > 0\,(k=1, 2, \ldots, K_1)$, $\mu_k > 0\,(k=1, 2, \ldots, K_2)$, $\omega_k > 0\,(k=1, 2, \ldots, K_3)$,

$\gamma_i > 0$ $(i = 1, 2, \ldots, n)$, $\xi_{ij}, \xi^*_{ij}, \eta_{ij}, \eta^*_{ij}, \rho_{ij}, \rho^*_{ij}, \sigma_{ij}, \sigma^*_{ij}, \alpha_{ij}, \alpha^*_{ij}, \beta_{ij}, \beta^*_{ij} \in R$ $(i, j = 1, 2, \ldots, n)$ *such that*

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{K_1} v_k|c_{ij}|^{r\xi_{ij}/v_k} G_j^{r\eta_{ij}/v_k} + \frac{\gamma_j}{\gamma_i}|c_{ji}|^{r\xi^*_{ji}} G_i^{r\eta^*_{ji}} + \sum_{k=1}^{K_2} \mu_k|d_{ij}|^{r\rho_{ij}/\mu_k} F_j^{r\sigma_{ij}/\mu_k} + \frac{\gamma_j}{\gamma_i}|d_{ji}|^{r\rho^*_{ji}} F_i^{r\sigma^*_{ji}} + \sum_{k=1}^{K_3} \omega_k|q_{ij}|^{r\alpha_{ij}/\omega_k} V_j^{r\beta_{ij}/\omega_k} + \frac{\gamma_j}{\gamma_i}|q_{ji}|^{r\alpha^*_{ji}} V_i^{r\beta^*_{ji}}\right) < rb_i, \tag{14}$$

where $r=\sum_{k=1}^{K_1} v_k + 1 = \sum_{k=1}^{K_2}\mu_k + 1 = \sum_{k=1}^{K_3}\omega_k + 1$; $K_1\xi_{ij}+\xi^*_{ij}=1$, $K_1\eta_{ij}+\eta^*_{ij}=1$, $K_2\rho_{ij}+\rho^*_{ij}=1$, $K_2\sigma_{ij}+\sigma^*_{ij}=1$, $K_3\alpha_{ij}+\alpha^*_{ij}=1$, $K_3\beta_{ij}+\beta^*_{ij}=1$.

Proof. We consider the following linear system

$$\frac{dz}{dt} = (-B + |C|G + |D|F + |Q|V)z. \tag{15}$$

Constructing a Lyapunov function

$$V(z) = \frac{1}{r}\sum_{i=1}^{n} \gamma_i |z_i|^r.$$

Calculating the right derivative D^+V of V along the solutions of (15), we get

$$\begin{aligned} D^+V(z) &= \sum_{i=1}^{n} \gamma_i |z_i(t)|^{r-1}\left\{\operatorname{sgn}(z_i(t))\left[-b_i z_i(t) + \sum_{j=1}^{n}(|c_{ij}|G_j + |d_{ij}|F_j + |q_{ij}|V_j)z_j(t)\right]\right\} \\ &\leqslant \sum_{i=1}^{n}\gamma_i\left\{-b_i|z_i(t)|^r + \sum_{j=1}^{n}(|c_{ij}|G_j + |d_{ij}|F_j + |q_{ij}|V_j)|z_i(t)|^{r-1}|z_j(t)|\right\}. \end{aligned} \tag{16}$$

From Lemma 2, we get

$$\begin{aligned} |c_{ij}|G_j|z_i(t)|^{r-1}|z_j(t)| &= \prod_{k=1}^{K_1}(|c_{ij}|^{\xi_{ij}/v_k} G_j^{\eta_{ij}/v_k}|z_i(t)|)^{v_k} \times |c_{ij}|^{\xi^*_{ij}} G_j^{\eta^*_{ij}}|z_j(t)| \\ &\leqslant \frac{1}{r}\left(\sum_{k=1}^{K_1} v_k|c_{ij}|^{r\xi_{ij}/v_k} G_j^{r\eta_{ij}/v_k}|z_i(t)|^r + |c_{ij}|^{r\xi^*_{ij}} G_j^{r\eta^*_{ij}}|z_j(t)|^r\right). \end{aligned} \tag{17}$$

Similarly, we have

$$|d_{ij}|F_j|z_i(t)|^{r-1}|z_j(t)| \leqslant \frac{1}{r}\left(\sum_{k=1}^{K_2} \mu_k|d_{ij}|^{r\rho_{ij}/\mu_k} F_j^{r\sigma_{ij}/\mu_k}|z_i(t)|^r + |d_{ij}|^{r\rho^*_{ij}} F_j^{r\sigma^*_{ij}}|z_j(t)|^r\right), \tag{18}$$

$$|q_{ij}|V_j|z_i(t)|^{r-1}|z_j(t)| \leqslant \frac{1}{r}\left(\sum_{k=1}^{K_3} \omega_k|q_{ij}|^{r\alpha_{ij}/\omega_k} V_j^{r\beta_{ij}/\omega_k}|z_i(t)|^r + |q_{ij}|^{r\alpha^*_{ij}} V_j^{r\beta^*_{ij}}|z_j(t)|^r\right). \tag{19}$$

By applying (17)–(19) to (16), according to the inequalities in (14), we get

$$
\begin{aligned}
D^+V(z) \leqslant{} & \frac{1}{r}\sum_{i=1}^{n}\gamma_i\left\{-rb_i+\sum_{j=1}^{n}\left(\sum_{k=1}^{K_1}v_k|c_{ij}|^{r\xi_{ij}/v_k}G_j^{r\eta_{ij}/v_k}+\frac{\gamma_j}{\gamma_i}|c_{ji}|^{r\xi_{ji}^*}G_i^{r\eta_{ji}^*}\right.\right.\\
&+\sum_{k=1}^{K_2}\mu_k|d_{ij}|^{r\rho_{ij}/\mu_k}F_j^{r\sigma_{ij}/\mu_k}+\frac{\gamma_j}{\gamma_i}|d_{ji}|^{r\rho_{ji}^*}F_i^{r\sigma_{ji}^*}\\
&\left.\left.+\sum_{k=1}^{K_3}\omega_k|q_{ij}|^{r\alpha_{ij}/\omega_k}V_j^{r\beta_{ij}/\omega_k}+\frac{\gamma_j}{\gamma_i}|q_{ji}|^{r\alpha_{ji}^*}V_i^{r\beta_{ji}^*}\right)\right\}|z_i(t)|^r\\
<{} & 0, \quad t>0.
\end{aligned}
\tag{20}
$$

From (15) and the Lyapunov stability theorem [21], we know that the zero solution of (15) is globally asymptotically stable, furthermore, the real parts of all eigenvalues of matrix $B-|C|G-|D|F-|Q|V$ are positive. Hence, $B-|C|G-|D|F-|Q|V$ is an M-matrix. From Theorem 1, we know that model (6) has a unique equilibrium point, which is globally exponentially stable. The proof is completed. □

Remark 4. It is difficult to check the condition of Corollary 1 in this paper, the condition of Theorem 1 in this paper is easy to test in practice.

When $Q=0$, model (6) becomes model (5); when $D=0$, model (6) becomes the following Cohen–Grossberg neural networks model with distributed delays:

$$
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}=-a_i(x_i(t))\left[b_i(x_i(t))-\sum_{j=1}^{n}c_{ij}g_j(x_j(t))-\sum_{j=1}^{n}q_{ij}\int_{-\infty}^{t}K_{ij}(t-s)v_j(s)\,\mathrm{d}s+I_i\right] \tag{21}
$$

for $i=1,2,\ldots,n$. For model (5) and (21), we have the following results.

Corollary 2. *Under assumptions* (H1), (H2) *and* (H3), *model* (5) *has a unique equilibrium point*, *which is globally exponentially stable if*

$$
W=B-|C|G-|D|F \tag{22}
$$

is an M-matrix, *where*

$$
B=\mathrm{diag}(b_1,b_2,\ldots,b_n), \quad C=(c_{ij})_{n\times n}, \quad D=(d_{ij})_{n\times n}.
$$

Corollary 3. *Under assumptions* (H1), (H2) *and* (H3), *model* (5) *has a unique equilibrium point*, *which is globally exponentially stable if there exist constants* $v_k>0\,(k=1,2,\ldots,K_1)$, $\mu_k>0\,(k=1,2,\ldots,K_2)$, $\gamma_i>0\,(i=1,2,\ldots,n)$, ξ_{ij}, ξ_{ij}^*, η_{ij}, η_{ij}^*, ρ_{ij}, ρ_{ij}^*, σ_{ij}, $\sigma_{ij}^*\in R$ $(i,j=1,2,\ldots,n)$ *such that*

$$
\sum_{j=1}^{n}\left(\sum_{k=1}^{K_1}v_k|c_{ij}|^{r\xi_{ij}/v_k}G_j^{r\eta_{ij}/v_k}+\frac{\gamma_j}{\gamma_i}|c_{ji}|^{r\xi_{ji}^*}G_i^{r\eta_{ji}^*}+\sum_{k=1}^{K_2}\mu_k|d_{ij}|^{r\rho_{ij}/\mu_k}F_j^{r\sigma_{ij}/\mu_k}+\frac{\gamma_j}{\gamma_i}|d_{ji}|^{r\rho_{ji}^*}F_i^{r\sigma_{ji}^*}\right)<rb_i, \tag{23}
$$

where $r=\sum_{k=1}^{K_1}v_k+1=\sum_{k=1}^{K_2}\mu_k+1$; $K_1\xi_{ij}+\xi_{ij}^*=1$, $K_1\eta_{ij}+\eta_{ij}^*=1$, $K_2\rho_{ij}+\rho_{ij}^*=1$, $K_2\sigma_{ij}+\sigma_{ij}^*=1$.

Remark 5. The activation functions of Corollary 3 in this paper is weaker than that of Theorem 3 in [3]. The condition ensuring global exponential stability of model in [3] can be derived from Corollary 3 in this paper. So, the result in [3] is a special case of Corollary 3 in this paper.

Corollary 4. *Under assumptions* (H1), (H2) *and* (H3), *model* (21) *has a unique equilibrium point*, *which is globally exponentially stable if*

$$W = B - |C|G - |Q|V \tag{24}$$

is an M*-matrix*, *where*

$$B = \mathrm{diag}(b_1, b_2, \ldots, b_n), \quad C = (c_{ij})_{n\times n}, \quad Q = (q_{ij})_{n\times n}.$$

Corollary 5. *Under assumptions* (H1), (H2) *and* (H3), *model* (21) *has a unique equilibrium point*, *which is globally exponentially stable if there exist constants* $v_k > 0\,(k=1,2,\ldots,K_1)$, $\omega_k > 0\,(k=1,2,\ldots,K_2)$, $\gamma_i > 0\,(i=1,2,\ldots,n)$, $\xi_{ij}, \xi^*_{ij}, \eta_{ij}, \eta^*_{ij}, \alpha_{ij}, \alpha^*_{ij}, \beta_{ij}, \beta^*_{ij} \in R$ $(i, j = 1, 2, \ldots, n)$ *such that*

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{K_1} v_k |c_{ij}|^{r\xi_{ij}/v_k} G_j^{r\eta_{ij}/v_k} + \frac{\gamma_j}{\gamma_i}|c_{ji}|^{r\xi^*_{ji}} G_i^{r\eta^*_{ji}} + \sum_{k=1}^{K_2} \omega_k |q_{ij}|^{r\alpha_{ij}/\omega_k} V_j^{r\beta_{ij}/\omega_k} + \frac{\gamma_j}{\gamma_i}|q_{ji}|^{r\alpha^*_{ji}} V_i^{r\beta^*_{ji}}\right) < rb_i, \tag{25}$$

where $r=\sum_{k=1}^{K_1} v_k + 1 = \sum_{k=1}^{K_2}\omega_k + 1$; $K_1\xi_{ij}+\xi^*_{ij}=1$, $K_1\eta_{ij}+\eta^*_{ij}=1$, $K_2\alpha_{ij}+\alpha^*_{ij}=1$, $K_2\beta_{ij}+\beta^*_{ij}=1$.

Remark 6. In [24], Wan and Sun have discussed the global asymptotic stability of model (21) when $c_{ij}=0$ and the activation functions were required to be monotone and smooth, but the exponential stability was not discussed.

4. Comparisons and examples

Model (6) in this paper is a quite general Cohen–Grossberg neural network model. For example, when $q_{ij}=0$ $(i, j = 1, 2, \ldots, n)$, model (6) includes the models from (1) to (5), which have been studied in [8,25–27,15,5,19,13,12,3,1,28,31]. In [8,25–27,15,5,19,13,12,3], the boundedness of the activation functions was required; in [1], the differentiability of the activation functions was required; in [1], the differentiability of the time-varying delays was also required. When $a_i(x_i(t)) = 1$ and $q_{ij}=0$, model (6) becomes the following model:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}g_j(x_j(t-\tau_{ij}(t))) - I_i \tag{26}$$

for $i=1,2,\ldots,n$, which is investigated by Huang and Cao [11]. In [11], $b_i(x_i(t))$ was required to satisfy $b_i(0)=0$, and the time-varying delays was required to satisfy $\tau_{ij}(t)=\tau_j(t)$ and $\mathrm{d}\tau_j(t)/\mathrm{d}t \leqslant 0$. Furthermore, when $b_i(x_i(t))=b_ix_i(t)$(b_i is a positive constant), model (26) becomes the recurrent neural network model with time-varying delays, which includes Hopfield neural network model and cellular neural network model. When $a_i(x_i(t))=1$, $b_i(x_i(t))=b_ix_i(t)$ (b_i is positive constant) and $v_i = f_i = g_i$ $(i = 1, 2, \ldots, n)$, model (6) becomes the following model:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = &-b_ix_i(t) + \sum_{j=1}^{n} c_{ij}g_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}g_j(x_j(t-\tau_{ij}(t)))\\ &+\sum_{j=1}^{n} q_{ij}\int_{-\infty}^{t} K_{ij}(t-s)g_j(s)\,\mathrm{d}s - I_i\end{aligned} \tag{27}$$

for $i = 1, 2, \ldots, n$, which is investigated in [29,23,30]. In [23,30], the monotonicity of the activation functions was required; in [29], the activation function was required to be the special function $g(\theta)=\frac{1}{2}(|\theta+1|-|\theta-1|)$, and $b_i=1$.

It is worth noting that we neither assume the boundedness and the differentiability of the activation functions, nor assume the differentiability of the time-varying delays, as needed in most other papers. In addition, the following examples show that the results obtained in this paper have a less restriction than those in the earlier results.

To compare with the earlier results, the results in [12,3,30] are restated as follows:

Theorem 2 (*Hwang et al. [12]*). *Under assumptions* (**H**1), (**H**2), (**H**3) *and the boundedness of the activation functions, the equilibrium point of model* (4) *is globally exponentially stable if*

$$\overline{G}(\|C\|_2 + \|D\|_2)\eta < 1, \tag{28}$$

where $\overline{G} = \max_{1 \leqslant i \leqslant n}\{G_i\}$, $\eta = \frac{\max_{1 \leqslant i \leqslant n}\{\overline{\alpha}_i\}}{\min_{1 \leqslant i \leqslant n}\{\underline{\alpha}_i \beta_i\}}$, $\|C\|_2 = (\lambda_{\max}(C^{\mathrm{T}}C))^{1/2}$ *and* $\|D\|_2 = (\lambda_{\max}(D^{\mathrm{T}}D))^{1/2}$.

Theorem 3 (*Cao and Liang [3]*). *Under assumptions* (H1), (H2), (H3) *and the boundedness of the activation functions, the equilibrium point of model* (4) *is globally exponentially stable if there exist constants* $\gamma_k > 0$ $(k = 1, 2, \ldots, K_1)$, $\mu_k > 0$ $(k = 1, 2, \ldots, K_2)$, $p_{ij}, p^*_{ij}, q_{ij}, q^*_{ij}, \xi_{ij}, \xi^*_{ij}, \eta_{ij}, \eta^*_{ij} \in R$ *such that*

$$\sigma_1 > \sigma_2 > 0, \tag{29}$$

where $r = \sum_{k=1}^{K_1}\gamma_k + 1 = \sum_{k=1}^{K_2}\mu_k + 1$; $K_1\xi_{ij} + \xi^*_{ij} = 1$, $K_1\eta_{ij} + \eta^*_{ij} = 1$, $K_2 p_{ij} + p^*_{ij} = 1$, $K_2 q_{ij} + q^*_{ij} = 1$, *and*

$$\begin{aligned}\sigma_1 = \min_{1 \leqslant i \leqslant n}&\left\{r\underline{\alpha}_i\beta_i - \overline{\alpha}_i\sum_{j=1}^{n}\sum_{k=1}^{K_1}\gamma_k|c_{ij}|^{r\xi_{ij}/\gamma_k}G_j^{r\eta_{ij}/\gamma_k} - \overline{\alpha}_i\sum_{j=1}^{n}\sum_{k=1}^{K_2}\mu_k|d_{ij}|^{rp_{ij}/\mu_k}G_j^{rq_{ij}/\mu_k}\right.\\ &\left.-\sum_{j=1}^{n}\overline{\alpha}_j|c_{ji}|^{r\xi^*_{ji}}G_i^{r\eta^*_{ji}}\right\},\\ \sigma_2 = \max_{1 \leqslant i \leqslant n}&\sum_{j=1}^{n}\overline{\alpha}_j|d_{ji}|^{rp^*_{ji}}G_i^{rq^*_{ji}}.\end{aligned}$$

Theorem 4 (*Zhang et al. [30]*). *If every activation function are partially Lipschitz continuous and monotone increasing function, then the model* (27) *has a unique equilibrium point which is absolute exponential stable if*

$$-(C^+ + |D| + |Q|) \in P_0. \tag{30}$$

Example 1. Consider the following model:

$$\begin{cases}\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -(1 + 0.2\cos x_1(t))[6x_1(t) - g_1(x_1(t)) + g_2(x_2(t)) - g_1(x_1(t - \tau(t))) + 2],\\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -(1 + 0.2\sin x_2(t))[4.5x_2(t) + g_2(x_2(t)) - g_1(x_1(t - \tau(t)))\\ \qquad -g_2(x_2(t - \tau(t)))) - 3],\end{cases} \tag{31}$$

where $g_1(u) = g_2(u) = \frac{1}{2}(|u + 1| - |u - 1|)$, $\tau(t) = 3|\cos t| + 1$.

Model (31) satisfies all assumptions (**H**1), (**H**2) and (**H**3) in this paper with

$$b_1 = 6, \quad b_2 = 4.5, \quad G_1 = G_2 = 1, \quad C = \begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}, \quad D = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$

$$\tau = 4, \quad I_1 = 2, \quad I_2 = -3,$$

then

$$B = \begin{pmatrix}6 & 0\\ 0 & 4.5\end{pmatrix}, \quad G = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

It is easy computing that

$$W = B - |C|G - |D|G = \begin{pmatrix}4 & -1\\ -1 & 2.5\end{pmatrix}$$

is an M-matrix. From Corollary 2, we know that model (31) has a unique equilibrium point which is globally exponentially stable.

Since $\underline{\alpha}_1=\underline{\alpha}_2=0.8$, $\overline{\alpha}_1=\overline{\alpha}_2=1.2$, $\beta_1=6$, $\beta_2=4.5$, we can get $\overline{G}=\max_{1\leqslant i\leqslant 2}\{G_i\}=1$, $\eta=\max_{1\leqslant i\leqslant 2}\{\overline{\alpha}_i\}/\min_{1\leqslant i\leqslant 2}\{\underline{\alpha}_i\beta_i\}=\frac{1}{3}$, $\|C\|_2=(\lambda_{\max}(C^{\mathrm{T}}C))^{1/2}=\sqrt{(3+\sqrt{5})/2}$, $\|D\|_2=(\lambda_{\max}(D^{\mathrm{T}}D))^{1/2}=\sqrt{(3+\sqrt{5})/2}$ in (28), hence $\overline{G}(\|C\|_2+\|D\|_2)\eta>1$. Thus, Theorem 2 is not hold, which means that Theorem 2 is not applicable to ascertain the stability of model (31).

On the other hand, if we take $r=4$ and $K_1=K_2=1$ in (29), then $\gamma_1=\mu_1=3$, $\xi_{ij}+\xi^*_{ij}=1$, $\eta_{ij}+\eta^*_{ij}=1$, $p_{ij}+p^*_{ij}=1$, $q_{ij}+q^*_{ij}=1$, $i, j=1,2$. It is easy computing that

$$\sigma_1=\min_{1\leqslant i\leqslant 2}\left\{r\underline{\alpha}_i\beta_i-\overline{\alpha}_i\sum_{j=1}^{2}\sum_{k=1}^{K_1}\gamma_k|c_{ij}|^{r\xi_{ij}/\gamma_k}G_j^{r\eta_{ij}/\gamma_k}-\overline{\alpha}_i\sum_{j=1}^{2}\sum_{k=1}^{K_2}\mu_k|d_{ij}|^{rp_{ij}/\mu_k}G_j^{rq_{ij}/\mu_k}-\sum_{j=1}^{2}\overline{\alpha}_j|c_{ji}|^{r\xi^*_{ji}}G_i^{r\eta^*_{ji}}\right\}=1.2,$$

$$\sigma_2=\max_{1\leqslant i\leqslant 2}\sum_{j=1}^{2}\overline{\alpha}_j|d_{ji}|^{rp^*_{ji}}G_i^{rq^*_{ji}}=2.4,$$

$\sigma_1<\sigma_2$. So Theorem 3 is not hold, which means that Theorem 3 is not applicable to ascertain the stability of model (31).

Since g_1 and g_2 are not differentiable, the results in [6] cannot apply to ascertain the stability of model (31).

Example 2. Consider the following model:

$$\begin{cases}\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t}=-7x_1(t)+g_1(x_1(t))+g_2(x_2(t))-2g_1(x_1(t-\tau(t)))\\ \qquad\qquad +2\int_{-\infty}^{t}K_{11}(t-s)g_1(s)\,\mathrm{d}s+\int_{-\infty}^{t}K_{12}(t-s)g_2(s)\,\mathrm{d}s+1,\\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t}=-9x_2(t)+g_2(x_2(t))-g_1(x_1(t-\tau(t)))+2g_2(x_2(t-\tau(t)))\\ \qquad\qquad -3\int_{-\infty}^{t}K_{21}(t-s)g_1(s)\,\mathrm{d}s+\int_{-\infty}^{t}K_{22}(t-s)g_2(s)\,\mathrm{d}s-3,\end{cases}\tag{32}$$

where $g_1(u)=g_2(u)=\frac{1}{2}(|u+1|-|u-1|)$, $K_{ij}(\theta)=(\frac{1}{3})^{m+1}(\theta^m\mathrm{e}^{-\theta/3})/m!$, $m=0,1,2,\ldots$; $i,j=1,2$; $\tau(t)=2|\sin t|$.

It is easily computing that $p_{ij}(\beta)=\int_0^\infty \mathrm{e}^{\beta\theta}K_{ij}(\theta)\,\mathrm{d}\theta=(1/(1-3\beta))^{m+1}$, $i,j=1,2$, $m=0,1,2,\ldots$, $p_{ij}(\beta)$ is continuous function in $[0,\frac{1}{3})$. Model (32) satisfies assumptions (H1), (H2) and (H3) in this paper with

$$B=\begin{pmatrix}7&0\\0&9\end{pmatrix},$$

$$G=F=V=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$C=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad D=\begin{pmatrix}-2&0\\-1&2\end{pmatrix},\quad Q=\begin{pmatrix}2&1\\-3&1\end{pmatrix}$$

in (7). It is easily computing that

$$B-|C|G-|D|F-|Q|V=\begin{pmatrix}2&-2\\-4&5\end{pmatrix}$$

is an M-matrix. From Theorem 1, we know that model (32) has a unique equilibrium point which is globally exponentially stable.

On the other hand, since

$$C^+=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad |D|=\begin{pmatrix}2&0\\1&2\end{pmatrix},\quad |Q|=\begin{pmatrix}2&1\\3&1\end{pmatrix}$$

in (30),

$$-(C^{+}+|D|+|Q|)=\begin{pmatrix}-5 & -2\\ -4 & -4\end{pmatrix}\notin P_0,$$

Theorem 4 is not hold, which means that Theorem 4 is not applicable to ascertain the stability of model (32).

5. Conclusions

In this paper, the Cohen–Grossberg neural network with both time-varying and distributed delays has been studied. This neural network is quite general, and can be used to describe some well-known neural networks, including Hopfield neural networks and cellular neural networks. Without assuming both global Lipschitz conditions on these activation functions and the differentiability on these time-varying delays, by constructing proper vector Lyapunov functions, using M-matrix theory, several new sufficient conditions have been obtained to ensure the existence, uniqueness, and global exponential stability of equilibrium for Cohen–Grossberg neural network with both time-varying and distributed delays. The sufficient conditions obtained are independent of amplification function and delays, which implies the strong self-regulation is dominant in the networks. It is worth noting that in this paper neither the activation functions are assumed to be bounded and differentiable, nor time-varying delays are assumed to be differentiable. Several previous results are improved and generalized, and two examples are given to show the effectiveness of obtained results.

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