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A model in a coupled system of simple neural oscillators with delays[☆]

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ABSTRACT

We consider a coupled system of simple neural oscillators. Using the symmetric functional differential equation theories of Wu [J. Wu, Symmetric functional differential equations and neural networks with memory, Transactions of the American Mathematical Society 350 (12) (1998) 4799–4838], we demonstrate the multiple Hopf bifurcations of the equilibrium at the origin. The existence of multiple branches of bifurcating periodic solution is obtained. Then some numerical simulations support our analysis results.

1. Introduction

Oscillation is a common feature of neural networks and has become an important aspect of neural information processing. There is a large number of results about the existence of periodic solutions of neural network model [4,9,10,12]. These theoretical results are helpful to understand the system's dynamics and are important complements to experimental and numerical investigations using analog circuits and digital computers.

Coupled networks of nonlinear dynamical systems have become a topic of considerable attention recently, mainly because a wide variety of physical and biological systems can naturally be modeled by such coupled networks. Coupling can lead an oscillators' synchronization, chaos, symmetric bifurcation and so on [1,7,8]. For example, networks of coupled dynamical systems have been used to model biological oscillators, Josephson junction arrays, genetic control and neural networks [2].

In the research of nonlinear dynamical systems, symmetric systems have become important topics. In general, the symmetry reflects a certain spatial invariant of the dynamical systems. Some bifurcations can have a smaller codimension in a class of systems with specified symmetries. In others, on the contrary, bifurcations may not occur in the presence of certain symmetries [3,11].

Time delays have been incorporated into symmetric models by many authors. There has been great interest in dynamical characteristics especially bifurcations where assumptions of symmetry properties of a delayed neural network [4–6,9] are given. For example, Wu [9] studied the local and global Hopf bifurcations of symmetric functional differential equations, which is a creative work. Guo [4] and Huang discussed the Hopf bifurcating periodic orbits in a ring of neurons with delays. Peng [6] discussed the D_n-equivariant Hopf, pitchfork bifurcations in continuous and discrete dynamical systems with delays [8]. However, in the case where coupled neural networks are not fully connected, they possess a structure that is independent of the symmetry which should naturally be taken into account when analyzing the (typical) dynamics of neural networks.

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A simple neural network composed of an excitatory and an inhibitory neurons oscillates easily by injecting a feedback signal from the excitatory neuron, which we call a simple neural oscillator. When this simple oscillator is coupled in a ring, new oscillatory models are generated uniquely by the effects of the connection. Hence sometimes, the models have many symmetric properties. In-phase oscillation, anti-phase oscillation and phase-shift oscillation are typical models observed in the system [7]. In this paper we consider a coupled system of simple neural oscillators described as:

$$\begin{cases} \dfrac{\mathrm{d}x_r(t)}{\mathrm{d}t} = -\mu x_r(t) + b\sigma(y_r(t)) - a\sigma(x_{r+1}(t-\tau)) - a\sigma(x_{r-1}(t-\tau)), \\ \dfrac{\mathrm{d}y_r(t)}{\mathrm{d}t} = -\mu y_r(t) - b\sigma(x_r(t)) + c\sigma(y_r(t)), \quad r = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

where $x_0 \equiv x_n$, μ, a, b and c are positive coupling coefficients, $\tau \geq 0$ is a delay. The output function σ is assumed as: $\sigma(x) = \tanh(x)$. Variables x_r and y_r are the state of inhibitory and excitatory neuron respectively. An excitatory neuron is assumed to have positive outputs. On the other hand, an inhibitory neuron always outputs negative signals. A pair of inhibitory neuron x_r and exhibitory neuron y_r $(r = 1, 2, \ldots, n)$ become a simple oscillator. We are interested in the consequences of the structure coupled neural network model (1.1) on local bifurcations.

In Section 2, some results on circle block matrix are given. Then in Section 3, by writing system (1.1) into its linearized system, we show that the structure of system (1.1) can be represented by a dihedral group D_n. There is a fully symmetric solution that loses stability as a parameter varies, and this loss of stability is due to the crossing of imaginary eigenvalues through the imaginary axis, and the Hopf bifurcation to periodic solutions appears. In Section 4, using a local symmetric Hopf bifurcation theorem of Wu [9], we obtain some important results about the spontaneous bifurcations of multiple branches of periodic solutions and their spatio-temporal patterns: mirror-reflecting waves, standing waves, and discrete waves which describe the oscillatory mode of each neuron.

2. Some results on circle block matrix

Consider a circle block matrix:

$$Circ(A_0, A_1, \ldots, A_{n-1}) = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ A_1 & A_{n-1} & A_{n-2} & \cdots & A_0 \end{pmatrix}$$

where A_r, $(r = 0, 1, 2, \ldots, n-1)$, is a $k \times k$ matrix.

In this section, we only consider the eigenvalues and eigenvectors of the matrix $Circ(A_0, A_1, \ldots, A_{n-1})$.

Let $u_r = \mathrm{e}^{\frac{2\pi r}{n}\mathrm{i}} (r = 1, 2, \ldots, n)$ be the roots of the equation $x^n = 1$. Define a function of matrices:

$$f(x) = A_0 + xA_1 + x^2A_2 + \cdots + x^{n-1}A_{n-1}.$$

Then we have the following conclusion of $Circ(A_0, A_1, \ldots, A_{n-1})$.

Lemma 2.1. (1) $\det(\lambda I_{kn} - Circ(A_0, A_1, \ldots, A_{n-1})) = \prod_{r=1}^{n} \det(\lambda I_k - f(u_r))$. *Therefore λ_0 is an eigenvalue of $Circ(A_0, A_1, \ldots, A_{n-1})$ if and only if λ_0 is an eigenvalue of $f(u_r)$ for some r $(r = 1, 2, \ldots, n)$.*

(2) If $x_0^{(r)}$ is an eigenvector of $f(u_r)$ about the eigenvalue $\lambda_0^{(r)}$, then

$$\begin{pmatrix} x_0^{(r)} \\ u_r x_0^{(r)} \\ \vdots \\ u_r^{n-1} x_0^{(r)} \end{pmatrix}$$

is an eigenvector of $Circ(A_0, A_1, \ldots, A_{n-1})$ about the eigenvalue $\lambda_0^{(r)}$ $(r = 1, 2, \ldots, n)$.

Proof. (1) Let $T = \begin{pmatrix} I_k & I_k & \cdots & I_k \\ u_1 I_k & u_2 I_k & \cdots & u_n I_k \\ \vdots & \vdots & & \vdots \\ u_1^{n-1} I_k & u_2^{n-1} I_k & \cdots & u_n^{n-1} I_k \end{pmatrix}$. Then

$$T^{-1} Circ(A_0, A_1, \ldots, A_{n-1}) T = \begin{pmatrix} f(u_1) & & & \\ & f(u_2) & & \\ & & \ddots & \\ & & & f(u_n) \end{pmatrix}.$$

Hence

$$\det(\lambda I_{kn} - Circ(A_0, A_1, \dots, A_{n-1})) = \prod_{r=1}^{n} \det(\lambda I_k - f(u_r)),$$

the conclusion (1) is obtained.

(2) Let $x_0^{(r)}$ be an eigenvector of $f(u_r)$ about eigenvalue $\lambda_0^{(r)}$ $(r = 1, 2, \dots, n)$. Then we have

$$\begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{n-1} \\ A_{n-1} & A_0 & A_1 & \cdots & A_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ A_1 & A_{n-1} & A_{n-2} & \cdots & A_0 \end{pmatrix} \begin{pmatrix} x_0^{(r)} \\ u_r x_0^{(r)} \\ \vdots \\ u_r^{n-1} x_0^{(r)} \end{pmatrix} = \begin{pmatrix} f(u_r) x_0^{(r)} \\ u_r f(u_r) x_0^{(r)} \\ \vdots \\ u_r^{n-1} f(u_r) x_0^{(r)} \end{pmatrix}$$

$$= \lambda_0^{(r)} \begin{pmatrix} x_0^{(r)} \\ u_r x_0^{(r)} \\ \vdots \\ u_r^{n-1} x_0^{(r)} \end{pmatrix},$$

the conclusion (2) is obtained. □

3. Linear stability analysis

It is clear that $(x_i, y_i) = (0, 0)$ $(i = 1, 2, \dots, n)$ is an equilibrium point of Eq. (1.1). The linearization of Eq. (1.1) at the origin leads to

$$\begin{cases} \dfrac{dx_r(t)}{dt} = -\mu x_r(t) + by_r(t) - ax_{r+1}(t-\tau) - ax_{r-1}(t-\tau), \\ \dfrac{dy_r(t)}{dt} = -\mu y_r(t) - bx_r(t) + cy_r(t), \quad r = 1, 2, \dots, n. \end{cases} \tag{3.1}$$

Since $\tanh(0) = 0$. Regarding τ as the parameter, we determine when the infinitesimal generator $A(\tau)$ of the C^0-semigroup generated by the linear system (3.1) has a pair of pure imaginary eigenvalues. The associated characteristic equation of Eq. (3.1) takes the form

$$\det(\Delta(\lambda, \tau)) = 0,$$

when

$$\Delta(\lambda, \tau) = \lambda I_{2n} - Circ(A_0, A_1, \dots, A_{n-1}),$$

and

$$A_0 = \begin{pmatrix} -\mu & b \\ -b & -\mu + c \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} e^{-\lambda\tau}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots,$$
$$A_{n-2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_{n-1} = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} e^{-\lambda\tau}.$$

Let $C([-\tau, 0], R^n)$ denote the Banach space of continuous mapping from $[-\tau, 0]$ into R^n equipped with the norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} \mid \varphi(\theta) \mid$ for $\varphi \in C([-\tau, 0], R^n)$. Let $\sigma \in R, A \ge 0, X : [\sigma - \tau, \sigma + A] \to R^n, t \in [\sigma, \sigma + A]$ be defined by $X_t(\theta) = X(t + \theta)$ for $-\tau \le \theta \le 0$. Define the mapping $f : C([-\tau, 0], R^{n\times 2}) \to R^{n\times 2}$ by

$$[f(\varphi)]_r = \begin{pmatrix} -\mu\varphi_r^{(1)}(0) + b\sigma(\varphi_r^{(2)}(0)) - a\sigma(\varphi_{r+1}^{(1)}(-\tau)) - a\sigma(\varphi_{r-1}^{(1)}(-\tau)) \\ -\mu\varphi_r^{(2)}(0) - b\sigma(\varphi_r^{(1)}(0)) + c\sigma(\varphi_r^{(2)}(0)) \end{pmatrix}^{T}, \tag{3.2}$$

where $\varphi = \left(\varphi^{(1)}, \varphi^{(2)}\right) \in C([-\tau, 0], R^{n\times 2})$, $\varphi^{(1)}, \varphi^{(2)} \in C([-\tau, 0], R^n)$.

Rewriting Eq. (3.1) as

$$\dot{U}(t) = LU(t) \tag{3.3}$$

with $L\varphi_r = \begin{pmatrix} -\mu\varphi_r^{(1)}(0) + b\varphi_r^{(2)}(0) - a\varphi_{r+1}^{(1)}(-\tau) - a\varphi_{r-1}^{(1)}(-\tau) \\ -\mu\varphi_r^{(2)}(0) - b\varphi_r^{(1)}(0) + c\varphi_r^{(2)}(0) \end{pmatrix}^{T}$.

The infinitesimal generator of the C_0-semigroup generated by the linear system (3.3) is $\mathcal{A}(\tau)$ with

$$\mathcal{A}(\tau)\phi = \dot{\phi}, \quad \phi \in Dom(\mathcal{A}(\tau)), \qquad Dom(\mathcal{A}(\tau)) = \{\phi \in C, \dot{\phi} \in C, \dot{\phi}(0) = L(\tau)\phi\}.$$

In the following, we explore the symmetry in Eqs. (1.1) and (3.1).

Let $\Gamma = D_n$ be the dihedral group. Γ acts on $R^{n\times 2}$ by

$$(\rho U)_r = ((\rho u)_r, (\rho v)_r) = (u_{r+2}, v_{r+2});$$
$$(\kappa U)_r = ((\kappa u)_r, (\kappa v)_r) = (u_{n+2-r}, v_{n+2-r}) \quad (r \bmod n),$$

for any $U = (u, v) \in R^{n\times 2}$, $u, v \in R^n$.

Theorem 3.1. *Both the system (1.1) and the linearized system (3.1) are D_n-equivalent.*

Proof.

$$\begin{aligned}
[f(\rho\varphi)]_r &= \begin{pmatrix} -\mu(\rho\varphi^{(1)})_r(0) + b\sigma(\rho\varphi^{(2)})_r(0) - a\sigma(\rho\varphi^{(1)})_{r+1}(-\tau) - a\sigma(\rho\varphi^{(1)})_{r-1}(-\tau) \\ -\mu(\rho\varphi^{(2)})_r(0) - b\sigma(\rho\varphi^{(1)})_r(0) + c\sigma(\rho\varphi^{(2)})_r(0) \end{pmatrix}^{\mathrm{T}} \\
&= \begin{pmatrix} -\mu\varphi^{(1)}_{r+2}(0) + b\sigma(\varphi^{(2)}_{r+2}(0)) - a\sigma(\varphi^{(1)}_{r+3}(-\tau)) - a\sigma(\varphi^{(1)}_{r+1}(-\tau)) \\ -\mu\varphi^{(2)}_{r+2}(0) - b\sigma(\varphi^{(1)}_{r+2}(0)) + c\sigma(\varphi^{(2)}_{r+2}(0)) \end{pmatrix}^{\mathrm{T}} \\
&= [(\rho f)(\varphi)]_r; \\
[f(\kappa\varphi)]_r &= \begin{pmatrix} -\mu(\kappa\varphi^{(1)})_r(0) + b\sigma(\kappa\varphi^{(2)})_r(0) - a\sigma(\kappa\varphi^{(1)})_{r+1}(-\tau) - a\sigma(\kappa\varphi^{(1)})_{r-1}(-\tau) \\ -\mu(\kappa\varphi^{(2)})_r(0) - b\sigma(\kappa\varphi^{(1)})_r(0) + c\sigma(\kappa\varphi^{(2)})_r(0) \end{pmatrix}^{\mathrm{T}} \\
&= \begin{pmatrix} -\mu\varphi^{(1)}_{n+2-r}(0) + b\sigma(\varphi^{(2)}_{n+2-r}(0)) - a\sigma(\varphi^{(1)}_{n+1-r}(-\tau)) - a\sigma(\varphi^{(1)}_{n+3-r}(-\tau)) \\ -\mu\varphi^{(2)}_{n+2-r}(0) - b\sigma(\varphi^{(1)}_{n+2-r}(0)) + c\sigma(\varphi^{(2)}_{n+2-r}(0)) \end{pmatrix}^{\mathrm{T}} \\
&= [(\kappa f)(\varphi)]_r.
\end{aligned}$$

Hence, f is D_n-equivalent. That is, the system (1.1) and (3.1) are D_n-equivalent. □

Using the Lemma 2.1, we can obtain the eigenvalues of $\Delta(\tau, \lambda)$:

$$\begin{aligned}
\det(\Delta(\lambda,\tau)) &= \prod_{r=0}^{n-1} \det(\lambda I_2 - f(u_r)) \\
&= \prod_{r=0}^{n-1} \det(\lambda I_2 - A_0 - u_r A_1 - u_r^{n-1} A_{n-1}) \\
&= \prod_{r=0}^{n-1} \det\begin{pmatrix} \lambda + \mu + 2a\mathrm{e}^{-\lambda\tau}\cos\frac{2\pi r}{n} & -b \\ b & \lambda+\mu-c \end{pmatrix} \\
&= \prod_{r=0}^{n-1}\left[\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 + 2a\cos\frac{2\pi r}{n}\lambda\mathrm{e}^{-\lambda\tau} + \left(2a\cos\frac{2\pi r}{n}\mu - 2a\cos\frac{2\pi r}{n}c\right)\mathrm{e}^{-\lambda\tau} - c\mu\right] \\
&= \begin{cases} (\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 + 2a(\lambda + \mu - c)\mathrm{e}^{-\lambda\tau} - c\mu) \\ \quad \prod_{r=0}^{\frac{n-1}{2}} \Big[\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 + 2a\cos\frac{2\pi r}{n}\lambda\mathrm{e}^{-\lambda\tau} \\ \quad + \left(2a\cos\frac{2\pi r}{n}\mu - 2a\cos\frac{2\pi r}{n}c\right)\mathrm{e}^{-\lambda\tau} - c\mu\Big]^2 = 0, \quad n \text{ is odd}; \\ (\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 - 2a(\lambda + \mu - c)\mathrm{e}^{-\lambda\tau} - c\mu) \\ \quad (\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 + \mu - c + 2a(\lambda +)\mathrm{e}^{-\lambda\tau} - c\mu) \\ \quad \prod_{r=0}^{\frac{n-2}{2}} \Big[\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 + 2a\cos\frac{2\pi r}{n}\lambda\mathrm{e}^{-\lambda\tau} \\ \quad + \left(2a\cos\frac{2\pi r}{n}\mu - 2a\cos\frac{2\pi r}{n}c\right)\mathrm{e}^{-\lambda\tau} - c\mu\Big]^2 = 0, \quad n \text{ is even}. \end{cases}
\end{aligned} \tag{3.4}$$

Here, $f(u_r) = A_0 + u_r A_1 + \cdots + u_r^{n-1} A_{n-1}$, $r = 1, 2, \ldots, n$.

We give a direct sum of $C^{n\times 2}$:

$$C^{n\times 2} = E_0 \oplus E_1 \oplus \cdots \oplus E_{n-1}$$

with $E_r = \{(V^{\mathrm{T}}, u_r V^{\mathrm{T}}, \cdots, u_r^{n-1} V^{\mathrm{T}})^{\mathrm{T}}, V^{\mathrm{T}} \in R^2\}$ and $r = 0, 1, \ldots, n-1$.

The condition for the Hopf bifurcation to occur in each invariant subspace E_r is that the equation

$$\lambda^2 + (2\mu - c)\lambda + \mu^2 + b^2 + 2a\cos\frac{2\pi r}{n}\lambda\mathrm{e}^{-\lambda\tau} + 2a\cos\frac{2\pi r}{n}(\mu - c)\mathrm{e}^{-\lambda\tau} - c\mu = 0 \tag{3.5}$$

has pure imaginary roots for $r = 0, 1, \ldots, n-1$.

We make the following assumptions:

$$(\mathrm{H}_1):\ \mu - c + 2a\cos\frac{2\pi r}{n} > 0,\ \mu^2 + b^2 + 2a\cos\frac{2\pi r}{n}(\mu - c) - c\mu < 0,$$
$$4a^2\cos^2\frac{2\pi r}{n} + 2(\mu^2 + b^2 - c\mu) - (2\mu - c)^2 < 0,\ (\mu^2 + b^2 - c\mu)^2 - 4a^2\cos^2\frac{2\pi r}{n}(\mu - c)^2 < 0$$
for all $r = 0, 1, \ldots, n-1$.
$$(\mathrm{H}_2):\ 4a^2\cos^2\frac{2\pi r}{n} + 2(\mu^2 + b^2 - c\mu) - (2\mu - c)^2 < 0;\ (\mu^2 + b^2 - c\mu)^2 - 4a^2\cos^2\frac{2\pi r}{n}(\mu - c)^2 > 0$$
for given $r \in \{1, 2, \ldots, n-1\} \setminus \left\{\left[\frac{n}{2}\right]\right\}$ (if n is even) or $r \in \{1, 2, \ldots, n-1\}$ (if n is odd).

Lemma 3.1. (1) *If the assumption* (H_1) *is satisfied, then the zero solution of Eq.* (1.1) *is stable for all* $\tau \geq 0$.

(2) *If the assumption* (H_2) *is satisfied, then the Eq.* (3.5) *has a pair of pure imaginary roots* $\pm \mathrm{i}\omega_0^{(r)}$, *where* $\omega_0^{(r)}$ *is determined by the following equation:*

$$\omega_0^{(r)} = \frac{\sqrt{2}}{2}\left[4a^2\cos^2\frac{2\pi r}{n} + 2(\mu^2 + b^2 - c\mu) - (2\mu - c)^2 + \sqrt{P}\right]^{\frac{1}{2}},$$
$$P = \left[4a^2\cos^2\frac{2\pi r}{n} + 2(\mu^2 + b^2 - c\mu) - (2\mu - c)^2\right]^2 - 4\left[(\mu^2 + b^2 - c\mu)^2 - 4a^2\cos^2\frac{2\pi r}{n}(\mu - c)^2\right],$$

when $\tau_s^{(r)} = \frac{1}{\omega_0^{(r)}}[\pi - \arccos\frac{\mu(\omega_0^{(r)})^2 + (\mu^2 + b^2 - c\mu)(\mu - c)}{2a\cos\frac{2\pi r}{n}[(\omega_0^{(r)})^2 + (\mu - c)^2]}] + \frac{2\pi s}{\omega_0^{(r)}}$, $s = 0, 1, 2, \ldots$.

Proof. (1) For $\tau = 0$, Eq. (3.5) becomes

$$\lambda^2 + \left(2\mu - c + 2a\cos\frac{2\pi r}{n}\right)\lambda + \mu^2 + b^2 + 2a\cos\frac{2\pi r}{n}(\mu - c) - c\mu = 0. \tag{3.6}$$

From (H_1), two roots of Eq. (3.6) have a negative real part.

(2) Let $\mathrm{i}\omega_0^{(r)}$ be a root of Eq. (3.5), for $r \in \{1, 2, \ldots, n-1\} \setminus \{[\frac{n}{2}]\}$ (if n is even) or $r \in \{1, 2, \ldots, n-1\}$ (if n is odd), then

$$-(\omega_0^{(r)})^2 + \mu^2 + b^2 - c\mu + 2a\cos\frac{2\pi r}{n}[\omega_0^{(r)}\sin\omega_0^{(r)}\tau + (\mu - c)\cos\omega_0^{(r)}\tau] = 0;$$
$$(2\mu - c)\omega_0^{(r)} + 2a\cos\frac{2\pi r}{n}[\omega_0^{(r)}\cos\omega_0^{(r)}\tau - (\mu - c)\sin\omega_0^{(r)}\tau] = 0.$$

Hence,

$$(\omega_0^{(r)})^4 - \left[4a^2\cos^2\frac{2\pi r}{n} + 2(\mu^2 + b^2 - c\mu) - (2\mu - c)^2\right](\omega_0^{(r)})^2 + (\mu^2 + b^2 - c\mu)^2 - 4a^2\cos^2\frac{2\pi r}{n}(\mu - c)^2 = 0.$$

It is easy to see that the conclusions (1), (2) are true. □

Theorem 3.2. *Let the assumption* (H_2) *be satisfied, then the* D_n*-equivariant Hopf bifurcations occur in system* (1.1) *when* $\tau = \tau_s^{(r)}$ *for* $r \in \{1, 2, \ldots, n-1\} \setminus \{[\frac{n}{2}]\}$ *(if* n *is even) or* $r \in \{1, 2, \ldots, n-1\}$ *(if* n *is odd) and* $s = 0, 1, 2, \ldots$.

Proof. By Eq. (3.5),

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{(2\lambda + 2\mu - c)\mathrm{e}^{\lambda\tau}}{2a\cos\frac{2\pi r}{n}(\lambda + \mu - c)\lambda} - \frac{\tau}{\lambda} + \frac{1}{\lambda(\lambda + \mu - c)}.$$

Hence,

$$Re\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau = \tau_s^{(r)}} = \frac{\sqrt{P}}{4a^2\cos^2\frac{2\pi r}{n}[(\omega_0^{(r)})^2 + (\mu - c)^2]} > 0.$$

Further, if $\tau = \tau_s^{(r)}$ for $r \in \{1, 2, \ldots, n-1\} \setminus \{[\frac{n}{2}]\}$ (if n is even) or $r \in \{1, 2, \ldots, n-1\}$ (if n is odd) and $s = 0, 1, 2, \ldots$, the Eq. (3.4) has multiple roots $\pm \mathrm{i}\omega_0^{(r)}$. And the system (3.1) is D_n-equivariant. Therefore, the system (1.1) exhibits multiple Hopf bifurcations. □

4. Multiple Hopf bifurcations

Firstly, we consider the generalized eigenspace corresponding to pure imaginary eigenvalues of $\mathcal{A}(\tau)$.

Let the assumption (H_2) hold and there exists some $r \in \{1, 2, \ldots, n-1\} \setminus \{[\frac{n}{2}]\}$ (if n is even) or $r \in \{1, 2, \ldots, n-1\}$ (if n is odd) such that Eq. (3.4) has multiple roots $\pm i\omega_0^{(r)}$ when $\tau = \tau_s^{(r)}$. From Section 2, we have:

Lemma 4.1. *For $r \in \{1, 2, \ldots, n-1\} \setminus \{[\frac{n}{2}]\}$ (if n is even) or $r \in \{1, 2, \ldots, n-1\}$ (if n is odd), the generalized eigenspace $U_{\pm i\omega_0^{(r)}}$ consisting of eigenvectors of $\mathcal{A}(\tau_s^{(r)})$ corresponding to $\pm i\omega_0^{(r)}$ is*

$$U_{\pm i\omega_0^{(r)}} = \left\{ \sum_{j=1}^{4} y_j \varepsilon_j, y_j \in R, j = 1, 2, 3, 4. \right\}$$

where

$$\begin{aligned}
\varepsilon_1(\theta) &= \cos(\omega_0^{(r)}\theta)\mathrm{Re}\{V_r\} - \sin(\omega_0^{(r)}\theta)\mathrm{Im}\{V_r\}, \\
\varepsilon_2(\theta) &= \sin(\omega_0^{(r)}\theta)\mathrm{Re}\{V_r\} + \cos(\omega_0^{(r)}\theta)\mathrm{Im}\{V_r\}, \\
\varepsilon_3(\theta) &= \cos(\omega_0^{(r)}\theta)\mathrm{Re}\{V_r\} + \sin(\omega_0^{(r)}\theta)\mathrm{Im}\{V_r\}, \\
\varepsilon_4(\theta) &= \sin(\omega_0^{(r)}\theta)\mathrm{Re}\{V_r\} - \cos(\omega_0^{(r)}\theta)\mathrm{Im}\{V_r\},
\end{aligned}$$

and $V_r = \begin{pmatrix} I_2 \\ u_r I_2 \\ \cdots \\ u_r^{n-1} I_2 \end{pmatrix}, \theta \in [-1, 0].$

In order to study the Hopf bifurcation of the origin, we consider the action of $D_n \times S^1$, where S^1 is the temporal. $D_n \times S^1$ acts by

$$(\Gamma, \theta)x(t) = \Gamma x(t+\theta), \qquad (\Gamma, \theta) \in D_n \times S^1.$$

According to the idea of Wu [9], we shall discuss the bifurcating periodic solution of Eq. (2.1). For $s \in N_0$, let $T = \frac{2\pi}{\omega_0^{(r)}}$. Denote P_T the Banach space of all continuous T-periodic solutions $X : R \to R^{n\times 2}$. Then $D_n \times S^1$ acts on P_T by

$$(r, \theta)x(t) = rx(t+\theta), \quad (r, \theta) \in D_n \times S^1, x \in P_T.$$

Denote by SP_T the subspace of P_T consisting of all T-periodic solutions of Eq. (1.1) with $\tau = \tau_s^{(r)}$. Then for each subgroup $\Sigma \le D_n \times S^1$, $\mathrm{Fix}(\Sigma, SP_T) = \{x \in SP_T; (r, \theta)x = x \text{ for all } (r, \theta) \in \Sigma\}$ is a subspace.

Consider the several subgroups of $D_n \times S^1$:

$$\begin{aligned}
\Sigma_1 &= \{(\kappa, 1), (1, 1)\}, \\
\Sigma_2 &= \{(\kappa, -1), (1, 1)\}, \\
\Sigma_3 &= \{(\rho, e^{i\frac{rT}{n}})\}, \\
\Sigma_4 &= \{(\rho, e^{-i\frac{rT}{n}})\}.
\end{aligned}$$

Lemma 4.2. (1) $\mathrm{Fix}(\Sigma_1, SP_T) = \{z_1(\varepsilon_1 + \varepsilon_3) + z_2(\varepsilon_2 + \varepsilon_4); z_1, z_2 \in R\}$,
(2) $\mathrm{Fix}(\Sigma_2, SP_T) = \{z_1(\varepsilon_1 - \varepsilon_3) + z_2(\varepsilon_2 - \varepsilon_4); z_1, z_2 \in R\}$,
(3) $\mathrm{Fix}(\Sigma_3, SP_T) = \{z_1\varepsilon_3 + z_2\varepsilon_4; z_1, z_2 \in R\}$,
(4) $\mathrm{Fix}(\Sigma_4, SP_T) = \{z_1\varepsilon_1 + z_2\varepsilon_2; z_1, z_2 \in R\}$.

Proof. (1) $x \in \mathrm{Fix}(\Sigma_1, SP_T)$ if and only if $\kappa x = x$. Let

$$\begin{aligned}
x(t) &= x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4 \\
&= x_1[\cos(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} - \sin(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] + x_2[\sin(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} + \cos(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] \\
&\quad + x_3[\cos(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} + \sin(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] + x_4[\sin(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} - \cos(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}],
\end{aligned}$$

then

$$\begin{aligned}
\kappa x(t) &= \kappa[x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4] \\
&= x_1[\cos(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} + \sin(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] + x_2[\sin(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} - \cos(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] \\
&\quad + x_3[\cos(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} - \sin(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] + x_4[\sin(\omega_0^{(r)}t)\mathrm{Re}\{V_r\} + \cos(\omega_0^{(r)}t)\mathrm{Im}\{V_r\}] \\
&= x_1\varepsilon_3 + x_2\varepsilon_4 + x_3\varepsilon_1 + x_4\varepsilon_2.
\end{aligned}$$

Hence, $\kappa x(t) = x(t)$ if and only if $x_1 = x_3, x_2 = x_4$. Therefore, $\mathrm{Fix}(\Sigma_1, SP_T)$ is spanded of $\varepsilon_1 + \varepsilon_3$ and $\varepsilon_2 + \varepsilon_4$, the conclusion (1) follows.

(2) $x \in \mathrm{Fix}(\Sigma_2, SP_T)$ if and only if $\kappa x = x\left(t + \frac{T}{2}\right)$ for all $t \in R$.

Let $x \in \mathrm{Fix}(\Sigma_2, SP_T)$,

$$\begin{aligned} x\left(t+\frac{T}{2}\right) &= x_1\varepsilon_1\left(t+\frac{T}{2}\right) + x_2\varepsilon_2\left(t+\frac{T}{2}\right) + x_3\varepsilon_3\left(t+\frac{T}{2}\right) + x_4\varepsilon_4\left(t+\frac{T}{2}\right) \\ &= -x_1\varepsilon_1 - x_2\varepsilon_2 - x_3\varepsilon_3 - x_4\varepsilon_4. \end{aligned}$$

Hence, $\kappa x(t) = x\left(t + \frac{T}{2}\right)$ if and only if $x_1 = -x_3, x_2 = -x_4$, and $\mathrm{Fix}(\Sigma_2, SP_T)$ is spanded of $\varepsilon_1 - \varepsilon_3$ and $\varepsilon_2 - \varepsilon_4$, the conclusion (2) follows.

(3) $x \in \mathrm{Fix}(\Sigma_3, SP_T)$ if and only if $\rho x = x(t - \frac{rT}{n})$.

Let

$$x(t) = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4,$$

then

$$\begin{aligned} x\left(t - \frac{rT}{n}\right) &= x_1\left[\cos\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Re}\{V_r\} - \sin\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Im}\{V_r\}\right] \\ &\quad + x_2\left[\sin\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Re}\{V_r\} + \cos\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Im}\{V_r\}\right] \\ &\quad + x_3\left[\cos\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Re}\{V_r\} + \sin\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Im}\{V_r\}\right] \\ &\quad + x_4\left[\sin\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Re}\{V_r\} - \cos\left(\omega_0^{(r)}t - \frac{2r\pi}{n}\right)\mathrm{Im}\{V_r\}\right] \\ &= \left(x_1\cos\frac{2r\pi}{n} - x_2\sin\frac{2r\pi}{n}\right)\varepsilon_1 + \left(x_1\sin\frac{2r\pi}{n} + x_2\cos\frac{2r\pi}{n}\right)\varepsilon_2 \\ &\quad + \left(x_3\cos\frac{2r\pi}{n} - x_4\sin\frac{2r\pi}{n}\right)\varepsilon_3 + \left(x_3\sin\frac{2r\pi}{n} + x_4\cos\frac{2r\pi}{n}\right)\varepsilon_4. \end{aligned}$$

$$\begin{aligned} \rho\varepsilon_1 &= \varepsilon_1\cos\frac{2r\pi}{n} - \varepsilon_2\sin\frac{2r\pi}{n}. \\ \rho\varepsilon_2 &= \varepsilon_1\sin\frac{2r\pi}{n} + \varepsilon_2\cos\frac{2r\pi}{n}, \\ \rho\varepsilon_3 &= \varepsilon_3\cos\frac{2r\pi}{n} + \varepsilon_4\sin\frac{2r\pi}{n}, \\ \rho\varepsilon_4 &= -\varepsilon_3\sin\frac{2r\pi}{n} + \varepsilon_4\cos\frac{2r\pi}{n}. \end{aligned}$$

Hence,

$$\begin{aligned} \rho x(t) &= \rho(x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 + x_4\varepsilon_4) \\ &= x_1\left(\varepsilon_1\cos\frac{2r\pi}{n} - \varepsilon_2\sin\frac{2r\pi}{n}\right) + x_2\left(\varepsilon_1\sin\frac{2r\pi}{n} + \varepsilon_2\cos\frac{2r\pi}{n}\right) \\ &\quad + x_3\left(\varepsilon_3\cos\frac{2r\pi}{n} + \varepsilon_4\sin\frac{2r\pi}{n}\right) + x_4\left(-\varepsilon_3\sin\frac{2r\pi}{n} + \varepsilon_4\cos\frac{2r\pi}{n}\right) \\ &= \left(x_1\cos\frac{2r\pi}{n} + x_2\sin\frac{2r\pi}{n}\right)\varepsilon_1 + \left(-x_1\sin\frac{2r\pi}{n} + x_2\cos\frac{2r\pi}{n}\right)\varepsilon_2 \\ &\quad + \left(x_3\cos\frac{2r\pi}{n} - x_4\sin\frac{2r\pi}{n}\right)\varepsilon_3 + \left(x_3\sin\frac{2r\pi}{n} + x_4\cos\frac{2r\pi}{n}\right)\varepsilon_4. \end{aligned}$$

So, $\rho x(t) = x\left(t - \frac{rT}{n}\right)$ if and only if $x_1 = x_2 = 0$, and $\mathrm{Fix}(\Sigma_3, SP_T)$ is spanded of $\varepsilon_3, \varepsilon_4$.

(4) $x \in \mathrm{Fix}(\Sigma_4, SP_T)$ if and only if $\rho x(t) = x(t + \frac{rT}{n})$.

Similar to the discussing of (3), we have: $\rho x(t) = x(t + \frac{rT}{n})$ if and only if $x_3 = x_4 = 0$, and $\mathrm{Fix}(\Sigma_4, SP_T)$ is spanded of $\varepsilon_1, \varepsilon_2$. $\square$

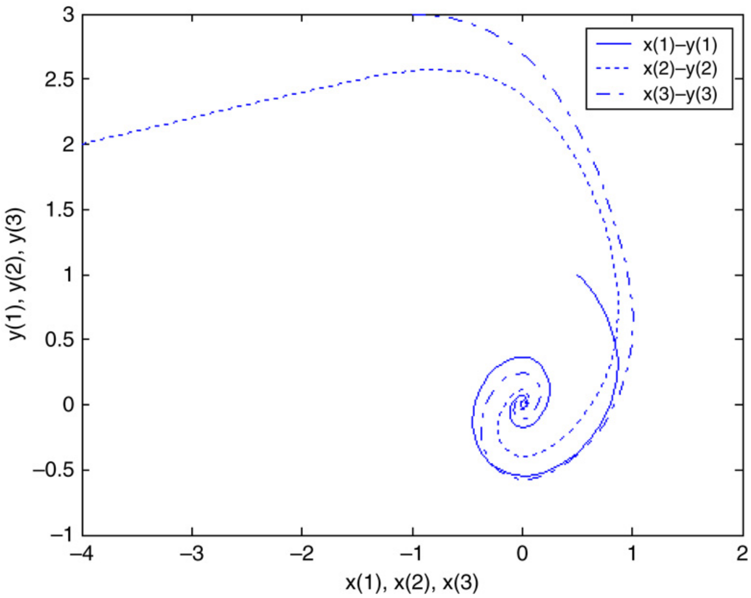

Fig. 1. Phase trajectories of system (1.1) with parameters $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 1$, $\tau = 0.1$.

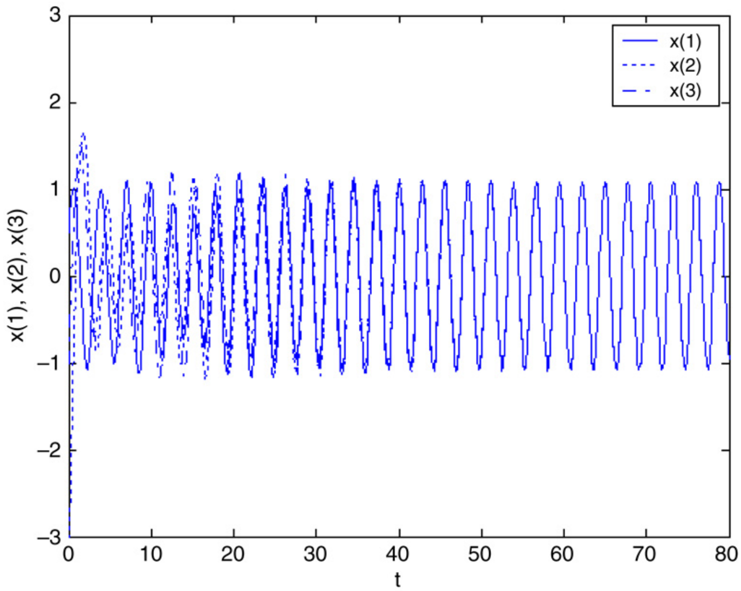

Fig. 2. Trajectories $x_1(t)$, $x_2(t)$ and $x_3(t)$ of system (1.1) with parameters $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 2$, $\tau = 1$.

Using the Theorem 4.1 of Wu [9] and Lemma 4.2, we have:

Theorem 4.1. *Assume that* (H_2) *holds. Fix an integer s such that* $\tau_s^{(r)} > 0$*. Then near* $\tau_s^{(r)}$ *there exist three branches of small-amplitude periodic solutions of Eq.* (1.1) *with period near* $\frac{2\pi}{w_0^{(r)}}$ *and satisfying*

(1) *discrete waves:*

$$x_i\left(t - \frac{rT}{n}\right) = x_{i-1}(t);$$

(2) *mirror-reflecting solution:*

$$x_i(t) = x_{n-i}(t);$$

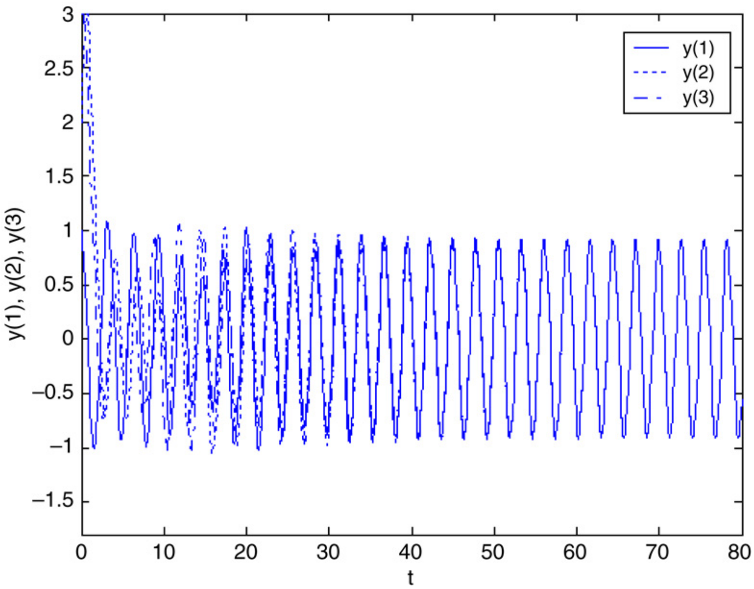

Fig. 3. Trajectories $y_1(t)$, $y_2(t)$ and $y_3(t)$ of system (1.1) with parameters $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 2$, $\tau = 1$.

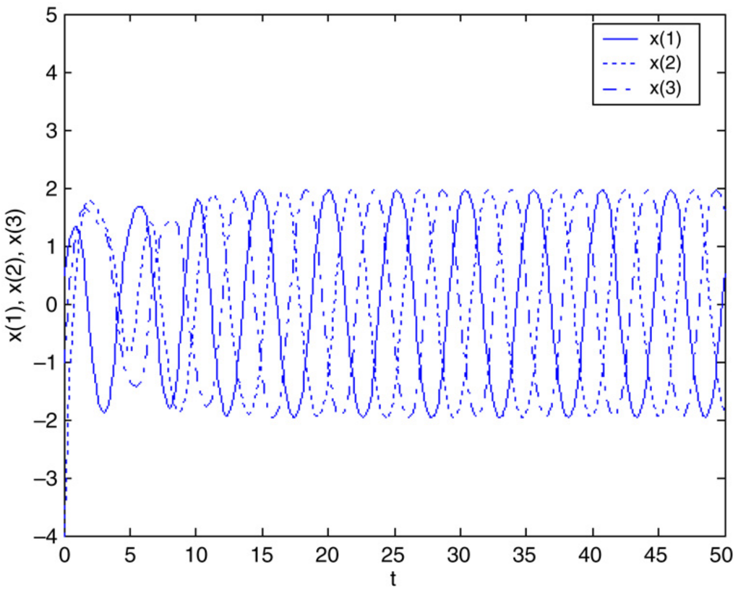

Fig. 4. Trajectories $x_1(t)$, $x_2(t)$ and $x_3(t)$ of system (1.1) with parameters $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 3.5$, $\tau = 1$.

(3) *standing waves:*

$$x_i\left(t - \frac{T}{n}\right) = x_{n-i}(t).$$

5. Computer simulation

To illustrate the analytical results found, let us consider the following particular case of Eq. (1.1):

Let $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 1$. Fig. 1 shows the equilibrium of Eq. (1.1) is stable when $\tau = 0.1$.

Let $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 2$. Figs. 2 and 3 show the synchronous periodic orbit occur when $\tau = 1$.

Let $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 3.5$. Figs. 4 and 5 show the periodic solutions separated into three locked-phased curves when $\tau = 1$.

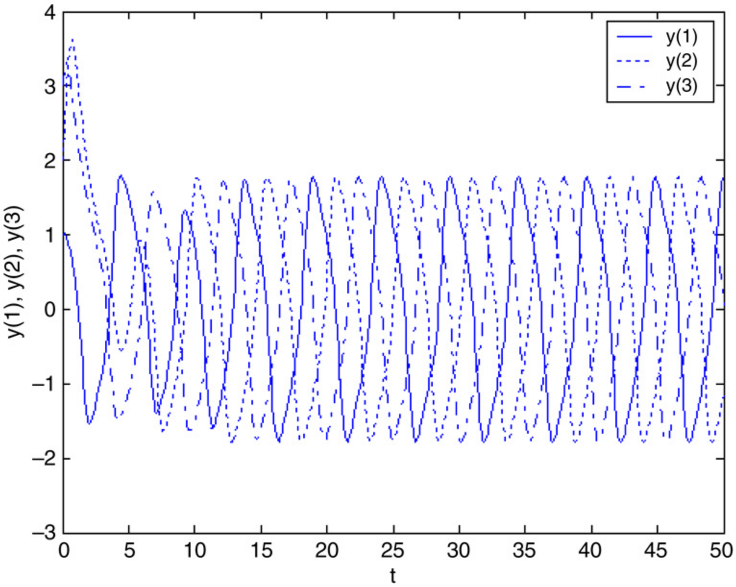

Fig. 5. Trajectories $y_1(t)$, $y_2(t)$ and $y_3(t)$ of system (1.1) with parameters $n = 3$, $\mu = 2.5$, $a = 0.5$, $b = 2.5$, $c = 3.5$, $\tau = 1$.

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