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The Laguerre spectral method for solving Neumann boundary value problems

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ABSTRACT

In this paper, we propose a Laguerre spectral method for solving Neumann boundary value problems. This approach differs from the classical spectral method in that the homogeneous boundary condition is satisfied exactly. Moreover, a tridiagonal matrix is employed, instead of the full stiffness matrix encountered in the classical variational formulation of such problems. For analyzing the numerical errors, some basic results on Laguerre approximations are established. The convergence is proved. The numerical results demonstrate the efficiency of this approach.

1. Introduction

The Laguerre spectral method has been used extensively for solving PDEs in unbounded domains. Usually, one only considers certain problems with Dirichlet boundary conditions; see, e.g., [1–16]. However, it is also interesting and important to consider various problems with Neumann boundary conditions. In a standard variational formulation, this kind of boundary condition is commonly imposed in a natural way. Unfortunately, this approach usually leads to a full stiffness matrix for approximating the second derivatives.

Recently, Auteri et al. [17] proposed a Legendre spectral method for solving Neumann boundary value problems in bounded domains. This method differs from the classical spectral method in that the Neumann boundary conditions are enforced according to an essential treatment, namely, the homogeneous Neumann boundary conditions are satisfied exactly for each basis. In particular, by taking the appropriate basis, such a treatment leads to sparse and better conditioned matrices. Wang and Wang [18] also analyzed the numerical errors of this algorithm.

In this paper, we shall focus on the Neumann problems in unbounded domains, using the Laguerre spectral method with the essential imposing of Neumann boundary conditions. For analyzing the numerical errors, we establish some basic results on the Laguerre approximations for Neumann problems, motivated by [19–21]. As examples, we consider two model problems. The related spectral schemes are proposed. The convergence is proved. In particular, by choosing appropriate base functions with zero slope at the endpoint, a tridiagonal matrix is employed for solving a one-dimensional problem, instead of the full stiffness matrix encountered in the classical Laguerre spectral method. We also present some numerical results in order to demonstrate the efficiency of this approach.

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This paper is organized as follows. In Section 2, we establish some related results on Laguerre approximations. In Section 3, we propose Laguerre spectral schemes with the essential imposing of Neumann boundary conditions in one and two space dimensions, and prove their convergence. We present some numerical results in Section 4. The final section is for some concluding remarks.

2. Laguerre approximations

In this section, we investigate Laguerre approximations, which form the mathematical foundation of the spectral method with the essential imposing of Neumann boundary conditions in unbounded domains.

2.1. One-dimensional Laguerre approximations

Let $\Lambda = (0, \infty)$ and $\chi(x)$ be a certain weight function. For integer $r \geq 0$, we have

$$H^r_\chi(\Lambda) = \{u \mid u \text{ is measurable on } \Lambda \text{ and } \|u\|_{r,\chi,\Lambda} < \infty\},$$

equipped with the following inner product, semi-norm and norm:

$$(u, v)_{r,\chi,\Lambda} = \sum_{0\leq k\leq r} \int_\Lambda \partial_x^k u(x)\partial_x^k v(x)\chi(x)\mathrm{d}x,$$

$$|u|_{r,\chi,\Lambda} = \left(\int_\Lambda (\partial_x^r u(x))^2 \chi(x)\mathrm{d}x\right)^{\frac{1}{2}}, \quad \|u\|_{r,\chi,\Lambda} = (u, u)^{\frac{1}{2}}_{r,\chi,\Lambda}.$$

In particular, we have $H^0_\chi(\Lambda) = L^2_\chi(\Lambda)$, with the inner product $(u, v)_{\chi,\Lambda}$ and the norm $\|u\|_{\chi,\Lambda}$. For any real $r > 0$, we define the space $H^r_\chi(\Lambda)$ and its norm $\|v\|_{r,\chi,\Lambda}$ by space interpolation as in [22]. We omit the subscript χ in the notation whenever $\chi(x) \equiv 1$.

Next let $\omega_{\alpha,\beta}(x) = x^\alpha \mathrm{e}^{-\beta x}$, $\alpha > -1$, $\beta > 0$. In particular, $\omega_\beta(x) = \omega_{0,\beta}(x) = \mathrm{e}^{-\beta x}$. The scaled Laguerre polynomial of degree l is defined by (cf. [11])

$$\mathcal{L}^{(\beta)}_l(x) = \frac{1}{l!}\mathrm{e}^{\beta x}\partial_x^l(x^l \mathrm{e}^{-\beta x}).$$

For any integer $N > 0$, we denote by $\mathcal{P}_N(\Lambda)$ the set of all polynomials of degree at most N, and

$${}_0H^1_{\omega_\beta}(\Lambda) = \{u \mid u \in H^1_{\omega_\beta}(\Lambda) \text{ and } u(0) = 0\}, \qquad {}_0\mathcal{P}_N(\Lambda) = \{\phi \mid \phi \in \mathcal{P}_N(\Lambda) \text{ and } \phi(0) = 0\}.$$

Throughout this paper, we denote by c a generic positive constant independent of N, β and any functions.

The orthogonal projection ${}_0P^1_{N,\beta,\Lambda} : {}_0H^1_{\omega_\beta}(\Lambda) \to {}_0\mathcal{P}_N(\Lambda)$ is defined by

$$(\partial_x u - \partial_x {}_0P^1_{N,\beta,\Lambda} u, \partial_x \phi)_{\omega_\beta,\Lambda} = 0, \quad \forall \phi \in {}_0\mathcal{P}_N(\Lambda).$$

Due to (i) of Lemma 2.2 in [11], for any $u \in {}_0H^1_{\omega_\beta}(\Lambda)$, we have

$$\|u\|_{\omega_\beta,\Lambda} \leq \frac{2}{\beta}\|\partial_x u\|_{\omega_\beta,\Lambda}. \tag{2.1}$$

Moreover, according to Theorem 2.3 of [11] with $\alpha = \gamma = 0$ and $\delta = \beta$, if $u \in L^2_{\omega_\beta}(\Lambda)$, $\partial_x^r u \in L^2_{\omega_{r-1,\beta}}(\Lambda)$ and $u(0) = 0$, then for integer $1 \leq r \leq N + 1$,

$$\|\partial_x^s({}_0P^1_{N,\beta,\Lambda} u - u)\|_{\omega_\beta,\Lambda} \leq c\beta^{s-1}(\beta N)^{\frac{1-r}{2}}\|\partial_x^r u\|_{\omega_{r-1,\beta},\Lambda}, \quad s = 0, 1. \tag{2.2}$$

To design a proper spectral method for solving Neumann problems, we use the scaled Laguerre functions as follows (cf. [23]):

$$\tilde{\mathcal{L}}^{(\beta)}_l(x) = \mathrm{e}^{-\frac{1}{2}\beta x}\mathcal{L}^{(\beta)}_l(x), \quad l = 0, 1, 2, \ldots.$$

As pointed out in [23], the set of $\tilde{\mathcal{L}}^{(\beta)}_l(x)$ forms a complete $L^2(\Lambda)$-orthogonal system.

In the spectral method for solving Neumann problems, we have to consider numerical solutions and their derivatives on the boundaries of domains. To do this, we introduce the spaces

$$\mathcal{F}(\Lambda) = H^1(\Lambda) \cap \{u \mid \text{ there exists finite trace of } \partial_x u \text{ at } x = 0\},$$
$${}^0\mathcal{F}(\Lambda) = \{u \mid u \in \mathcal{F}(\Lambda), \partial_x u(0) = 0\}, \qquad {}_{00}\mathcal{F}(\Lambda) = \{u \mid u \in \mathcal{F}(\Lambda), u(0) = \partial_x u(0) = 0\},$$
$$\mathcal{Q}_{N,\beta}(\Lambda) = \left\{\mathrm{e}^{-\frac{1}{2}\beta x}\psi \mid \psi \in \mathcal{P}_N(\Lambda)\right\}, \qquad {}^0\mathcal{Q}_{N,\beta}(\Lambda) = \{\phi \mid \phi \in \mathcal{Q}_{N,\beta}(\Lambda), \partial_x\phi(0) = 0\},$$
$${}_{00}\mathcal{Q}_{N,\beta}(\Lambda) = \{\phi \mid \phi \in \mathcal{Q}_{N,\beta}(\Lambda), \phi(0) = \partial_x\phi(0) = 0\}.$$

The set $\mathcal{F}(\Lambda)$ is meaningful (cf. [19–21]). For instance, if $u \in H^1(\Lambda)$ and $\partial_x u$ is continuous near the point $x = 0$, then $u \in \mathcal{F}(\Lambda)$. Moreover, it is clear that $H^2(\Lambda) \subset \mathcal{F}(\Lambda)$.

Next for any $u \in {}^0\mathcal{F}(\Lambda)$, we set

$$\tilde{u}(x) = u(x) - \left(\frac{1}{2}\beta x + 1\right) u(0)\mathrm{e}^{-\frac{1}{2}\beta x}.$$

It can be verified readily that $\tilde{u} \in {}_{00}\mathcal{F}(\Lambda)$. In order to obtain precise error estimates, we introduce a mapping ${}_{00}\tilde{P}^1_{N,\beta,\Lambda} : {}_{00}\mathcal{F}(\Lambda) \to {}_{00}\mathcal{Q}_{N,\beta}(\Lambda)$, defined by

$$ {}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u}(x) = \mathrm{e}^{-\frac{1}{2}\beta x} \int_0^x {}_0P^1_{N-1,\beta,\Lambda}\partial_\xi \left(\mathrm{e}^{\frac{1}{2}\beta\xi}\tilde{u}(\xi)\right) \mathrm{d}\xi \in {}_{00}\mathcal{Q}_{N,\beta}(\Lambda). \tag{2.3}$$

Accordingly, we define another mapping ${}^0\tilde{P}^1_{N,\beta,\Lambda} : {}^0\mathcal{F}(\Lambda) \to {}^0\mathcal{Q}_{N,\beta}(\Lambda)$ as follows:

$$ {}^0\tilde{P}^1_{N,\beta,\Lambda}u(x) = {}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u}(x) + \left(\frac{1}{2}\beta x + 1\right) u(0)\mathrm{e}^{-\frac{1}{2}\beta x}.$$

Lemma 2.1. *If $u \in {}^0\mathcal{F}(\Lambda)$, $\partial_x^r \left(\mathrm{e}^{\frac{1}{2}\beta x}u\right) \in L^2_{\omega_{r-2,\beta}}(\Lambda)$ for integer $2 \le r \le N+1$, then*

$$\|\partial_x^s({}^0\tilde{P}^1_{N,\beta,\Lambda}u - u)\|_\Lambda \le c\beta^{s-2}(\beta N)^{1-\frac{r}{2}} \left\|\partial_x^r \left(\mathrm{e}^{\frac{1}{2}\beta x}u\right)\right\|_{\omega_{r-2,\beta},\Lambda}, \quad s = 0, 1. \tag{2.4}$$

Proof. We first estimate $\|\partial_x({}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u})\|_\Lambda$. By (2.1) and (2.3) we have

$$\begin{aligned}
\|\partial_x({}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u})\|_\Lambda &= \left\|\mathrm{e}^{-\frac{1}{2}\beta x}\partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}{}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right) - \frac{1}{2}\beta({}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u})\right\|_\Lambda \\
&\le \left\|\partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}{}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_\beta,\Lambda} + \frac{1}{2}\beta \left\|\mathrm{e}^{\frac{1}{2}\beta x}{}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right\|_{\omega_\beta,\Lambda} \\
&\le 2\left\|\partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}{}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_\beta,\Lambda} = 2\left\|{}_0P^1_{N-1,\beta,\Lambda}\partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right) - \partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_\beta,\Lambda}.
\end{aligned}$$

Further using (2.2) with $s = 0$, we obtain that for integer $2 \le r \le N+1$,

$$\|\partial_x({}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u})\|_\Lambda \le c\beta^{-1}(\beta N)^{1-\frac{r}{2}} \left\|\partial_x^r \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_{r-2,\beta},\Lambda}. \tag{2.5}$$

We next estimate $\|{}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u}\|_\Lambda$. For simplicity of statements, let

$$\phi(x) = \int_0^x {}_0P^1_{N-1,\beta,\Lambda}\partial_\xi \left(\mathrm{e}^{\frac{1}{2}\beta\xi}\tilde{u}(\xi)\right) \mathrm{d}\xi.$$

Then by virtue of (2.3), (2.1) and (2.2) with $s = 0$, we get that for integer $2 \le r \le N+1$,

$$\begin{aligned}
\|{}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u}\|_\Lambda &= \left\|\phi - \mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right\|_{\omega_\beta,\Lambda} \le \frac{2}{\beta}\left\|\partial_x(\phi - \mathrm{e}^{\frac{1}{2}\beta x}\tilde{u})\right\|_{\omega_\beta,\Lambda} \\
&= \frac{2}{\beta}\left\|{}_0P^1_{N-1,\beta,\Lambda}\partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right) - \partial_x \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_\beta,\Lambda} \\
&\le c\beta^{-2}(\beta N)^{1-\frac{r}{2}} \left\|\partial_x^r \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_{r-2,\beta},\Lambda}.
\end{aligned} \tag{2.6}$$

Since ${}^0\tilde{P}^1_{N,\beta,\Lambda}u - u = {}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u}$, we have from (2.5) and (2.6) that for integer $2 \le r \le N+1$,

$$\begin{aligned}
\|\partial_x^s({}^0\tilde{P}^1_{N,\beta,\Lambda}u - u)\|_\Lambda &= \|\partial_x^s({}_{00}\tilde{P}^1_{N,\beta,\Lambda}\tilde{u} - \tilde{u})\|_\Lambda \le c\beta^{s-2}(\beta N)^{1-\frac{r}{2}} \left\|\partial_x^r \left(\mathrm{e}^{\frac{1}{2}\beta x}\tilde{u}\right)\right\|_{\omega_{r-2,\beta},\Lambda} \\
&= \beta^{s-2}(\beta N)^{1-\frac{r}{2}} \left\|\partial_x^r \left(\mathrm{e}^{\frac{1}{2}\beta x}u\right)\right\|_{\omega_{r-2,\beta},\Lambda}, \quad s = 0, 1.
\end{aligned}$$

This ends the proof. □

Remark 2.1. It is pointed out that the main idea and techniques used in this proof come from [19–21].

In the numerical analysis of the one-dimensional Laguerre spectral method with the essential imposing of Neumann boundary conditions, we need the following orthogonal projection. To this end, we define the bilinear form $a_{1,\lambda}(u,v)$ as follows:

$$a_{1,\lambda}(u,v)=(\partial_x u,\partial_x v)_\Lambda+\lambda(u,v)_\Lambda,\quad \lambda\ge 0,\ \forall u,v\in {}^0\mathcal{F}(\Lambda).$$

The orthogonal projection ${}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}:{}^0\mathcal{F}(\Lambda)\to {}^0\mathcal{Q}_{N,\beta}(\Lambda)$ is defined by

$$a_{1,\lambda}({}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}u-u,\phi)=0,\quad \forall\phi\in {}^0\mathcal{Q}_{N,\beta}(\Lambda).$$

By the projection theorem, for any $\phi\in {}^0\mathcal{Q}_{N,\beta}(\Lambda)$, we have

$$\|\partial_x({}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}u-u)\|^2_\Lambda+\lambda\|{}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}u-u\|^2_\Lambda\le\|\partial_x(\phi-u)\|^2_\Lambda+\lambda\|\phi-u\|^2_\Lambda.$$

Take $\phi={}^0\tilde{P}^1_{N,\beta,\Lambda}u\in {}^0\mathcal{Q}_{N,\beta}(\Lambda)$. Then with the aid of (2.4), we verify:

Theorem 2.1. *For any $u\in {}^0\mathcal{F}(\Lambda)$, $\partial_x^r\left(e^{\frac{1}{2}\beta x}u\right)\in L^2_{\omega_{r-2,\beta}}(\Lambda)$, $\lambda\ge 0$ and integer $2\le r\le N+1$,*

$$\|\partial_x({}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}u-u)\|^2_\Lambda+\lambda\|{}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}u-u\|^2_\Lambda\le c(\beta^{-2}+\lambda\beta^{-4})(\beta N)^{2-r}\left\|\partial_x^r\left(e^{\frac{1}{2}\beta x}u\right)\right\|^2_{\omega_{r-2,\beta},\Lambda}. \tag{2.7}$$

2.2. Two-dimensional Laguerre approximations

We now turn to two-dimensional Laguerre approximations. Let $\Lambda_1=\Lambda_2=\Lambda$ and $\Omega=\{(x,y)\mid x\in\Lambda_1,y\in\Lambda_2\}$. Define $\omega^1_{\alpha,\beta}(x)=\omega_{\alpha,\beta}(x)$ and $\omega^2_{\alpha,\beta}(y)=\omega_{\alpha,\beta}(y)$. For a certain weight function $\chi(x,y)$, we define the weighted spaces $L^2_\chi(\Omega)$ and $H^1_\chi(\Omega)$ with their inner products and norms in the usual way. We omit the subscript χ in the notation whenever $\chi(x,y)\equiv 1$.

Now let

$${}^0\mathcal{F}(\Omega)={}^0\mathcal{F}(\Lambda_2;{}^0\mathcal{F}(\Lambda_1)),\qquad {}^0Q_{N,\beta}(\Omega)={}^0\mathcal{Q}_{N,\beta}(\Lambda_1)\otimes{}^0\mathcal{Q}_{N,\beta}(\Lambda_2).$$

The definition of the space ${}^0\mathcal{F}(\Omega)$ means that for any $u(x,y)\in {}^0\mathcal{F}(\Omega)$, we have that

$$\|u(\cdot,y)\|_{H^1(\Lambda_1)}\in {}^0\mathcal{F}(\Lambda_2)\quad\text{with } u(x,\cdot)\in {}^0\mathcal{F}(\Lambda_1),$$

where the integral in the norm is the Bochner integral.

We now introduce a mapping ${}^0\tilde{P}^1_{N,\beta,\Omega}:{}^0\mathcal{F}(\Omega)\to {}^0Q_{N,\beta}(\Omega)$ as follows:

$${}^0\tilde{P}^1_{N,\beta,\Omega}u={}^0\tilde{P}^1_{N,\beta,\Lambda_1}\cdot{}^0\tilde{P}^1_{N,\beta,\Lambda_2}u.$$

For estimating $\|{}^0\tilde{P}^1_{N,\beta,\Omega}u-u\|_{1,\Omega}$, we use the notation that for integer $2\le r\le N+1$,

$$\begin{aligned}\mathbb{A}^r_{\beta,\Omega}(u)=&(\beta^{-2}+\beta^{-4})\left(\int_{\Lambda_1}\left\|\partial_y^r\left(e^{\frac{1}{2}\beta y}u\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx+\int_{\Lambda_2}\left\|\partial_x^r\left(e^{\frac{1}{2}\beta x}u\right)\right\|^2_{\omega^1_{r-2,\beta},\Lambda_1}dy\right)\\&+\beta^{-4}\left(\int_{\Lambda_1}\left\|\partial_y^r\left(e^{\frac{1}{2}\beta y}\partial_x u\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx+\int_{\Lambda_2}\left\|\partial_x^r\left(e^{\frac{1}{2}\beta x}\partial_y u\right)\right\|^2_{\omega^1_{r-2,\beta},\Lambda_1}dy\right)\\&+(\beta^{-6}+\beta^{-8})\int_{\Lambda_1}e^{-\beta x}\left\|\partial_x^2\left(e^{\frac{1}{2}\beta x}\partial_y^r\left(e^{\frac{1}{2}\beta y}u\right)\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx.\end{aligned}$$

Lemma 2.2. *For any $u\in {}^0\mathcal{F}(\Omega)$ and integer $2\le r\le N+1$,*

$$\|{}^0\tilde{P}^1_{N,\beta,\Omega}u-u\|^2_{1,\Omega}\le c(\beta N)^{2-r}\mathbb{A}^r_{\beta,\Omega}(u), \tag{2.8}$$

provided that $\mathbb{A}^r_{\beta,\Omega}(u)$ is finite.

Proof. Since ${}^0\tilde{P}^1_{N,\beta,\Omega}u={}^0\tilde{P}^1_{N,\beta,\Lambda_1}\cdot{}^0\tilde{P}^1_{N,\beta,\Lambda_2}u$, we have that

$$\|{}^0\tilde{P}^1_{N,\beta,\Omega}u-u\|^2_{1,\Omega}\le F_1(u)+F_2(u),$$

where

$$\begin{aligned}F_1(u)=&2\|\partial_x({}^0\tilde{P}^1_{N,\beta,\Lambda_1}u-u)\|^2_\Omega+2\|\partial_y({}^0\tilde{P}^1_{N,\beta,\Lambda_1}u-u)\|^2_\Omega+2\|{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u-u\|^2_\Omega,\\F_2(u)=&2\|\partial_x({}^0\tilde{P}^1_{N,\beta,\Lambda_2}\cdot{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u-{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u)\|^2_\Omega+2\|\partial_y({}^0\tilde{P}^1_{N,\beta,\Lambda_2}\cdot{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u-{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u)\|^2_\Omega\\&+2\|{}^0\tilde{P}^1_{N,\beta,\Lambda_2}\cdot{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u-{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u\|^2_\Omega.\end{aligned}$$

Thanks to (2.4), we deduce that

$$F_1(u) \le c(\beta^{-2}+\beta^{-4})(\beta N)^{2-r}\int_{\Lambda_2}\left\|\partial_x^r\left(e^{\frac{1}{2}\beta x}u\right)\right\|^2_{\omega^1_{r-2,\beta},\Lambda_1}dy + c\beta^{-4}(\beta N)^{2-r}\int_{\Lambda_2}\left\|\partial_x^r\left(e^{\frac{1}{2}\beta x}\partial_y u\right)\right\|^2_{\omega^1_{r-2,\beta},\Lambda_1}dy.$$

Similarly

$$\begin{aligned}F_2(u) \le\ & c\beta^{-4}(\beta N)^{2-r}\int_{\Lambda_1}\left\|\partial_y^r\left(e^{\frac{1}{2}\beta y}\partial_x {}^0\tilde{P}^1_{N,\beta,\Lambda_1}u\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx\\ &+c(\beta^{-2}+\beta^{-4})(\beta N)^{2-r}\int_{\Lambda_1}\left\|\partial_y^r\left(e^{\frac{1}{2}\beta y}{}^0\tilde{P}^1_{N,\beta,\Lambda_1}u\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx.\end{aligned}$$

Further, by using (2.4) with $s = 0, 1$ and $r = 2$, we obtain that

$$\begin{aligned}F_2(u) \le\ & c\beta^{-4}(\beta N)^{2-r}\int_{\Lambda_1}\left\|\partial_y^r\left(e^{\frac{1}{2}\beta y}\partial_x u\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx + c(\beta^{-2}+\beta^{-4})(\beta N)^{2-r}\int_{\Lambda_1}\left\|\partial_y^r\left(e^{\frac{1}{2}\beta y}u\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx\\ &+c(\beta^{-6}+\beta^{-8})(\beta N)^{2-r}\int_{\Lambda_1}e^{-\beta x}\left\|\partial_x^2\left(e^{\frac{1}{2}\beta x}\partial_y^r\left(e^{\frac{1}{2}\beta y}u\right)\right)\right\|^2_{\omega^2_{r-2,\beta},\Lambda_2}dx.\end{aligned}$$

A combination of previous statements leads to the desired result. □

In the numerical analysis of the two-dimensional Laguerre spectral method with the essential imposing of Neumann boundary conditions, we need another orthogonal projection. To this end, we introduce the bilinear form $a_{2,\lambda}(u, w)$ as follows:

$$a_{2,\lambda}(u, v) = (\partial_x u, \partial_x v)_\Omega + (\partial_y u, \partial_y v)_\Omega + \lambda(u, v)_\Omega, \quad \lambda \ge 0,\ \forall u, v \in {}^0\mathcal{F}(\Omega).$$

The orthogonal projection ${}^0\tilde{P}^1_{N,\beta,\lambda,\Omega} : {}^0\mathcal{F}(\Omega) \to {}^0Q_{N,\beta}(\Omega)$ is defined by

$$a_{2,\lambda}({}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u - u, \phi) = 0, \quad \forall\phi \in {}^0Q_{N,\beta}(\Omega).$$

By the projection theorem,

$$\begin{aligned}&\|\partial_x({}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u-u)\|^2_\Omega + \|\partial_y({}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u-u)\|^2_\Omega + \lambda\|{}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u-u\|^2_\Omega\\ &\quad\le \|\partial_x(\phi-u)\|^2_\Omega + \|\partial_y(\phi-u)\|^2_\Omega + \lambda\|\phi-u\|^2_\Omega, \quad \forall\phi\in {}^0Q_{N,\beta}(\Omega).\end{aligned}$$

Take $\phi = {}^0\tilde{P}^1_{N,\beta,\Omega}u \in {}^0Q_{N,\beta}(\Omega)$. Then with the aid of (2.8), we obtain:

Theorem 2.2. *For any $u \in {}^0\mathcal{F}(\Omega)$ and integers $2 \le r \le N+1$,*

$$\|\partial_x({}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u-u)\|^2_\Omega + \|\partial_y({}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u-u)\|^2_\Omega + \lambda\|{}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}u-u\|^2_\Omega \le c(\beta N)^{2-r}\mathbb{A}^r_{\beta,\Omega}(u), \tag{2.9}$$

provided that $\mathbb{A}^r_{\beta,\Omega}(u)$ is finite.

3. The Laguerre spectral method

In this section, we consider the Laguerre spectral method with the essential imposing of Neumann boundary conditions in one and two space dimensions.

3.1. The one-dimensional problem

We first consider an ordinary differential equation

$$\begin{cases}-\partial_x^2 U(x) + \lambda U(x) = f(x), & x \in \Lambda,\\ \partial_x U(0) = \lim\limits_{x\to\infty} U(x) = \lim\limits_{x\to\infty}\partial_x U(x) = 0,\end{cases} \tag{3.1}$$

where $\lambda \ge 0$. A weak formulation of (3.1) is to seek a solution $U \in {}^0\mathcal{F}(\Lambda)$ such that

$$a_{1,\lambda}(U, v) = (f, v)_\Lambda, \quad \forall v \in {}^0\mathcal{F}(\Lambda). \tag{3.2}$$

The Laguerre spectral scheme for (3.2) aims to find $u_N \in {}^0\mathcal{Q}_{N,\beta}(\Lambda)$ such that

$$a_{1,\lambda}(u_N, \phi) = (f, \phi)_\Lambda, \quad \forall\phi \in {}^0\mathcal{Q}_{N,\beta}(\Lambda). \tag{3.3}$$

We next deal with the convergence of scheme (3.3). Let $U_N = {}^0\tilde{P}^1_{N,\beta,\lambda,\Lambda}U$. We derive from (3.2) that

$$a_{1,\lambda}(U_N, \phi) = (f, \phi)_\Lambda, \quad \forall\phi \in {}^0\mathcal{Q}_{N,\beta}(\Lambda). \tag{3.4}$$

Setting $\widetilde{U}_N = u_N - U_N$, and subtracting (3.4) from (3.3), we obtain that

$$a_{1,\lambda}(\widetilde{U}_N, \phi) = 0, \quad \forall \phi \in {}^0\mathcal{Q}_{N,\beta}(\Lambda).$$

Taking $\phi = \widetilde{U}_N$, we assert that $a_{1,\lambda}(\widetilde{U}_N, \widetilde{U}_N) = 0$. Finally, we use (2.7) to obtain that for any $U \in {}^0\mathcal{F}(\Lambda)$, $\partial_x^r\left(e^{\frac{1}{2}\beta x}U\right) \in L^2_{\omega_{r-2,\beta}}(\Lambda)$, $\lambda \geq 0$ and integer $2 \leq r \leq N+1$,

$$\|\partial_x(U-u_N)\|^2_\Lambda + \lambda\|U-u_N\|^2_\Lambda \leq c(\beta^{-2}+\lambda\beta^{-4})(\beta N)^{2-r}\left\|\partial_x^r\left(e^{\frac{1}{2}\beta x}U\right)\right\|^2_{\omega_{r-2,\beta},\Lambda}. \tag{3.5}$$

3.2. The two-dimensional problem

We next consider the elliptic equation

$$\begin{cases} -\Delta U(x,y) + \lambda U(x,y) = f(x,y), & (x,y) \in \Omega, \\ \partial_n U(x,y) = 0, & (x,y) \text{ on } \partial\Omega \\ U(x,y) \to 0, \qquad \partial_x U(x,y) \to 0, \qquad \partial_y U(x,y) \to 0, & x \text{ or } y \to \infty, \end{cases} \tag{3.6}$$

where $\lambda \geq 0$. A weak formulation of (3.6) is to seek a solution $U \in {}^0\mathcal{F}(\Omega)$ such that

$$a_{2,\lambda}(U, v) = (f, v)_\Omega, \quad \forall v \in {}^0\mathcal{F}(\Omega). \tag{3.7}$$

The Laguerre spectral scheme for (3.7) aims to find $u_N \in {}^0Q_{N,\beta}(\Omega)$ such that

$$a_{2,\lambda}(u_N, \phi) = (f, \phi)_\Omega, \quad \forall \phi \in {}^0Q_{N,\beta}(\Omega). \tag{3.8}$$

We next deal with the convergence of scheme (3.8). Let $U_N = {}^0\tilde{P}^1_{N,\beta,\lambda,\Omega}U$. We derive from (3.7) that

$$a_{2,\lambda}(U_N, \phi) = (f, \phi)_\Omega, \quad \forall \phi \in {}^0Q_{N,\beta}(\Omega). \tag{3.9}$$

Setting $\widetilde{U}_N = u_N - U_N$ and subtracting (3.9) from (3.8), we obtain that

$$a_{2,\lambda}(\widetilde{U}_N, \phi) = 0, \quad \forall \phi \in {}^0Q_{N,\beta}(\Omega).$$

Taking $\phi = \widetilde{U}_N$, we assert that $a_{2,\lambda}(\widetilde{U}_N, \widetilde{U}_N) = 0$. Finally, we use (2.9) to obtain that for any $U \in {}^0\mathcal{F}(\Omega)$, $\lambda \geq 0$ and integer $2 \leq r \leq N+1$,

$$\|\partial_x(U-u_N)\|^2_\Omega + \|\partial_y(U-u_N)\|^2_\Omega + \lambda\|U-u_N\|^2_\Omega \leq c(\beta N)^{2-r}\mathbb{A}^r_{\beta,\Omega}(U), \tag{3.10}$$

provided that $\mathbb{A}^r_{\beta,\Omega}(U)$ is finite.

4. Numerical results

In this section, we describe the numerical implementations and present some numerical results.

4.1. The one-dimensional problem

We first consider spectral scheme (3.3). We take the basis functions

$$\eta_l^{(\beta)}(x) = \tilde{\mathcal{L}}_l^{(\beta)}(x) - \frac{2l+1}{2l+3}\tilde{\mathcal{L}}_{l+1}^{(\beta)}(x), \quad 0 \leq l \leq N-1.$$

Since $\partial_x \tilde{\mathcal{L}}_l^{(\beta)}(0) = -\beta\left(l+\frac{1}{2}\right)$, we have that $\partial_x \eta_l^{(\beta)}(0) = 0,\ 0 \leq l \leq N-1$.

In actual computation, we expand the numerical solution as

$$u_N(x) = \sum_{k=0}^{N-1} a_l \eta_l(x).$$

Let

$$f_{l'} = (f, \eta_{l'})_\Lambda, \quad 0 \leq l' \leq N-1,$$

and define the vectors

$$\overrightarrow{X} = (a_0, a_1, \ldots, a_{N-1})^T, \qquad \overrightarrow{F} = (f_0, f_1, \ldots, f_{N-1})^T.$$

Taking $\phi = \eta_{l'}(x)$, $0 \leq l' \leq N-1$ in (3.3), we find that (3.3) is equivalent to the following system:

$$(A + \lambda B)\overrightarrow{X} = \overrightarrow{F}, \tag{4.1}$$

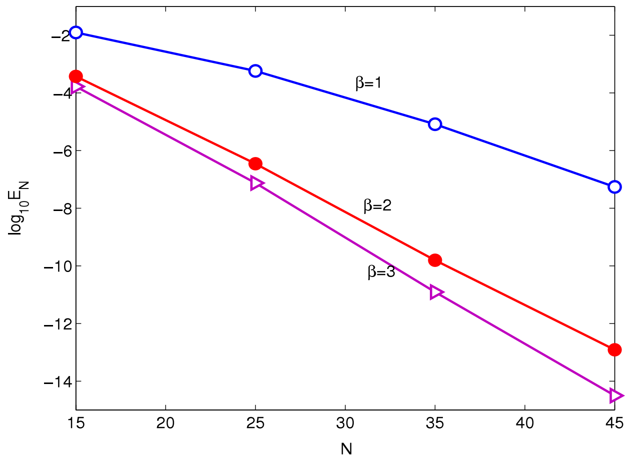

Fig. 1. The discrete $L^2(\Lambda)$-errors of scheme (3.3) with $\lambda = 1.5$, $\beta = 1, 2, 3$ and various values of N.

where the matrices are given by

$$A = (a_{l'l}), \qquad B = (b_{l'l}), \quad 0 \le l', l \le N-1,$$

with the entries

$$a_{l'l} = \int_\Lambda \partial_x \eta_l(x) \partial_x \eta_{l'}(x) \mathrm{d}x, \qquad b_{l'l} = \int_\Lambda \eta_l(x)\eta_{l'}(x)\mathrm{d}x, \quad 0 \le l, l' \le N-1.$$

Due to (18) and (21) of [14], we obtain that for $0 \le l',\ l \le N-1$,

$$a_{l'l} = \begin{cases} \dfrac{\beta(2l'+1)}{4(2l'+3)}, & l' = l-1, \\ \dfrac{\beta(4l'^2+8l'+1)}{2(2l'+3)^2}, & l' = l, \\ \dfrac{\beta(2l+1)}{4(2l+3)}, & l' = l+1, \\ 0, & \text{otherwise}, \end{cases} \qquad b_{l'l} = \begin{cases} -\dfrac{2l'+1}{\beta(2l'+3)}, & l' = l-1, \\ \dfrac{1}{\beta} + \dfrac{(2l'+1)^2}{\beta(2l'+3)^2}, & l' = l, \\ -\dfrac{2l+1}{\beta(2l+3)}, & l' = l+1, \\ 0, & \text{otherwise}. \end{cases}$$

Next we denote by E_N the discrete $L^2(\Lambda)$-error. Let $\lambda = 1.5$ in (4.1) and take the test function

$$U(x) = (1 + x + x^2)\mathrm{e}^{-x}\cos x.$$

In Fig. 1, we plot the numerical errors $\log_{10} E_N$ versus N with various values of β. It is seen that the errors decay exponentially as N increases. It seems that scheme (3.3) with suitable bigger parameter β provides more accurate numerical results. How to choose the best parameter β is an open problem. Roughly speaking, a reasonable choice of parameter β depends mainly on two factors. The first is how fast the solution varies near the point $x = 0$. The second is how rapidly the solution decays as $x \to \infty$. In our example, the solution changes rapidly near the point $x = 0$, and decays rapidly as $x \to \infty$, so it is better to take bigger β.

As pointed out in this paper, the Neumann condition in our method (LM) is enforced according to an essential treatment. In particular, a tridiagonal matrix is employed; see, e.g., (4.1). But in a standard variational formulation, this kind of boundary condition is commonly imposed in a natural way. Moreover, the classical approach (CLM) employs the scaled Laguerre functions $\tilde{\mathscr{L}}_l^{(\beta)}(x)$ as the orthogonal basis functions. This approach leads to a full stiffness matrix for approximating the second derivative with respect to x. In Table 1, we compare the condition numbers of the corresponding coefficient matrices of Problem (3.1) for two different methods with $\lambda = 1.5$ and various parameters β. We find that the condition numbers of our method are much smaller than that of the classical method.

4.2. The two-dimensional problem

We next consider spectral scheme (3.8). We expand the numerical solution as

$$u_N(x, y) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} u_{k,l}\eta_k(x)\eta_l(y).$$

Table 1
The condition numbers.

N	$\beta = 1$		$\beta = 2$		$\beta = 3$	
	LM	CLM	LM	CLM	LM	CLM
15	3.997	1.040e2	2.806	4.143e2	5.362	9.282e2
25	4.189	2.747e2	2.882	1.090e3	5.570	2.460e3
35	4.269	5.260e2	2.915	2.100e3	5.673	4.721e3
45	4.313	8.583e2	2.934	3.420e3	5.729	7.712e3

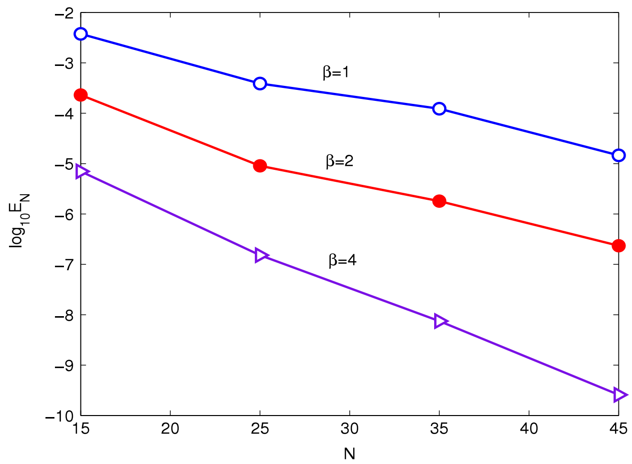

Fig. 2. The discrete $L^2(\Lambda)$-errors of scheme (3.8) with $\lambda = 1$, $\beta = 1, 2, 4$ and various values of N.

Let

$$f_{k'l'} = (f, \eta_{k'}\eta_{l'})_\Omega, \quad 0 \le k', l' \le N-1,$$

and define the vectors

$$\overrightarrow{X} = (u_{0,0}, u_{1,0}, \ldots, u_{N-1,0}, u_{0,1}, u_{1,1}, \ldots, u_{N-1,1}, \ldots, u_{0,N-1}, u_{1,N-1}, \ldots, u_{N-1,N-1})^T,$$
$$\overrightarrow{F} = (f_{0,0}, f_{1,0}, \ldots, f_{N-1,0}, f_{0,1}, f_{1,1}, \ldots, f_{N-1,1}, \ldots, f_{0,N-1}, f_{1,N-1}, \ldots, f_{N-1,N-1})^T.$$

Taking $\phi = \eta_{k'}(x)\eta_{l'}(y)$ in (3.8) for $0 \le k', l' \le N-1$, we find that (3.8) is equivalent to the following system:

$$(A \otimes B + B \otimes A + \lambda B \otimes B)\overrightarrow{X} = \overrightarrow{F}, \tag{4.2}$$

where the matrices A and B are the same as before.

We still denote by E_N the discrete $L^2(\Omega)$-error. Let $\lambda = 1$ in (4.2) and take the test function

$$U(x, y) = (1 + \cos x + \cos y) \exp(-\sqrt{1 + x^2 + y^2}).$$

In Fig. 2, we plot the numerical errors $\log_{10} E_N$ versus N with various values of β. It is again seen that the errors decay exponentially as N increases. It seems that scheme (3.8) with suitable bigger parameter β provides more accurate numerical results.

5. Concluding remarks

In this paper, we proposed a new Laguerre spectral method for solving Neumann boundary value problems and established some basic results on Laguerre approximations, which formed the mathematical foundation of a Laguerre spectral method with the essential imposing of Neumann boundary conditions. We also analyzed the numerical errors of the proposed spectral schemes. In particular, by choosing appropriate basis functions with zero slope at the endpoint, a tridiagonal matrix is employed, instead of the full stiffness matrix encountered in the classical variational formulation. The numerical results demonstrated the spectral accuracy and coincided well with the theoretical analysis.

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