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FFT based option pricing under a mean reverting process with stochastic volatility and jumps

E. Pillay [a], J.G. O'Hara [b,*]

[a] School of Statistics and Actuarial Science, University of KwaZulu-Natal, Private Bag X540001, Durban 4000, South Africa

[b] Centre for Computational Finance and Economic Agents, University of Essex, Colchester CO4 3SQ, UK

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ABSTRACT

Numerous studies present strong empirical evidence that certain financial assets may exhibit mean reversion, stochastic volatility or jumps. This paper explores the valuation of European options when the underlying asset follows a mean reverting log-normal process with stochastic volatility and jumps. A closed form representation of the characteristic function of the process is derived for the computation of European option prices via the fast Fourier transform.

1. Introduction

It is well known that the prices of certain financial asset classes show evidence of mean reversion. As a result, the pricing of options on these asset classes has become an important topic in quantitative finance. For instance, commodity markets often show prices to fluctuate randomly around an equilibrium level. This is due to the impact of relative prices on the supply of the commodity in question (see [1]). Commodities such as oil also exhibit jumps linked to abnormal shocks either in production or demand. Currency markets also show evidence of mean reversion. For instance, see [2] for an empirical study of mean reversion in real exchange rates over a flexible exchange rate period. Interestingly, empirical studies also indicate that certain stock prices may exhibit mean reversion as well; *e.g.*, see [3].

Another significant aspect of modeling financial assets is the idea of stochastic volatility. The notion of allowing volatility to follow a stochastic process has become popular for option pricing over the past ten years, as the existence of a non-flat implied volatility surface (or term structure) has become even more prominent, especially since the 1987 stock market crash.

By combining these fundamental ideas of asset mean reversion and stochastic volatility, Wong and Lo [4] propose a new model for option pricing, whereby the underlying asset price is assumed to follow the Schwartz [1] mean reverting log-normal process governed by a single Brownian motion, with the volatility process following the Heston model [5], where the volatility is also driven by a single Brownian motion process. Both the asset price process and the volatility process include time dependent coefficients and are correlated by a constant correlation coefficient. A powerful technique is derived for attaining the characteristic function of the logarithm of the underlying asset spot price, from which they are able to apply the fast Fourier transform (FFT) for the computation of European option prices.

In addition to the ideas of mean reversion and stochastic volatility, we include the notion that financial assets show evidence of price jumps. For example, Jorion [6] investigates the existence of discontinuities in the sample paths of exchange

* Corresponding author.
E-mail addresses: evashun.pillay@standardbank.co.za (E. Pillay), johara@essex.ac.uk (J.G. O'Hara).

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rates and stock market indices whilst Seifert and Uhrig-Homburg [7] provide empirical evidence of jumps occurring in electricity prices. We then go on to apply the FFT method as introduced in [8,9] to compute European option prices. Finally we compare the FFT results against a Monte Carlo simulation for the pricing of a European call option under our model.

2. Adding jumps to the mean reverting asset price process

Consider the following two-factor model:

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space on which are defined two Brownian motion processes W_t^1 and W_t^2, $0 \leq t \leq T$. Let $\mathcal{F}_t$ be the filtration generated by these Brownian motions. Suppose that $\mathbb{Q}$ is a risk neutral probability under which the asset price process S_t and volatility process v_t are governed by the following dynamics:

$$dS_t = \kappa\left(\theta(t) - \ln S_t\right) S_t dt + \sqrt{v_t} dW_t^1, \tag{2.1}$$

$$dv_t = b(a(t) - v_t)dt + \sigma\sqrt{v_t} dW_t^2, \tag{2.2}$$

$$dW_t^1 dW_t^2 = \rho dt, \tag{2.3}$$

where $\theta(t)$ is a deterministic function that represents the equilibrium mean level of the asset against time, κ is the mean reverting intensity of the asset, $a(t)$ is a deterministic function that describes the equilibrium mean level of the volatility process against time and b is the mean reversion speed of the volatility process. The constant σ is the volatility coefficient of the volatility process and W_t^1 and W_t^2 are correlated with correlation coefficient ρ.

Suppose that on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ we now define a Poisson process N_t for all $0 \leq t \leq T$, with a constant intensity parameter $\lambda > 0$. Furthermore, we assume that the Poisson process N_t is independent of both Brownian motion processes W_t^1 and W_t^2. We also define a sequence of random variables e^{J_i}, for all $1 \leq i \leq N_t$, which represent the jump sizes of the Poisson process. Each of the e^{J_i} are log-normally, identically and independently distributed over time, where $J_i \sim N(\mu, \gamma^2)$ and $\gamma > 0$. Since the e^{J_i} are i.i.d., we suppress the i subscript for notational convenience. Define the following mean reverting asset price process governed by both a Brownian motion process and compound Poisson process:

$$dS_t = \kappa\left(\theta(t) - \ln S_t - \frac{\lambda m}{\kappa}\right) S_t dt + \sqrt{v_t} S_t dW_t^1 + \left(e^J - 1\right) S_{t-} dN_t, \tag{2.4}$$

where

$$m = \mathbb{E}^{\mathbb{Q}}\left[e^J - 1\right] = e^{\mu + \frac{1}{2}\gamma^2} - 1, \tag{2.5}$$

and S_{t-} is the value of the process S_t immediately before a jump.

We now define the following process:

$$X_t = \ln S_t. \tag{2.6}$$

Applying the Itô–Doeblin formula to (2.6) we obtain

$$X_t = X_0 + \int_0^t \frac{1}{S_u} dS_u^c - \frac{1}{2}\int_0^t \frac{1}{S_u^2} dS_u^c dS_u^c + \sum_{0 \leq u \leq t} (X_u - X_{u-}) \tag{2.7}$$

where S_t^c refers to the continuous component of SDE (2.4). X_{t-} is the value of the process X_t immediately before a jump. Consider the last term in (2.7). We know that should a jump occur, the size of the jump will be e^J. Hence, if a jump occurs at time u, it follows that $X_u = e^J X_{u-}$. Therefore

$$X_u - X_{u-} = \left(e^J - 1\right) X_{u-}, \tag{2.8}$$

whenever there is a jump at time u, and of course $X_u - X_{u-} = 0$ if there is no jump at u. In either case we have

$$X_u - X_{u-} = \left(e^J - 1\right) X_{u-} \Delta N_u.$$

This observation allows us to express the last term in (2.7) as

$$\sum_{0 \leq u \leq t} (X_u - X_{u-}) = \sum_{0 \leq u \leq t} \left(e^J - 1\right) X_{u-} \Delta N_u = \int_0^t \left(e^J - 1\right) X_{u-} dN_u. \tag{2.9}$$

Simplifying (2.7) together with (2.9) and expressing the result in differential form, we obtain

$$dX_t = \kappa\left(\theta(t) - \frac{v_t}{2\kappa} - \frac{\lambda m}{\kappa} - X_t\right) dt + \sqrt{v_t} dW_t^1 + \left(e^J - 1\right) X_{t-} dN_t. \tag{2.10}$$

For mathematical convenience, we allow the long term mean reverting equilibriums of both the asset price process (2.10) and the volatility process (2.2) to be constant for all $0 \leq t \leq T$ (i.e. $\theta(t) = \theta$ for all $0 \leq t \leq T$, and $a(t) = a$ for all $0 \leq t \leq T$). We now define the following two-factor mean reverting process with stochastic volatility and jumps:

$$dX_t = \kappa \left(\theta - \frac{v_t}{2\kappa} - \frac{\lambda m}{\kappa} - X_t\right) dt + \sqrt{v_t} dW_t^1 + \left(e^J - 1\right) X_{t-} dN_t, \tag{2.11}$$

$$dv_t = b(a - v_t)dt + \sigma \sqrt{v_t} dW_t^2, \tag{2.12}$$

$$dW_t^1 dW_t^2 = \rho dt. \tag{2.13}$$

3. Deriving the characteristic function

The characteristic function of the process (2.11) is defined as

$$\phi_t(u) \equiv \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_T} | \mathcal{F}_t\right]. \tag{3.1}$$

We now apply the method of Wong and Lo [4] to compute the characteristic function of the process (2.11).

Lipton [10] and Duffie et al. [11] introduce a generalized Feynman–Kac theorem for affine jump diffusion processes. On defining the following function:

$$f(u; t, x, v) = \mathbb{E}^{\mathbb{Q}}\left[e^{iuX_T} | X_t = x, v_t = v\right] \tag{3.2}$$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[e^{rT} e^{iuX_T} | X_t = x, v_t = v\right], \tag{3.3}$$

which can be viewed as a contingent claim that pays e^{rT+iux} at time T, where r is a constant interest rate, X_t is the mean reverting asset price process with jumps specified by (2.11) and v_t is the volatility process specified by (2.12), the generalized Feynman–Kac theorem implies that $f(u; t, x)$ solves the following partial integro-differential equation (PIDE):

$$\frac{\partial f}{\partial t} + \kappa \left(\theta - \frac{v}{2\kappa} - \frac{\lambda m}{\kappa} - x\right) \frac{\partial f}{\partial x} + \frac{1}{2} v \frac{\partial^2 f}{\partial x^2} + b(a - v) \frac{\partial f}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f}{\partial v^2} + \rho\sigma v \frac{\partial^2 f}{\partial x \partial v}$$
$$+ \lambda \int_{-\infty}^{\infty} [f(u; t, x + J, v) - f(u; t, x, v)] q(J) dJ = 0, \tag{3.4}$$

where $q(J)$ is the distribution function of the random variable J and $\lambda > 0$ is the constant intensity parameter of the Poisson process N_t.

The coefficients, $\kappa\left(\theta - \frac{v}{2\kappa} - \frac{\lambda m}{\kappa} - x\right)$ and $\sqrt{v_t}$, of the mean reverting asset price process (2.11) and the coefficients, $b(a - v_t)$ and $\sigma\sqrt{v_t}$, of the volatility process (2.12) are all affine in nature. It follows that the function $f(u; t, x, v)$ is of exponential affine form. Hence the solution of (3.4) has the form

$$f(u; t, x, v) = \mathrm{e}^{B(t,T)+C(t,T)x+D(t,T)v+\mathrm{i}ux}, \tag{3.5}$$

where $B(t, T)$, $C(t, T)$ and $D(t, T)$ are deterministic functions of t. From (3.2), it is clear that

$$f(u; T, x, v) = \mathrm{e}^{iux}, \tag{3.6}$$

which is the boundary condition of PIDE (3.4). This implies that

$$B(T, T) = 0, \qquad C(T, T) = 0, \qquad D(T, T) = 0. \tag{3.7}$$

We now consider the integral term in (3.4) (omitting the conditional portion of the expectations for notational convenience):

$$\begin{aligned}
\lambda \int_{-\infty}^{\infty} [f(u; t, x + J, v) - f(u; t, x, v)] q(J) dJ &= \lambda \int_{-\infty}^{\infty} \left[\mathbb{E}^{\mathbb{Q}}[e^{iu(X_T+J)}] - \mathbb{E}^{\mathbb{Q}}[e^{iuX_T}]\right] q(J) dJ \\
&= \lambda \int_{-\infty}^{\infty} \left[\mathbb{E}^{\mathbb{Q}}[e^{iuX_T}(e^{iuJ} - 1)]\right] q(J) dJ \\
&= \lambda \int_{-\infty}^{\infty} \mathbb{E}^{\mathbb{Q}}[e^{iuX_T}]\mathbb{E}^{\mathbb{Q}}[(e^{iuJ} - 1)] q(J) dJ \\
&= f(u; t, x, v)\Lambda(u),
\end{aligned} \tag{3.8}$$

where $\Lambda(u) = \lambda\left(e^{iu\mu - \frac{1}{2}u^2\gamma^2} - 1\right)$. The above result is based on the fact that the jump size J is independent of the process X_t, and $J \sim N(\mu, \gamma^2)$.

Substituting the result (3.8) and the partial derivatives in (3.4) and rearranging we have

$$[B_t(t,T) + (\kappa\theta - \lambda m)(iu + C(t,T)) + abD(t,T) + \Lambda(u)] + [C_t(t,T) - \kappa(iu + C(t,T))]x + \left[D_t(t,T) + \frac{1}{2}(iu + C(t,T))(iu + C(t,T) - 1) - bD(t,T) + \frac{1}{2}\sigma^2 D^2(t,T) + \rho\sigma D(t,T)(iu + C(t,T))\right]v = 0. \tag{3.9}$$

Since Eq. (3.9) holds for all t, x and v we can conclude that the three terms in square brackets on the left-hand side must vanish. This reduces the problem to one of solving three, much simpler, ordinary differential equations:

$$B_t(t,T) + (\kappa\theta - \lambda m)(iu + C(t,T)) + abD(t,T) + \Lambda(u) = 0, \tag{3.10}$$

$$C_t(t,T) - \kappa(iu + C(t,T)) = 0, \tag{3.11}$$

$$D_t(t,T) + \frac{1}{2}(iu + C(t,T))(iu + C(t,T) - 1) - bD(t,T) + \frac{1}{2}\sigma^2 D^2(t,T) + \rho\sigma D(t,T)(iu + C(t,T)) = 0, \tag{3.12}$$

subject to boundary conditions (3.7).

The solution to Eq. (3.11) with the boundary condition $C(T,T) = 0$ is

$$C(t,T) = iu\left(e^{-\kappa(T-t)} - 1\right). \tag{3.13}$$

We now consider Eq. (3.12). Substituting (3.13) in (3.12) we have

$$D_t(t,T) = -\frac{1}{2}\sigma^2 D^2(t,T) + \left(b - \rho\sigma iue^{-\kappa(T-t)}\right)D(t,T) + \frac{1}{2}\left(u^2 e^{-2\kappa(T-t)} + iue^{-\kappa(T-t)}\right). \tag{3.14}$$

We now consider a transformation of the independent variable t. Letting $\tau = e^{-\kappa(T-t)}$ and defining $\widetilde{D}(\tau,T) = D(t,T)$ it follows that

$$\begin{aligned}\frac{\partial D(t,T)}{\partial t} &= \frac{\partial \widetilde{D}(\tau,T)}{\partial \tau}\frac{\partial \tau}{\partial t} \\ &= \kappa e^{-\kappa(T-t)}\frac{\partial \widetilde{D}(\tau,T)}{\partial \tau}.\end{aligned} \tag{3.15}$$

Substituting (3.15) in (3.14), we obtain the following Riccati equation:

$$\frac{\partial \widetilde{D}(\tau,T)}{\partial \tau} = -\frac{\sigma^2}{2\kappa\tau}\widetilde{D}^2(\tau,T) + \left(\frac{b}{\kappa\tau} - \frac{\rho\sigma iu}{\kappa}\right)\widetilde{D}(t,T) + \frac{1}{2\kappa}\left(\frac{u^2}{\tau} + iu\right), \tag{3.16}$$

with the initial condition, $\widetilde{D}(1,T) = 0$.

To solve (3.16), we need a particular solution from which the general solution can be derived. We apply the following well known transformation for Riccati equations:

$$\widetilde{D}(\tau,T) = \frac{2\kappa\tau w'(\tau)}{\sigma^2 w(\tau)}. \tag{3.17}$$

The derivative of (3.17) with respect to τ is

$$\frac{\partial \widetilde{D}(\tau,T)}{\partial \tau} = \frac{\left(2\kappa w'(\tau) + 2\kappa\tau w''(\tau)\right)\sigma^2 w(\tau) - 2\kappa\tau\sigma^2(w'(\tau))^2}{\sigma^4 w^2(\tau)}. \tag{3.18}$$

Substituting (3.17) and (3.18) in (3.16) and simplifying we are left with

$$\tau w''(\tau) - \left[\left(\frac{b}{\kappa} - 1\right) - \tau\left(\frac{\rho\sigma iu}{\kappa}\right)\right]w'(\tau) + \left(\frac{u^2\sigma^2}{4\kappa^2} + \frac{iu}{4\kappa^2}\right)w(\tau) = 0. \tag{3.19}$$

The above ordinary differential equation has a general solution of the form (see [12])

$$w(\tau) = e^{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma u}{2\kappa}\tau}\left[C_2\Phi\left(a^*, b^*, \frac{\tau}{\zeta}\right) + C_3\tau^{1-b^*}\Phi\left(a^* - b^* + 1, 2 - b^*, \frac{x}{\zeta}\right)\right], \tag{3.20}$$

where

$$a^* = \frac{(\sqrt{\rho^2 - 1} + \rho)\frac{b^*}{2} + \frac{\sigma}{4\kappa}}{\sqrt{\rho^2 - 1}}, \tag{3.21}$$

$$b^* = 1 - \frac{b}{\kappa}, \tag{3.22}$$

and

$$\zeta = \frac{-\kappa}{\sigma u\sqrt{1-\rho^2}}. \tag{3.23}$$

C_2 and C_3 are constants to be determined from the boundary conditions and a^* is a complex function. $\Phi(a^*, b^*, z)$ is the degenerate hypergeometric function, which has the following Kummer series expansion:

$$\Phi(a, b, z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k z^k}{(b)_k k!}, \tag{3.24}$$

where

$$(a)_k = a(a+1)\cdots(a+k-1). \tag{3.25}$$

If we let $C_2 = 1$ and $C_3 = 0$ in (3.20), it follows that a particular solution for (3.19) is

$$w(\tau) = e^{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma u}{2\kappa}\tau}\left[\Phi\left(a^*, b^*, \frac{\tau}{\zeta}\right)\right]. \tag{3.26}$$

Using the transformation (3.17), Wong and Lo [4] show that a particular solution to (3.16) is

$$U(\tau) = \frac{2\kappa\tau}{\sigma^2}\,\frac{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma u}{2\kappa}\Phi\left(a^*, b^*, \frac{\tau}{\zeta}\right) + \frac{a^*}{b^*\zeta}\Phi\left(a^*+1, b^*+1, \frac{\tau}{\zeta}\right)}{\Phi\left(a^*, b^*, \frac{\tau}{\zeta}\right)}, \tag{3.27}$$

which is used in obtaining the general solution to (3.16):

$$\widetilde{D}(\tau) = U(\tau) + \frac{\frac{\Phi^2(a^*,b^*,\frac{1}{\zeta})}{\Phi^2(a^*,b^*,\frac{\tau}{\zeta})}\tau^{\frac{b}{\kappa}} e^{-2(\sqrt{1-\rho^2})\frac{\sigma u}{2\kappa}(\tau-1)}}{-\frac{1}{U(1)} + \frac{\sigma^2}{2\kappa}\int_1^{\tau} \frac{\Phi^2(a^*,b^*,\frac{1}{\zeta})}{\Phi^2(a^*,b^*,\frac{\tau}{\zeta})}\eta^{\frac{b}{\kappa}-1} e^{-2(\sqrt{1-\rho^2})\frac{\sigma u}{2\kappa}(\eta-1)}d\eta}. \tag{3.28}$$

We now consider the final ordinary differential equation (3.10). Substituting (3.13) and (3.28) in (3.10) we have

$$B_t(t, T) = (\lambda m - \kappa\theta)iue^{-\kappa(T-t)} - abD(t, T) - \Lambda(u). \tag{3.29}$$

Integrating both sides and invoking the condition $B(T, T) = 0$, we obtain

$$B(t, T) = \left(\frac{\lambda m}{\kappa} - \theta\right)iu(e^{-\kappa(T-t)} - 1) - ab\int_t^T D(s, T)ds + \Lambda(u)(T-t). \tag{3.30}$$

We can conclude that the characteristic function of the mean reverting process (2.11) with stochastic volatility (2.12) is

$$\phi_t(u) = e^{B(t,T)+C(t,T)x+D(t,T)v+iux}, \tag{3.31}$$

where

$$\begin{aligned}
B(t, T) &= \left(\frac{\lambda m}{\kappa} - \theta\right)iu(e^{-\kappa(T-t)} - 1) - ab\int_t^T D(s, T)ds + \Lambda(u)(T-t),\\
C(t, T) &= iu\left(e^{-\kappa(T-t)} - 1\right),\\
D(t, T) &= U(e^{-\kappa(T-t)}) + \frac{e^{-b(T-t)}V(e^{-\kappa(T-t)})}{\frac{-1}{U(1)} + \frac{\sigma^2}{2\kappa}\int_1^{e^{-\kappa(T-t)}} \tau^{\frac{b}{\kappa}-1}V(\tau)d\tau},\\
U(\tau) &= \frac{2\kappa\tau}{\sigma^2}\,\frac{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma u}{2\kappa}\Phi\left(a^*, b^*, \frac{\tau}{\zeta}\right) + \frac{a^*}{b^*\zeta}\Phi\left(a^*+1, b^*+1, \frac{\tau}{\zeta}\right)}{\Phi\left(a^*, b^*, \frac{\tau}{\zeta}\right)},\\
V(\tau) &= \frac{\Phi^2(a^*, b^*, \frac{1}{\zeta})}{\Phi^2(a^*, b^*, \frac{\tau}{\zeta})}e^{\frac{\sigma u}{\kappa}(1-\tau)\sqrt{1-\rho^2}},\\
a^* &= \frac{(\sqrt{\rho^2-1}+\rho)\frac{b^*}{2} + \frac{\sigma}{4\kappa}}{\sqrt{\rho^2-1}},\\
b^* &= 1 - \frac{b}{\kappa},\\
\zeta &= \frac{-\kappa}{\sigma u\sqrt{1-\rho^2}}.
\end{aligned} \tag{3.32}$$

4. European option pricing using the fast Fourier transform

Let K be the strike price and T the expiration of a European call option with terminal asset price S_T, where S_T is governed by the dynamics (2.11) with stochastic volatility (2.12). The price of a European call option is computed as the discounted risk neutral conditional expectation of the terminal payoff $(S_T - K)^+ = \max(S_T - K, 0)$:

$$C(t, S_T) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+|\mathcal{F}_t\right], \tag{4.1}$$

where r is the constant interest rate. Assume that for simplicity (without loss of "generality") we let $t = 0$ and define $X_t = \ln S_t$ and $k = \ln K$. Furthermore, we express the call option pricing function (4.1) as a function of the log strike k rather than the terminal log asset price X_T:

$$C_T(k) = e^{-rT}\int_k^{\infty}(e^{X_T} - e^k)q_T(X_T)dX_T, \tag{4.2}$$

where $q_T(X_T)$ is the density function of the process X_T.

The call price function (4.2) is not square-integrable because $C_T(k)$ converges to S_0 for $k \to -\infty$. Hence, Carr and Madan [8] introduce a modified call price function $c_T(k) = e^{\alpha k}C_T(k)$ for $\alpha > 0$. The European call price can be easily recovered by applying the FFT:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\frac{e^{-rT}\phi_T(u - i(\alpha+1))}{(\alpha + iu)(\alpha + iu + 1)}du. \tag{4.3}$$

In this instance, an efficient implementation of the FFT requires a closed form representation of the characteristic function $\phi_T(u)$. We have seen that the asset price dynamics (2.11) with stochastic volatility (2.12) does indeed have an analytical characteristic function (3.31) with deterministic functions described by (3.32).

Following the method of Dempster and Hong [9], we evaluate the integral (4.3). We approximate the Fourier integral in (4.3) by the sum

$$C_T(k) \approx \frac{e^{-\alpha k}}{2\pi}\sum_{j=0}^{N-1} e^{-iu_j k}\psi_T(u_j)\Delta, \tag{4.4}$$

where Δ denotes the integration steps and

$$u_j := \left(j - \frac{N}{2}\right)\Delta \qquad j = 0, \ldots, N-1. \tag{4.5}$$

Now, a one-dimensional FFT computes, for any complex (input) array $\{Z[n] \in \mathbb{C}|n = 0, \ldots, N-1\}$, the following (output) array of identical structure:

$$Y[\ell] := \sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}n\ell}Z[n], \tag{4.6}$$

for all $\ell = 0, \ldots, N-1$. In order to apply the above algorithm to evaluate the sum (4.4), we define a grid of size $N \times 1$, $L := \{k_u : 0 \le u \le N-1\}$, with $\omega > 0$ denoting the distance between the grid points, where

$$k_u = \left(u - \frac{N}{2}\right)\omega, \tag{4.7}$$

and evaluate the sum

$$\Gamma(k) = \sum_{j=0}^{N-1} e^{-iu_j k}\psi_T(u_j). \tag{4.8}$$

Choosing $\omega\Delta = \frac{2\pi}{N}$ gives the following values of $\Gamma(k_u)$ on L:

$$\Gamma(k_u) = \sum_{j=0}^{N-1} e^{-iu_j k_u}\psi_T(u_j). \tag{4.9}$$

Substituting (4.5) and (4.7) in (4.9) and using our choice of $\omega\Delta$ we get

$$\begin{aligned}\Gamma(k_u) &= \sum_{j=0}^{N-1} e^{-i(j-\frac{N}{2})\Delta(u-\frac{N}{2})\omega}\psi_T(u_j)\\ &= (-1)^u i^N \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}uj}(-1)^j\psi_T(u_j).\end{aligned} \tag{4.10}$$

Table 1
European call option prices (with jumps): FFT vs. Monte Carlo.

Strike price	FFT	Monte Carlo	% difference
0.37466	1.05578	1.04881	0.660%
0.45594	0.97285	0.97056	0.236%
0.55486	0.88356	0.87773	0.659%
0.67523	0.76517	0.76145	0.486%
0.82173	0.62933	0.62362	0.908%
1	0.43771	0.45381	0.853%
1.21695	0.25555	0.25399	0.611%
1.48097	0.05526	0.05576	0.912%
1.80227	0.00153	0.00155	0.998%

This can be computed from the definition of the FFT (4.6) by taking the input array as

$$Z[j] := (-1)^j \psi_T(u_j) \qquad \forall j = 0, \dots, N-1. \tag{4.11}$$

The result is an approximation for the option price at $N \times 1$ different (log) strikes given by

$$C_T(k_u) \approx \frac{e^{-\alpha k}}{2\pi} \Gamma(k_u) \Delta. \tag{4.12}$$

5. Numerical results

In this section we present a numerical comparison between the FFT and a Monte Carlo simulation option pricing described by Broadie and Kaya (see [13]). We apply the two techniques for the pricing of a European call option under our mean reverting model with stochastic volatility and jumps with a view to comparing the performances of the two models. The codes for the two models were written in Matlab which includes the fast Fourier routine FFTW (the Fastest Fourier Transform in the West) of Frigo and Johnson [14]. The experiments were conducted on an Intel Core 2 Duo 2.00 GHz machine running under Windows Vista Service Pack 1, with 2.00 GB RAM.

We implement our FFT scheme with $\Delta = 0.25$ and $N = 128$, which leads to a log strike spacing of $\omega = \frac{8\pi}{128}$. The modified call price coefficient is set to $\alpha = 1.5$. For the Monte Carlo simulation we use 50,000 sample paths with $M = 365$. The non-jump related parameters for our model are taken from a similar comparison carried out by Wong and Lo [4], where they compare the FFT to Monte Carlo simulation for pricing European call options under a mean reverting process with stochastic volatility and no jumps: $\theta = 4.0399$, $\kappa = 10$, $a = 0.5328$, $b = 3.33$, $\sigma = 0.04$, $\rho = 0.9$, $r = 0.05$, $S_0 = 1.3$, $v_0 = 0.18$ and $T = 1$. For the jump related parameters, we use values given by Broadie and Kaya [13], where they compare their exact Monte Carlo approach to the non-exact Monte Carlo approach for the pricing of European call options under a non-mean reverting process with stochastic volatility and jumps: $\lambda = 0.11$, $\mu = 0.12$ and $\gamma = 0.15$.

The FFT takes about 2 s to produce 128 option prices corresponding to different strike prices. The Monte Carlo simulation takes about two and a half minutes to produce a single option price. Table 1 compares the pricing accuracy between the two methods across a range of strike prices. It is clear that the relative percentage differences are all less than 1%. If we consider the Monte Carlo simulation to be the benchmark, not only does this numerical example confirm that our FFT analytical solution is correct, it also illustrates how much more efficient this technique is.

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