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A new structure-preserving method for quaternion Hermitian eigenvalue problems

Zhigang Jia [a], Musheng Wei [b,*], Sitao Ling [c]

[a] School of Mathematical Sciences, Jiangsu Normal University, Jiangsu, 221116, PR China
[b] College of Mathematics and Science, Shanghai Normal University, Shanghai 200234, PR China
[c] College of Science, China University of Mining and Technology, Jiangsu, 221116, PR China

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ABSTRACT

In this paper we propose a novel structure-preserving algorithm for solving the right eigenvalue problem of quaternion Hermitian matrices. The algorithm is based on the structure-preserving tridiagonalization of the real counterpart for quaternion Hermitian matrices by applying orthogonal *JRS*-symplectic matrices. The algorithm is numerically stable because we use orthogonal transformations; the algorithm is very efficient, it costs about a quarter arithmetical operations, and a quarter to one-eighth CPU times, comparing with standard general-purpose algorithms. Numerical experiments are provided to demonstrate the efficiency of the structure-preserving algorithm.

1. Introduction

In 1843, Sir William Rowan Hamilton (1805–1865) introduced quaternions as he tried to extend complex numbers to higher spatial dimensions, and then he spent the rest of his life obsessed with them and their applications [1]. Nevertheless he probably never thought that one day in the future the quaternion he had invented would be used in quaternionic quantum mechanics (qQM) and many other fields.

About 100 years later, Finkelstein et al. [2–5] built the foundations of qQM and gauge theories. Their fundamental works led to a renewed interest in algebrization and geometrization of physical theories by non-commutative fields [6,7]. Among the numerous references on this subject, the important paper of Horwitz and Biedenharn [8] showed that the assumption of a complex projection of the scalar product, also called complex geometry [9], permits the definition of a suitable tensor product [10] between single-particle quaternionic wavefunctions. Quaternions become to play a more and more important role in many application fields, such as special relativity [11], group representations [12–15], non-relativistic [16,17] and relativistic dynamics [18,19], field theory [20], Lagrangian formalism [21], electro weak model [22], grand unification theories [23] and the preonic model [24]. A clear and detailed discussion of qQM together with possible topics for future developments in field theory and particle physics is found in the recent book by Adler [25].

Many papers have addressed to clarify the proper choice of the quaternionic eigenvalue equation within qQM with complex or quaternionic geometry. For example, we find the works on the quaternionic right eigenvalue equation [26], quaternionic eigenvalues and the characteristic equation [27,28], diagonalization of matrices [29], the Jordan form and q-determinant [30,31]. Most recently, many papers studied the (right) eigenvalue equation for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ linear quaternionic operators [32,33,26,34,35]. Interesting applications are found in solving differential equations within quaternionic formulations of quantum mechanics [36,37]. There are two obstacles in discussing quaternion eigenproblems. The first one

* Corresponding author.
E-mail addresses: zhgjia@jsnu.edu.cn (Z. Jia), mwei@shnu.edu.cn (M. Wei).

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is related to the difficulty in obtaining a suitable definition of the determinant for quaternionic matrices, the second one is represented by the loss, for non-commutative fields, of the fundamental theorem of the algebra. To overcome these difficulties, one can study the eigenvalue problem for $4n$-dimensional real matrices or $2n$-dimensional complex matrices obtained from n-dimensional quaternionic matrices. For example, [33,38] and many others gave the spectral theorem for a quaternion matrix and some numerical algorithms by real or complex counterpart methods.

In analogy with complex quantum mechanics, observable in quaternionic quantum mechanics will be represented by quaternion Hermitian operators, and quaternion Hermitian operators have special structures and properties ((2.11 g) and pp. 27–29 of [25]). In this paper we will use these features to compute eigenvalues and corresponding eigenvectors of quaternion Hermitian operators, by proposing a structure-preserving algorithm. At first, we will characterize structures of the real counterpart of a quaternion Hermitian operator: symmetry and *JRS*-symmetry (for definition of *JRS*-symmetry, see Section 2). These structures are unchanged under orthogonal *JRS*-symplectic transformations. That makes it possible to derive a structure-preserving method for the tridiagonalization of the real counterpart. This structure-preserving method only costs about a quarter of arithmetic operations of the Householder tridiagonalization algorithm for the real symmetric counterpart, and what is more important is that the tridiagonal matrix obtained by the structure-preserving method is still symmetric and *JRS*-symmetric, and has the following form

$$\begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix}$$

where D is an $n \times n$ real symmetric tridiagonal matrix. Therefore to evaluate the eigenvalues of the quaternion Hermitian matrix, we only need to compute the eigenvalues of a single D.

This paper is organized as follows. In Section 2, we will mention some preliminary results used in the paper. In Section 3, we will propose a structure-preserving algorithm for solving the right eigenvalue problem for quaternion Hermitian operators. In Section 4, we will provide three experiments to compare this algorithm with two other standard algorithms, and apply it to compute eigenfaces for face recognition in color. Finally in Section 5 we will make some concluding remarks.

2. Preliminaries

In this section we recall some basic information about quaternions and quaternionic matrices for completeness. A quaternion $q \in \mathbb{H}$ is expressed as

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$, and three imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

The quaternion skew-field $\mathbb{H}$ is an associative but non-commutative algebra of rank four over $\mathbb{R}$, endowed with an involutory antiautomorphism

$$q \to \overline{q} = a - bi - cj - dk.$$

Every non-zero quaternion is invertible, and the unique inverse is given by $1/q = \overline{q}/|q|^2$, where the quaternionic norm $|q|$ is defined by

$$|q|^2 = \overline{q}q = a^2 + b^2 + c^2 + d^2.$$

Two quaternions q and p belong to the same eigenclass when the following relation

$$q = s^{-1}ps, \quad s \in \mathbb{H}$$

holds. Quaternions belonging to the same eigenclass have the same real part and the same norm,

$$\mathrm{Re}(q) = \mathrm{Re}(s^{-1}ps) = \mathrm{Re}(p), \qquad |q| = |s^{-1}ps| = |p|,$$

and consequently, they have the same absolute value of the imaginary part.

Quaternionic right eigenvalue problem. The states of qQM will be described by vectors, $|\psi\rangle$, of a quaternionic Hilbert space, $\mathbb{H}^n$. To solve the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi = O_{\mathbb{H}}\Psi, \tag{2.1}$$

where $O_{\mathbb{H}}$ is a quaternionic linear operator, we need to study the quaternionic eigenequation

$$O_{\mathbb{H}}|\psi\rangle = |\psi\rangle q, \quad |\psi\rangle \in \mathbb{H}^n, q \in \mathbb{H}. \tag{2.2}$$

By adopting quaternionic scalar products in $\mathbb{H}^n$, we find states in one-to-one correspondence with unit rays of the form

$$|r\rangle = \{|\psi\rangle u\}$$

where $|\psi\rangle$ is a normalized vector and u is a quaternionic phase of unity magnitude. The state vector, $|\psi\rangle u$, corresponding to the same physical state $|\psi\rangle$, is an $O_{\mathbb{H}}$-eigenvector with eigenvalue $\bar{u}qu$

$$O_{\mathbb{H}}|\psi\rangle u = |\psi\rangle u(\bar{u}qu).$$

For real values of q, we find only one eigenvalue, otherwise quaternionic linear operators will be characterized by an infinite eigenvalue spectrum

$$\{q, \bar{u}_1 q u_1, \ldots, \bar{u}_l q u_l, \ldots\}$$

with u_l unitary quaternions. The related set of eigenvectors

$$\{|\psi\rangle, |\psi\rangle u_1, \ldots, |\psi\rangle u_l, \ldots\}$$

represents a ray. We can characterize the spectrum by choosing a representative ray

$$|\psi\rangle = |\psi\rangle u_\lambda$$

so that the corresponding eigenvalue $\lambda = \bar{u}_\lambda q u_\lambda$ is complex. For example, if $q = a + bi + cj + dk$ with $c^2 + d^2 \neq 0$ then we can choose $u_\lambda = x/|x|$ with $x = b + \sqrt{b^2 + c^2 + d^2} - dj + ck$ such that $\lambda = \bar{u}_\lambda q u_\lambda = a + \sqrt{b^2 + c^2 + d^2}i$. For this state the right eigenvalue equation (2.2) becomes

$$O_{\mathbb{H}}|\psi\rangle = |\psi\rangle \lambda \tag{2.3}$$

with $|\psi\rangle \in \mathbb{H}^n$ and $\lambda \in \mathbb{C}$.

Real counterpart method. The real counterpart method is to solve the n-dimensional quaternionic eigenvalue problem by solving an equivalent $4n$-dimensional real eigenvalue problem.

Let $\mathbb{1}_n$ denote the $n \times n$ unit matrix, and the superscripts ^{T} and * denote the transpose and the conjugate transpose, respectively. Define an unitary quaternion matrix

$$U = \frac{1}{2}\begin{bmatrix} \mathbb{1}_n & -j\mathbb{1}_n & -i\mathbb{1}_n & -k\mathbb{1}_n \\ \mathbb{1}_n & j\mathbb{1}_n & -i\mathbb{1}_n & k\mathbb{1}_n \\ \mathbb{1}_n & -j\mathbb{1}_n & i\mathbb{1}_n & k\mathbb{1}_n \\ \mathbb{1}_n & j\mathbb{1}_n & i\mathbb{1}_n & -k\mathbb{1}_n \end{bmatrix}. \tag{2.4}$$

For any quaternion matrix $Q = Q_0 + Q_1 i + Q_2 j + Q_3 k$, where $Q_0, Q_1, Q_2, Q_3 \in \mathbb{R}^{n\times n}$, its real counterpart can be defined as

$$\Upsilon_Q \equiv \begin{bmatrix} Q_0 & Q_2 & Q_1 & Q_3 \\ -Q_2 & Q_0 & Q_3 & -Q_1 \\ -Q_1 & -Q_3 & Q_0 & Q_2 \\ -Q_3 & Q_1 & -Q_2 & Q_0 \end{bmatrix}. \tag{2.5}$$

Indeed, if we denote $Q = A + Bj$, where $A = Q_0 + Q_1 i$, $B = Q_2 + Q_3 i \in \mathbb{C}^{n\times n}$, there is

$$\Upsilon_Q = U^* \begin{bmatrix} A + Bj & 0 & 0 & 0 \\ 0 & A - Bj & 0 & 0 \\ 0 & 0 & \bar{A} + \bar{B}j & 0 \\ 0 & 0 & 0 & \bar{A} - \bar{B}j \end{bmatrix} U. \tag{2.6}$$

From (2.6), the eigenvalue problem of Q is equivalent to the eigenvalue problem of Υ_Q.

Now we study the structural properties of Υ_Q. Define four unitary matrices

$$P_n = \begin{bmatrix} \mathbb{1}_n & 0 & 0 & 0 \\ 0 & -\mathbb{1}_n & 0 & 0 \\ 0 & 0 & \mathbb{1}_n & 0 \\ 0 & 0 & 0 & -\mathbb{1}_n \end{bmatrix}, \qquad J_n = \begin{bmatrix} 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & 0 & 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & \mathbb{1}_n & 0 & 0 \end{bmatrix}, \tag{2.7}$$

$$R_n = \begin{bmatrix} 0 & -\mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_n \\ 0 & 0 & -\mathbb{1}_n & 0 \end{bmatrix}, \qquad S_n = \begin{bmatrix} 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

A real matrix $M \in \mathbb{R}^{4n\times 4n}$ is called *JRS-symmetric* if $J_n M J_n^T = M$, $R_n M R_n^T = M$ and $S_n M S_n^T = M$. A matrix $O \in \mathbb{R}^{4n\times 4n}$ is called *JRS-symplectic* if $OJ_nO^T = J_n$, $OR_nO^T = R_n$ and $OS_nO^T = S_n$. A matrix $W \in \mathbb{R}^{4n\times 4n}$ is called *orthogonal JRS-symplectic* if it is orthogonal and *JRS*-symplectic. One can see that an orthogonal *JRS*-symplectic matrix must be orthogonal symplectic, but the converse is not always true. By simple computations, we can obtain the following properties of the real counterparts of quaternion matrices, some of them can be found in [33,39].

Lemma 2.1. *Let Q and H be two $n \times n$ quaternion matrices, $\alpha \in \mathbb{R}$. Then we have the following.*

(1) $\Upsilon_{Q+H} = \Upsilon_Q + \Upsilon_H$. $\Upsilon_{\alpha Q} = \alpha \Upsilon_Q$. $\Upsilon_{QH} = \Upsilon_Q \Upsilon_H$.
(2) *Υ_Q is JRS-symmetric.*
(3) *Q is unitary if and only if Υ_Q is orthogonal.*
(4) *If Υ_Q is orthogonal then it must be orthogonal JRS-symplectic.*

Now we recall two results from Proposition 4.1 and Theorem 5.1 in [33].

Theorem 2.2. *Let $Q \in \mathbb{H}^{n\times n}$. Then the real eigenvalues of Υ_Q appear in fours; the purely imaginary eigenvalues of Υ_Q appear in pairs and in conjugate pairs.*

Theorem 2.3. *Let $Q \in \mathbb{H}^{n\times n}$. Then Q is a diagonalizable quaternion matrix if and only if its real counterpart Υ_Q is a diagonalizable real matrix.*

The complex eigenvalue spectrum $\{\lambda_1, \ldots, \lambda_n\}$ of $Q \in \mathbb{H}^{n\times n}$ can be obtained from the $4n$ dimensional eigenvalue spectrum of $\Upsilon_Q \in \mathbb{R}^{4n\times 4n}$,

$$\{\lambda_1, \overline{\lambda}_1, \lambda_1, \overline{\lambda}_1, \ldots, \lambda_n, \overline{\lambda}_n, \lambda_n, \overline{\lambda}_n\}.$$

Corresponding to two different complex eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2 \neq \overline{\lambda}_1$, two quaternionic eigenvectors of Q are linearly independent.

From above observations we see that, if we want to compute the eigenvalues of one $n \times n$ quaternionic matrix Q, then we can deal with the eigenvalue problem of the $4n \times 4n$ real counterpart matrix Υ_Q. In general, that will cost more computation workload, computational time and storage spaces. We find that all these costs will be greatly reduced if the structures of Υ_Q are fully considered.

3. Structure-preserving algorithm

In this section, we propose a structure-preserving algorithm for solving the eigenvalue problem of the Hermitian quaternionic matrix.

By Lemma 2.1, the real counterpart of an $n \times n$ Hermitian quaternionic matrix has double structures: symmetry and *JRS*-symmetry, and depends on $2n^2 - n$ parameters compared to $16n^2$ parameters of a general $4n \times 4n$ real matrix. Therefore, a structure-preserving algorithm can be expected to be much more efficient than a general-purpose method.

An ideal method tailored to the matrix structure would be

- strongly backward stable, i.e., the computed solution is the exact solution corresponding to a nearby matrix with the same structure;
- reliable, i.e., capable to solve all eigenvalue problems in the considered matrix class;
- requiring $O(n^3)$ floating point operations (flops), preferably much less than a competitive general-purpose method.

In the following subsections we will propose a new structure-preserving algorithm for solving right eigenvalue problems of the Hermitian quaternionic matrix.

We refer the reader to [40–46] for more information on structure-preserving methods for other problems.

3.1. Orthogonal JRS-symplectic matrices

Recalling the definition of orthogonal *JRS*-symplectic matrix in Section 2, every orthogonal *JRS*-symplectic matrix W has the block structure

$$W = \begin{bmatrix} W_0 & W_2 & W_1 & W_3 \\ -W_2 & W_0 & W_3 & -W_1 \\ -W_1 & -W_3 & W_0 & W_2 \\ -W_3 & W_1 & -W_2 & W_0 \end{bmatrix}, \quad W_1, W_2, W_2, W_3 \in \mathbb{R}^{n\times n}. \tag{3.1}$$

Two types of elementary orthogonal matrices have this form. One is a $4n \times 4n$ generalized symplectic Givens rotation defined as

$$G_l = \begin{bmatrix}
\mathbb{1}_{l-1}, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & \cos\alpha_0, & 0, & 0, & \cos\alpha_2, & 0, & 0, & \cos\alpha_1, & 0, & 0, & \cos\alpha_3, & 0 \\
0, & 0, & \mathbb{1}_{n-l}, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 0, & \mathbb{1}_{l-1}, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & -\cos\alpha_2, & 0, & 0, & \cos\alpha_0, & 0, & 0, & \cos\alpha_3, & 0, & 0, & -\cos\alpha_1, & 0 \\
0, & 0, & 0, & 0, & 0, & \mathbb{1}_{n-l}, & 0, & 0, & 0, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & \mathbb{1}_{l-1}, & 0, & 0, & 0, & 0, & 0 \\
0, & -\cos\alpha_1, & 0, & 0, & -\cos\alpha_3, & 0, & 0, & \cos\alpha_0, & 0, & 0, & \cos\alpha_2, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \mathbb{1}_{n-l}, & 0, & 0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \mathbb{1}_{l-1}, & 0, & 0 \\
0, & -\cos\alpha_3, & 0, & 0, & \cos\alpha_1, & 0, & 0, & -\cos\alpha_2, & 0, & 0, & \cos\alpha_0, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \mathbb{1}_{n-l}
\end{bmatrix} \tag{3.2}$$

where $1 \le l \le n, \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in [-\pi/2, \pi/2)$ and $\cos^2\alpha_0 + \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1$. Notice that if $\alpha_2 \equiv 0$ and $\alpha_3 \equiv 0$ then (3.2) is a $4n \times 4n$ symplectic Givens rotation $J_s(i, \alpha)$ defined by Eq. (37) in [40]. Another one is the direct sum of four identical $n \times n$ Householder matrices

$$H_l = \begin{bmatrix} \mathbb{1}_n - \beta vv^T & 0 & 0 & 0 \\ 0 & \mathbb{1}_n - \beta vv^T & 0 & 0 \\ 0 & 0 & \mathbb{1}_n - \beta vv^T & 0 \\ 0 & 0 & 0 & \mathbb{1}_n - \beta vv^T \end{bmatrix},$$

where v is a vector of length n with its first $l-1$ elements equal to zero and β a scalar satisfying $\beta(\beta v^T v - 2) = 0$.

3.2. Structure-preserving tridiagonalization

Suppose that W is a $4n \times 4n$ orthogonal JRS-symplectic matrix, if Ω is JRS-symmetric then

$$J_n(W^T\Omega W)J_n^T = (J_nW^T)\Omega(J_nW^T)^T = W^T\Omega W,$$
$$R_n(W^T\Omega W)R_n^T = (R_nW^T)\Omega(R_nW^T)^T = W^T\Omega W,$$
$$S_n(W^T\Omega W)S_n^T = (S_nW^T)\Omega(S_nW^T)^T = W^T\Omega W,$$

i.e., $W^T\Omega W$ is still JRS-symmetric. Therefore, orthogonal JRS-symplectic equivalence transformations preserve JRS-symmetric structures.

Recall that if $Q = Q_0 + Q_1i + Q_2j + Q_3k$ is Hermitian then Q_0 is symmetric and Q_1, Q_2, Q_3 are skew-symmetric, and consequently Υ_Q defined by (2.5) is symmetric. In this case Υ_Q has double structures: symmetry and JRS-symmetry. Now we deduce the standard form of the real counterpart of a Hermitian quaternion matrix under the orthogonal JRS-symplectic transformations.

Theorem 3.1. *Suppose that $Q \in \mathbb{H}^{n\times n}$ is Hermitian and Υ_Q is the real counterpart of Q. Then there exists an orthogonal JRS-symplectic matrix $W \in \mathbb{R}^{4n\times 4n}$ such that*

$$W\Upsilon_Q W^T = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix}, \tag{3.3}$$

where $D \in \mathbb{R}^{n\times n}$ is a symmetric tridiagonal matrix.

Proof. Since Q is Hermitian, Υ_Q has the form in (2.5) with $Q_0^T = Q_0, Q_1^T = -Q_1, Q_2^T = -Q_2, Q_3^T = -Q_3$. Now we prove the assertion by induction on the order n of Q. For $n = 1$, it is clear that the theorem is true. Suppose that for the case $1 \le n < m$, there exists an orthogonal JRS-symplectic matrix $\widetilde{W} \in \mathbb{R}^{4n\times 4n}$ such that

$$\widetilde{W}\Upsilon_Q\widetilde{W}^T = \begin{bmatrix} D_0 & 0 & 0 & 0 \\ 0 & D_0 & 0 & 0 \\ 0 & 0 & D_0 & 0 \\ 0 & 0 & 0 & D_0 \end{bmatrix} \in \mathbb{R}^{4n\times 4n},$$

where $D_0 \in \mathbb{R}^{n\times n}$ is a symmetric tridiagonal matrix. For $n = m$, denote

$$Q_0 = \begin{bmatrix} \omega_{11}^{(0)} & \omega_{12}^{(0)} & \omega_{13}^{(0)} & \Omega_{14}^{(0)} \\ \omega_{12}^{(0)} & \omega_{22}^{(0)} & \omega_{23}^{(0)} & \Omega_{24}^{(0)} \\ \omega_{13}^{(0)} & \omega_{23}^{(0)} & \omega_{33}^{(0)} & \Omega_{34}^{(0)} \\ (\Omega_{14}^{(0)})^T & (\Omega_{24}^{(0)})^T & (\Omega_{34}^{(0)})^T & \Omega_{44}^{(0)} \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & \omega_{12}^{(1)} & \omega_{13}^{(1)} & \Omega_{14}^{(1)} \\ -\omega_{12}^{(1)} & 0 & \omega_{23}^{(1)} & \Omega_{24}^{(1)} \\ -\omega_{13}^{(1)} & -\omega_{23}^{(1)} & 0 & \Omega_{34}^{(1)} \\ -(\Omega_{14}^{(1)})^T & -(\Omega_{24}^{(1)})^T & -(\Omega_{34}^{(1)})^T & \Omega_{44}^{(1)} \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 0 & \omega_{12}^{(2)} & \omega_{13}^{(2)} & \Omega_{14}^{(2)} \\ -\omega_{12}^{(2)} & 0 & \omega_{23}^{(2)} & \Omega_{24}^{(2)} \\ -\omega_{13}^{(2)} & -\omega_{23}^{(2)} & 0 & \Omega_{34}^{(2)} \\ -(\Omega_{14}^{(2)})^T & -(\Omega_{24}^{(2)})^T & -(\Omega_{34}^{(2)})^T & \Omega_{44}^{(2)} \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & \omega_{12}^{(3)} & \omega_{13}^{(3)} & \Omega_{14}^{(3)} \\ -\omega_{12}^{(3)} & 0 & \omega_{23}^{(3)} & \Omega_{24}^{(3)} \\ -\omega_{13}^{(3)} & -\omega_{23}^{(3)} & 0 & \Omega_{34}^{(3)} \\ -(\Omega_{14}^{(3)})^T & -(\Omega_{24}^{(3)})^T & -(\Omega_{34}^{(3)})^T & \Omega_{44}^{(3)} \end{bmatrix},$$

in which $\omega_{st}^{(r)} \in \mathbb{R}$ for $r = 0, 1, 2, 3, s, t = 1, 2, 3, s \le t, \Omega_{s4}^{(r)} \in \mathbb{R}^{1\times(m-3)}$ for $s = 1, 2, 3, \Omega_{44}^{(0)}$ is an $(m-3)\times(m-3)$ symmetric matrix, and $\Omega_{44}^{(r)}$ for $r = 1, 2, 3$ are $(m-3)\times(m-3)$ skew-symmetric matrices.

Denote $\gamma_{12} = \sqrt{(\omega_{12}^{(0)})^2 + (\omega_{12}^{(2)})^2 + (\omega_{12}^{(1)})^2 + (\omega_{12}^{(3)})^2}$, and we take a $4m \times 4m$ generalized symplectic Givens rotation G_2 defined as in (3.2) with $\cos\alpha_s = \omega_{12}^{(s)}/\gamma_{12}$ ($s = 0, 1, 2, 3$), such that

$$\widetilde{\Upsilon}_Q \equiv G_2 \Upsilon_Q G_2^T = \begin{bmatrix} \widetilde{Q}_0 & \widetilde{Q}_2 & \widetilde{Q}_1 & \widetilde{Q}_3 \\ -\widetilde{Q}_2 & \widetilde{Q}_0 & \widetilde{Q}_3 & -\widetilde{Q}_1 \\ -\widetilde{Q}_1 & -\widetilde{Q}_3 & \widetilde{Q}_0 & \widetilde{Q}_2 \\ -\widetilde{Q}_3 & \widetilde{Q}_1 & -\widetilde{Q}_2 & \widetilde{Q}_0 \end{bmatrix} \tag{3.4}$$

with

$$\widetilde{Q}_0 = \begin{bmatrix} \omega_{11}^{(0)} & \gamma_{12} & \omega_{13}^{(0)} & \Omega_{14}^{(0)} \\ \gamma_{12} & \widetilde{\omega}_{22}^{(0)} & \widetilde{\omega}_{23}^{(0)} & \widetilde{\Omega}_{24}^{(0)} \\ \omega_{13}^{(0)} & \widetilde{\omega}_{23}^{(0)} & \omega_{33}^{(0)} & \Omega_{34}^{(0)} \\ (\Omega_{14}^{(0)})^T & (\widetilde{\Omega}_{24}^{(0)})^T & (\Omega_{34}^{(0)})^T & \Omega_{44}^{(0)} \end{bmatrix}, \quad \widetilde{Q}_1 = \begin{bmatrix} 0 & 0 & \omega_{13}^{(1)} & \Omega_{14}^{(1)} \\ 0 & 0 & \widetilde{\omega}_{23}^{(1)} & \widetilde{\Omega}_{24}^{(1)} \\ -\omega_{13}^{(1)} & -\widetilde{\omega}_{23}^{(1)} & 0 & \Omega_{34}^{(1)} \\ -(\Omega_{14}^{(1)})^T & -(\widetilde{\Omega}_{24}^{(1)})^T & -(\Omega_{34}^{(1)})^T & \Omega_{44}^{(1)} \end{bmatrix},$$

$$\widetilde{Q}_2 = \begin{bmatrix} 0 & 0 & \omega_{13}^{(2)} & \Omega_{14}^{(2)} \\ 0 & 0 & \widetilde{\omega}_{23}^{(2)} & \widetilde{\Omega}_{24}^{(2)} \\ -\omega_{13}^{(2)} & -\widetilde{\omega}_{23}^{(2)} & 0 & \Omega_{34}^{(2)} \\ -(\Omega_{14}^{(2)})^T & -(\widetilde{\Omega}_{24}^{(2)})^T & -(\Omega_{34}^{(2)})^T & \Omega_{44}^{(2)} \end{bmatrix}, \quad \widetilde{Q}_3 = \begin{bmatrix} 0 & 0 & \omega_{13}^{(3)} & \Omega_{14}^{(3)} \\ 0 & 0 & \widetilde{\omega}_{23}^{(3)} & \widetilde{\Omega}_{24}^{(3)} \\ -\omega_{13}^{(3)} & -\widetilde{\omega}_{23}^{(3)} & 0 & \Omega_{34}^{(3)} \\ -(\Omega_{14}^{(3)})^T & -(\widetilde{\Omega}_{24}^{(3)})^T & -(\Omega_{34}^{(3)})^T & \Omega_{44}^{(3)} \end{bmatrix}.$$

In a similar manner we can find generalized symplectic Givens rotations $G_3, G_4, \ldots, G_m$ such that

$$\widehat{\Upsilon}_Q \equiv G_m \cdots G_3 (G_2 \Upsilon_Q G_2^T)\,(G_m \cdots G_3)^T = \begin{bmatrix} \widehat{Q}_0 & \widehat{Q}_2 & \widehat{Q}_1 & \widehat{Q}_3 \\ -\widehat{Q}_2 & \widehat{Q}_0 & \widehat{Q}_3 & -\widehat{Q}_1 \\ -\widehat{Q}_1 & -\widehat{Q}_3 & \widehat{Q}_0 & \widehat{Q}_2 \\ -\widehat{Q}_3 & \widehat{Q}_1 & -\widehat{Q}_2 & \widehat{Q}_0 \end{bmatrix}$$

with

$$\widehat{Q}_0 = \begin{bmatrix} \omega_{11}^{(0)} & \gamma_{12} & \gamma_{13} & \Gamma_{14} \\ \gamma_{12} & \widehat{\omega}_{22}^{(0)} & \widehat{\omega}_{23}^{(0)} & \widehat{\Omega}_{24}^{(0)} \\ \gamma_{13} & \widehat{\omega}_{23}^{(0)} & \widehat{\omega}_{33}^{(0)} & \widehat{\Omega}_{34}^{(0)} \\ (\Gamma_{14})^T & (\widehat{\Omega}_{24}^{(0)})^T & (\widehat{\Omega}_{34}^{(0)})^T & \widehat{\Omega}_{44}^{(0)} \end{bmatrix}, \quad \widehat{Q}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\omega}_{23}^{(1)} & \widehat{\Omega}_{24}^{(1)} \\ 0 & -\widehat{\omega}_{23}^{(1)} & 0 & \widehat{\Omega}_{34}^{(1)} \\ 0 & -(\widehat{\Omega}_{24}^{(1)})^T & -(\widehat{\Omega}_{34}^{(1)})^T & \widehat{\Omega}_{44}^{(1)} \end{bmatrix},$$

$$\widehat{Q}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\omega}_{23}^{(2)} & \widehat{\Omega}_{24}^{(2)} \\ 0 & -\widehat{\omega}_{23}^{(2)} & 0 & \widehat{\Omega}_{34}^{(2)} \\ 0 & -(\widehat{\Omega}_{24}^{(2)})^T & -(\widehat{\Omega}_{34}^{(2)})^T & \widehat{\Omega}_{44}^{(2)} \end{bmatrix}, \quad \widehat{Q}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\omega}_{23}^{(3)} & \widehat{\Omega}_{24}^{(3)} \\ 0 & -\widehat{\omega}_{23}^{(3)} & 0 & \widehat{\Omega}_{34}^{(3)} \\ 0 & -(\widehat{\Omega}_{24}^{(3)})^T & -(\widehat{\Omega}_{34}^{(3)})^T & \widehat{\Omega}_{44}^{(3)} \end{bmatrix},$$

where $\Gamma_{14} = [\gamma_{14}, \ldots, \gamma_{1m}]$, $\gamma_{1s} = \sqrt{(\omega_{1s}^{(0)})^2 + (\omega_{1s}^{(2)})^2 + (\omega_{12}^{(1)})^2 + (\omega_{1s}^{(3)})^2}$ ($s = 3, \ldots, m$).

It is obvious that there exists a Householder matrix $H_2 \in \mathbb{R}^{m \times m}$ such that

$$\overset{\approx}{\Upsilon}_Q \equiv \begin{bmatrix} H_2 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & H_2 \end{bmatrix} \widehat{\Upsilon}_Q \begin{bmatrix} H_2 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & H_2 \end{bmatrix}^T = \begin{bmatrix} \overset{\approx}{Q}_0 & \overset{\approx}{Q}_2 & \overset{\approx}{Q}_1 & \overset{\approx}{Q}_3 \\ -\overset{\approx}{Q}_2 & \overset{\approx}{Q}_0 & \overset{\approx}{Q}_3 & -\overset{\approx}{Q}_1 \\ -\overset{\approx}{Q}_1 & -\overset{\approx}{Q}_3 & \overset{\approx}{Q}_0 & \overset{\approx}{Q}_2 \\ -\overset{\approx}{Q}_3 & \overset{\approx}{Q}_1 & -\overset{\approx}{Q}_2 & \overset{\approx}{Q}_0 \end{bmatrix}$$

with

$$\overset{\approx}{Q}_0 = \begin{bmatrix} \omega_{11}^{(0)} & \widetilde{\gamma}_{12} & 0 & 0 \\ \widetilde{\gamma}_{12} & \overset{\approx}{\omega}_{22}^{(0)} & \overset{\approx}{\omega}_{23}^{(0)} & \overset{\approx}{\Omega}_{24}^{(0)} \\ 0 & \overset{\approx}{\omega}_{23}^{(0)} & \overset{\approx}{\omega}_{33}^{(0)} & \overset{\approx}{\Omega}_{34}^{(0)} \\ 0 & (\overset{\approx}{\Omega}_{24}^{(0)})^T & (\overset{\approx}{\Omega}_{34}^{(0)})^T & \overset{\approx}{\Omega}_{44}^{(0)} \end{bmatrix}, \quad \overset{\approx}{Q}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \overset{\approx}{\omega}_{23}^{(1)} & \overset{\approx}{\Omega}_{24}^{(1)} \\ 0 & -\overset{\approx}{\omega}_{23}^{(1)} & 0 & \overset{\approx}{\Omega}_{34}^{(1)} \\ 0 & -(\overset{\approx}{\Omega}_{24}^{(1)})^T & -(\overset{\approx}{\Omega}_{34}^{(1)})^T & \overset{\approx}{\Omega}_{44}^{(1)} \end{bmatrix},$$

$$\overset{\approx}{\widehat{Q}}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \overset{\approx}{\widehat{\omega}}{}^{(2)}_{23} & \overset{\approx}{\widehat{\Omega}}{}^{(2)}_{24} \\ 0 & -\overset{\approx}{\widehat{\omega}}{}^{(2)}_{23} & 0 & \overset{\approx}{\widehat{\Omega}}{}^{(2)}_{34} \\ 0 & -(\overset{\approx}{\widehat{\Omega}}{}^{(2)}_{24})^T & -(\overset{\approx}{\widehat{\Omega}}{}^{(2)}_{34})^T & \widehat{\Omega}^{(2)}_{44} \end{bmatrix}, \qquad \overset{\approx}{\widehat{Q}}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \overset{\approx}{\widehat{\omega}}{}^{(3)}_{23} & \overset{\approx}{\widehat{\Omega}}{}^{(3)}_{24} \\ 0 & -\overset{\approx}{\widehat{\omega}}{}^{(3)}_{23} & 0 & \overset{\approx}{\widehat{\Omega}}{}^{(3)}_{34} \\ 0 & -(\overset{\approx}{\widehat{\Omega}}{}^{(3)}_{24})^T & -(\overset{\approx}{\widehat{\Omega}}{}^{(3)}_{34})^T & \overset{\approx}{\widehat{\Omega}}{}^{(3)}_{44} \end{bmatrix},$$

in which $\widetilde{\gamma}_{12} = \sqrt{\gamma_{12}^2 + \cdots + \gamma_{1m}^2}$.

Let Ω denote the submatrix of $\overset{\approx}{\widehat{\Upsilon}}_Q$ by deleting the $1, m+1, 2m+1, 3m+1$ rows and columns. Then Ω is a $4(m-1) \times 4(m-1)$ symmetric and JRS-symmetric matrix as defined in (2.5), and it must be a real counterpart of an $(m-1) \times (m-1)$ quaternion matrix. By the introduction assumption, there exists a $4(m-1) \times 4(m-1)$ orthogonal JRS-symplectic matrix

$$\widetilde{W} = \begin{bmatrix} \widetilde{W}_0 & \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_3 \\ -\widetilde{W}_2 & \widetilde{W}_0 & \widetilde{W}_3 & -\widetilde{W}_1 \\ -\widetilde{W}_1 & -\widetilde{W}_3 & \widetilde{W}_0 & \widetilde{W}_2 \\ -\widetilde{W}_3 & \widetilde{W}_1 & -\widetilde{W}_2 & \widetilde{W}_0 \end{bmatrix}, \qquad \widetilde{W}_s \in \mathbb{R}^{(m-1)\times(m-1)} \quad (s = 0, 1, 2, 3)$$

such that

$$\widetilde{W} \Omega \widetilde{W}^T = \begin{bmatrix} D_0 & 0 & 0 & 0 \\ 0 & D_0 & 0 & 0 \\ 0 & 0 & D_0 & 0 \\ 0 & 0 & 0 & D_0 \end{bmatrix} \in \mathbb{R}^{4(m-1)\times 4(m-1)}, \tag{3.5}$$

in which $D_0 \in \mathbb{R}^{(m-1)\times(m-1)}$ is a symmetric tridiagonal matrix. Define

$$W = \mathcal{F} \begin{bmatrix} H_2 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & H_2 \end{bmatrix} G_m \cdots G_2 \in \mathbb{R}^{4m\times 4m}, \tag{3.6}$$

where $\mathcal{F} \in \mathbb{R}^{4m\times 4m}$ is an orthogonal JRS-symplectic matrix dependent on $\widetilde{W}$,

$$\mathcal{F} \equiv \begin{bmatrix} \mathcal{F}_0 & \mathcal{F}_2 & \mathcal{F}_1 & \mathcal{F}_3 \\ -\mathcal{F}_2 & \mathcal{F}_0 & \mathcal{F}_3 & -\mathcal{F}_1 \\ -\mathcal{F}_1 & -\mathcal{F}_3 & \mathcal{F}_0 & \mathcal{F}_2 \\ -\mathcal{F}_3 & \mathcal{F}_1 & -\mathcal{F}_2 & \mathcal{F}_0 \end{bmatrix}, \qquad \mathcal{F}_s = \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{W}_s \end{bmatrix} \in \mathbb{R}^{k\times k} \quad (s = 0, 1, 2, 3).$$

Then for $n = m$, we have

$$W \Upsilon_Q W^T = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix}$$

in which $D \in \mathbb{R}^{m\times m}$ is a symmetric tridiagonal matrix. Therefore for $n = m$, the assertion of the theorem also holds, and we have completed the proof of the theorem. $\square$

3.3. Right eigenvalue problem

From Theorem 3.1, for a Hermitian quaternionic matrix Q, the eigen-information about Υ_Q can be obtained by computing the eigen-information of a symmetric tridiagonal matrix D. By applying Lemma 2.1, (3.1) and (3.3),

$$\Upsilon_D = \Upsilon_V \Upsilon_Q \Upsilon_{V^*} = \Upsilon_{VQV^*},$$

where $V = W_0 + W_1 i + W_2 j + W_3 k$ is a unitary matrix in $\mathbb{H}^{n\times n}$. Then we have

$$VQV^* = D. \tag{3.7}$$

We call (3.7) the real tridiagonalization of the Hermitian quaternion matrix Q. We formulate this result in the following theorem.

Theorem 3.2. *Let $Q \in \mathbb{H}^{n\times n}$ be Hermitian. Then there exists an unitary quaternion matrix $V \in \mathbb{H}^{n\times n}$ such that $VQV^* = D$ where D is a real symmetric tridiagonal matrix. Furthermore,*

(1) if (λ, x) is an eigenpair of D then λ is a right eigenvalue of Q and V^*x is the corresponding eigenvector;

(2) if D has a diagonalization $D = X\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)X^T$, where $X \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\lambda_s \in \mathbb{R}(s = 1, 2, \ldots, n)$, then Q has a diagonalization

$$Q = Z\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)Z^*, \quad Z = V^*X.$$

Note that the diagonalization of a Hermitian quaternion matrix has been studied in the literature (see [33,35], etc.), here in Theorems 3.1–3.2, we emphasize that the structures are preserved at each step of the diagonalization process, and the eigenvectors can be computed directly.

3.4. Structure-preserving algorithm

In this subsection we propose a structure-preserving method for solving the right eigenvalue problem of a Hermitian quaternion matrix $Q = Q_0+Q_1i+Q_2j+Q_3k$, $Q_s \in \mathbb{R}^{n\times n}$, $s = 0, 1, 2, 3$. In many applications, the dimension of the Hermitian quaternion matrix Q is very large. If we directly apply standard algorithms for solving real symmetrical eigenvalue problems to the real counterpart Υ_Q, of which the dimension is four times large as the dimension of Q, then the algorithm would be inefficient. So we develop a structure-preserving algorithm by applying orthogonal *JRS*-symplectic matrices, therefore the algorithm is numerically stable, efficient, and the real counterpart needs not to be generated.

We first discuss two nice properties of a *JRS*-symplectic Givens matrix G_l defined by (3.2) applied to similarity transforms of the real counterpart Υ_Q of a Hermitian quaternion matrix Q.

Property 1. *For given $g = g_0 + g_1i + g_2j + g_3k \in \mathbb{H}$ for $g_0, g_1, g_2, g_3 \in \mathbb{R}$, if $g \neq 0$, then we can take G_1, the condensed form of JRS-symplectic Givens matrix G_l,*

$$G_1 = \begin{bmatrix} \cos\alpha_0 & \cos\alpha_2 & \cos\alpha_1 & \cos\alpha_3 \\ -\cos\alpha_2 & \cos\alpha_0 & \cos\alpha_3 & -\cos\alpha_1 \\ -\cos\alpha_1 & -\cos\alpha_3 & \cos\alpha_0 & \cos\alpha_2 \\ -\cos\alpha_3 & \cos\alpha_1 & -\cos\alpha_2 & \cos\alpha_0 \end{bmatrix} = \begin{bmatrix} g_0 & g_2 & g_1 & g_3 \\ -g_2 & g_0 & g_3 & -g_1 \\ -g_1 & -g_3 & g_0 & g_2 \\ -g_3 & g_1 & -g_2 & g_0 \end{bmatrix} \Big/ |g| = \Upsilon_g/|g|,$$

then

$$G_1^T\Upsilon_g = \Upsilon_g G_1^T = |g|\mathbb{I}_4.$$

Property 2. *Because Q is Hermitian, the diagonal elements of Q are all real numbers. Therefore, for any JRS-symplectic Givens matrix G_l, with the condensed form G_1 of G_l, we always have*

$$G_1\Upsilon_{Q(l,l)}G_1^T = G_1Q_0(l,l)\mathbb{I}_4G_1^T = Q_0(l,l)\mathbb{I}_4 = \Upsilon_{Q(l,l)}.$$

With the above observations, we now propose the following algorithms.

Algorithm 3.3. A method for generating a 4×4 generalized symplectic Givens rotation.

Function: $G_1 = JRSGivens(g_0, g_1, g_2, g_3)$

If $g_1 = g_2 = g_3 = 0$,

$$G_1 = \mathbb{I}_4;$$

else

$$G_1 = \begin{bmatrix} g_0 & g_2 & g_1 & g_3 \\ -g_2 & g_0 & g_3 & -g_1 \\ -g_1 & -g_3 & g_0 & g_2 \\ -g_3 & g_1 & -g_2 & g_0 \end{bmatrix} \Big/ \sqrt{g_0^2 + g_1^2 + g_2^2 + g_3^2}. \tag{3.8}$$

This algorithm costs 24 flops, including 1 square root operation. Notice that the transformation G_1 acts as a four-dimensional Givens rotation [47,48]. We refer the reader to [45,46] for a backward stable implementation of (3.8) and more Givens-like actions.

Now we propose an algorithm for the tridiagonalization of the real counterpart of a Hermitian quaternionic matrix.

Algorithm 3.4. For given Hermitian quaternionic matrix $Q = Q_0+Q_1i+Q_2j+Q_3k \in \mathbb{H}^{n\times n}$, where $Q_s \in \mathbb{R}^{n\times n}$, $s = 0, 1, 2, 3$. This algorithm presents an orthogonal *JRS*-symplectic matrix $W \in \mathbb{R}^{4n\times 4n}$ and a symmetric tridiagonal matrix $D \in \mathbb{R}^{n\times n}$ satisfying (3.3).

Function: $[W, D] = Trihermq\,(Q_0, Q_1, Q_2, Q_3)$.

Set $W = \mathbb{I}_{4n}$

1. for $r = 1 : n - 1$
2. for $t = r + 1 : n$

3. Generate a generalized symplectic Givens rotation for the (t, r)-th elements of Q_s by

$$G_1 = JRSGivens(Q_0(t, r), Q_1(t, r), Q_2(t, r), Q_3(t, r)).$$

4. Denote $a_s = Q_s([r : t-1, t+1 : n], t)$. Then

$$Z = [a_0\ a_2\ a_1\ a_3]G_1.$$

5. Update the t-th rows of Q_s by

$$\begin{aligned} &Q_0([r : t-1, t+1 : n], t) = Z(:, 1),\\ &Q_2([r : t-1, t+1 : n], t) = Z(:, 2),\\ &Q_1([r : t-1, t+1 : n], t) = Z(: 3),\\ &Q_3([r : t-1, t+1 : n], t) = Z(:, 4). \end{aligned}$$

6. Update the t-th columns of Q_s by

$$\begin{aligned} &Q_0(t, [r : t-1, t+1 : n]) = Q_0([r : t-1, t+1 : n], t)^T,\\ &Q_s(t, [r : t-1, t+1 : n]) = -Q_s([r : t-1, t+1 : n], t)^T, \quad s = 1, 2, 3. \end{aligned}$$

7. Generate the orthogonal *JRS*-symplectic matrix $W \in \mathbb{R}^{4n\times 4n}$ by

$$W(:, [t, n+t, 2n+t, 3n+t]) = W(:, [t, n+t, 2n+t, 3n+t])G_1.$$

8. end
9. Generate a Householder matrix $H = \text{house}\,(Q_0(r+1 : n, r))$, then for $s = 0, 1, 2, 3$ do

$$\begin{aligned} &Q_s(r+1 : n, :) = HQ_s(r+1 : n, :),\\ &Q_s(:, r+1 : n) = Q_s(:, r+1 : n)H. \end{aligned}$$

10. Update the orthogonal *JRS*-symplectic matrix W,

$$\begin{aligned} &W(:, [r+1 : n, n+r+1 : 2n, 2n+r+1 : 3n, 3n+r+1 : 4n])\\ &\quad = W(:, [r+1 : n, n+r+1 : 2n, 2n+r+1 : 3n, 3n+r+1 : 4n])\,\text{diag}\,(H, H, H, H). \end{aligned}$$

11. end.

Algorithm 3.4 takes about $12n^3$ flops for the tridiagonalization for the real counterpart $\Upsilon_Q \in \mathbb{R}^{4n\times 4n}$ of an $n \times n$ Hermitian quaternionic matrix Q. There needs to be extra $50n^3/3$ flops for computing the product of all orthogonal *JRS*-symplectic matrices. Recall that the Householder tridiagonalization of $4n \times 4n$ symmetric matrix Υ_Q needs about $2(4n)^3/3 = 128n^3/3$ flops, and there needs to be extra $128n^3/3$ flops for computing the product of all Householder matrices. What is more important is that Algorithm 3.4 is strongly backward stable, since in every step the structures of Υ_Q are preserved.

By Theorem 3.2, we now present a method for the right eigenvalue problem of a Hermitian quaternion matrix based on the structure-preserving tridiagonalization of its real counterpart.

Algorithm 3.5. A structure-preserving method for the computation of the right eigenvalues and corresponding eigenvectors of a Hermitian quaternion matrix $Q = Q_0 + Q_1 i + Q_2 j + Q_3 k \in \mathbb{H}^{n\times n}$, where $Q_s \in \mathbb{R}^{n\times n}$, $s = 0, 1, 2, 3$.

Function: $[V, \Lambda] = eighermq(Q_0, Q_1, Q_2, Q_3)$

1. Compute the tridiagonalization of the real counterpart of Q by Algorithm 3.4.

$$[W, D] = Trihermq(Q_0, Q_1, Q_2, Q_3).$$

2. Compute the eigenvalues and corresponding eigenvectors of D by the MATLAB order *eig*.

$$[V_D, \Lambda_D] = eig(D).$$

3. Recalling (3.7) and Theorem 3.2, let

$$\begin{aligned} &W_0 = W(1 : n, 1 : n), \qquad W_1 = W(1 : n, n+1 : 2n),\\ &W_2 = W(1 : n, 2n+1 : 3n), \qquad W_3 = W(1 : n, 3n+1 : 4n), \end{aligned}$$

then

$$V = V_D^T(W_0 + W_1 i + W_2 j + W_3 k), \quad \Lambda = \Lambda_D.$$

In Algorithm 3.5 V_D and $W_s, s = 0, 1, 2, 3$, are $n\times n$ real matrices, V and Λ can be computed by and only by real operations. So in many cases the calculation may cost less CPU time by Algorithm 3.5 than by the order *eig* of the famous Quaternion toolbox for MATLAB, developed by Sangwine and Le Bihan [49].

4. Numerical experiment

In this section we present three numerical examples. In the first one, we apply Algorithm 3.5 to solve the right eigenvalue problem of a Hermitian quaternionic linear operator from quaternionic quantum mechanics, and compare our algorithm with the famous Quaternion toolbox for Matlab [49]. In the second one we deal with the tridiagonalization of a random Hermitian quaternionic matrix. In the third one we apply our algorithms to compute eigenfaces for face recognition in color. All these experiments are performed on an AMD Athlon(tm) 64×2 Dual Core 2.11 GHz/1.00 GB computer using MATLAB 7.10, and the tolerance is taken to be $tol = 10^{-15}$.

Example 4.1. In a quaternionic Hilbert space $\mathbb{H}^n$, an $n \times n$ Hermitian quaternionic linear operator $O_{\mathbb{H}}$ as in (2.2) is defined by

$$O_{\mathbb{H}} = \begin{bmatrix} 3 & -2i & -j & -2k & 0 & \cdots & 0 \\ 2i & 3 & -2i & \ddots & \ddots & \ddots & \vdots \\ j & 2i & \ddots & \ddots & \ddots & \ddots & 0 \\ 2k & \ddots & \ddots & \ddots & \ddots & \ddots & -2k \\ 0 & \ddots & \ddots & \ddots & \ddots & -2i & -j \\ \vdots & \ddots & \ddots & \ddots & 2i & 3 & -2i \\ 0 & \cdots & 0 & 2k & j & 2i & 3 \end{bmatrix}.$$

1. When $n = 5$, compute the eigenvalues and corresponding eigenvectors of $O_{\mathbb{H}}$ by Algorithm 3.5.
2. Make a comparison between Algorithm 3.5 and the order *eig* of Quaternion toolbox for Matlab [49].

Solution. 1. Since $O_{\mathbb{H}}$ is Hermitian, by Algorithm 3.5 we get

$$\Lambda = \{-1.2426, 0.0000, 3.0000, 6.0000, 7.2426\},$$
$$V = V_0 + V_1 i + V_2 j + V_3 k,$$

where

$$V_0 = \begin{bmatrix} 0.3557 & 0 & -0.4943 & 0 & 0.1996 \\ 0.4969 & 0 & 0.1988 & 0 & -0.3975 \\ 0.5031 & 0 & 0.3926 & 0 & 0.5031 \\ 0.4969 & 0 & -0.0994 & 0 & -0.3975 \\ -0.3557 & 0 & 0.1998 & 0 & -0.1996 \end{bmatrix}^T, \quad V_1 = \begin{bmatrix} 0 & -0.5895 & 0 & 0.3118 & 0 \\ 0 & -0.1739 & 0 & -0.3230 & 0 \\ 0 & 0.0245 & 0 & 0.0245 & 0 \\ 0 & 0.5714 & 0 & 0.2733 & 0 \\ 0 & 0.0005 & 0 & 0.2771 & 0 \end{bmatrix}^T,$$

$$V_2 = \begin{bmatrix} 0 & 0 & 0.0936 & 0 & 0.2945 \\ 0 & 0 & -0.0994 & 0 & -0.2981 \\ 0 & 0 & -0.4908 & 0 & -0.0000 \\ 0 & 0 & 0.1988 & 0 & 0.2981 \\ 0 & 0 & -0.4617 & 0 & 0.2945 \end{bmatrix}^T, \quad V_3 = \begin{bmatrix} 0 & -0.0647 & 0 & 0.2119 & 0 \\ 0 & 0.2236 & 0 & 0.5217 & 0 \\ 0 & -0.2209 & 0 & -0.2209 & 0 \\ 0 & -0.2236 & 0 & -0.0745 & 0 \\ 0 & -0.3770 & 0 & 0.5243 & 0 \end{bmatrix}^T.$$

In fact $O_{\mathbb{H}}$ is similar to a real symmetric tridiagonal matrix

$$D = \begin{bmatrix} 3.0000 & 3.0000 & 0 & 0 & 0 \\ 3.0000 & 3.0000 & 2.1344 & 0 & 0 \\ 0 & 2.1344 & 3.0000 & 2.1082 & 0 \\ 0 & 0 & 2.1082 & 3.0000 & 3.0000 \\ 0 & 0 & 0 & 3.0000 & 3.0000 \end{bmatrix},$$

which can be computed by Algorithm 3.4. Now we make an error analysis,

$$\max_{\widetilde{\lambda} \in \widetilde{\Lambda}} \left\{ \min_{\lambda \in \Lambda} |\lambda - \widetilde{\lambda}| \right\} = 1.998401444325282e - 015,$$

$$\min_{\widetilde{\lambda} \in \widetilde{\Lambda}} \left\{ \min_{\lambda \in \Lambda} |\lambda - \widetilde{\lambda}| \right\} = 2.220446049250313e - 016,$$

where

$$\widetilde{\Lambda} = \{3 - 3\sqrt{2}, 0, 3, 6, 3 + 3\sqrt{2}\}$$

is the set of all explicit eigenvalues of $O_{\mathbb{H}}$. Above results show that our algorithms are reliable.

2. For $n = 10: 300$, we apply Algorithm 3.5 and the order *eig* of Quaternion toolbox for Matlab [49] to $O_{\mathbb{H}}$, respectively. In Fig. 1, the solid line and the dash-dotted line show the CPU times costed by Algorithm 3.5 and the order *eig* of Quaternion

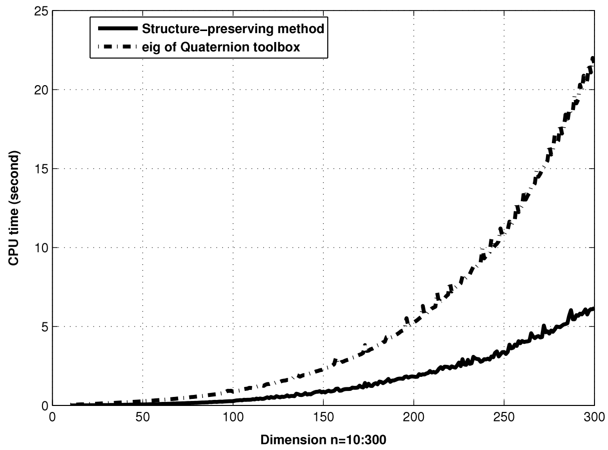

Fig. 1. CPU times for computing eigenvalues and eigenvectors.

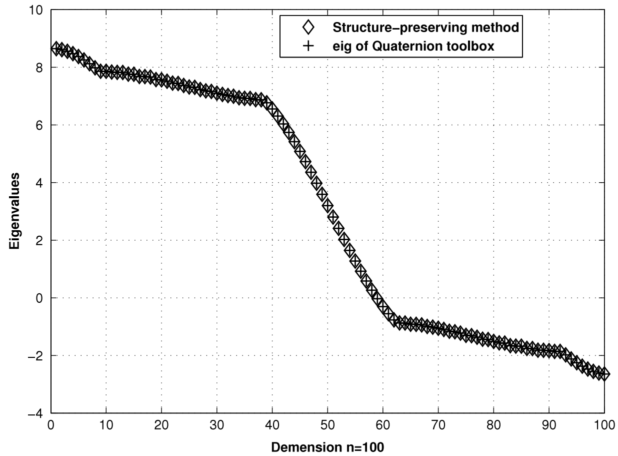

Fig. 2. Eigenvalues in descending order.

toolbox, respectively. When $n \leq 20$, the CPU time costed by Algorithm 3.5 is about one-eighth of that by the order *eig* of Quaternion toolbox; when n is large the CPU time costed by Algorithm 3.5 is about a quarter of that by the order *eig* of Quaternion toolbox. Fig. 2 shows that the computational accuracy of Algorithm 3.5 is almost the same as the order *eig* of Quaternion toolbox [49].

Example 4.2. In this example we compare Algorithm 3.4 with the Householder tridiagonalization algorithm (for example, see Algorithm 1.1 on p. 161 of [50]) and the order *tridiagonalize* of Quaternion toolbox [49]. Given a Hermitian quaternion matrix $Q = Q_0 + Q_1 i + Q_2 j + Q_3 k$ where $Q_0 \in \mathbb{R}^{n\times n}$ is a random symmetric matrix, $Q_1, Q_2, Q_3 \in \mathbb{R}^{n\times n}$ are three random skew-symmetric matrices.

In Fig. 3, the solid line, the "+" line and the dash-dotted line show the CPU times costed by Algorithm 3.4, the order tridiagonalize of Quaternion toolbox and the Householder tridiagonalization algorithm, respectively. When the order n is large, the CPU time costed by Algorithm 3.4 is about one-eighth of that by the Householder tridiagonalization algorithm, and about a quarter of that by the order tridiagonalize of Quaternion toolbox [49]. Notice that the operations of Algorithm 3.4 are about a quarter of that of the Householder tridiagonalization algorithm, and the dimension of the matrix processed in each step of Algorithm 3.4 is also about a quarter of that of the Householder tridiagonalization algorithm.

Example 4.3. Color can be used to improve the performance of face recognition as it has plenty of discriminative information. It is well known that a color image for face recognition can be represented by a quaternion matrix. But the algorithms tend to be more computationally expensive. Applying our algorithms will overcome this major drawback.

Calculating eigenfaces is the core work of face recognition in color. It needs to compute the eigenvalues and corresponding eigenvectors of a covariance matrix. Now we apply Algorithm 3.5 to do this. In this experiment, we take 16 faces in color from Faces95 [51], and compute their eigenfaces in color as shown in Fig. 4. The process takes about 0.015 s. It should be noted that the *eig* order of Quaternion toolbox will take about 0.203 s to do the same work.

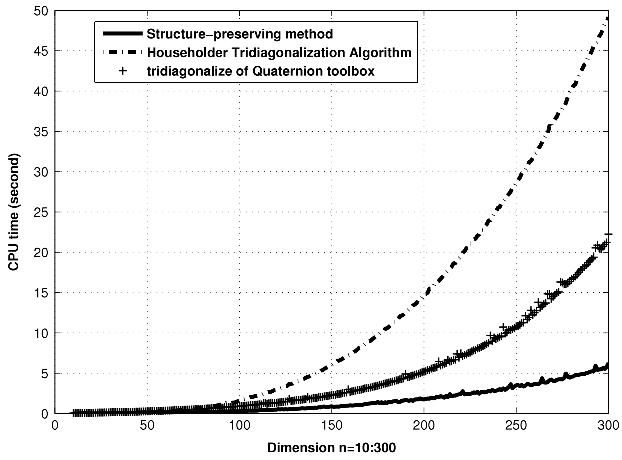

Fig. 3. CPU time of tridiagonalization.

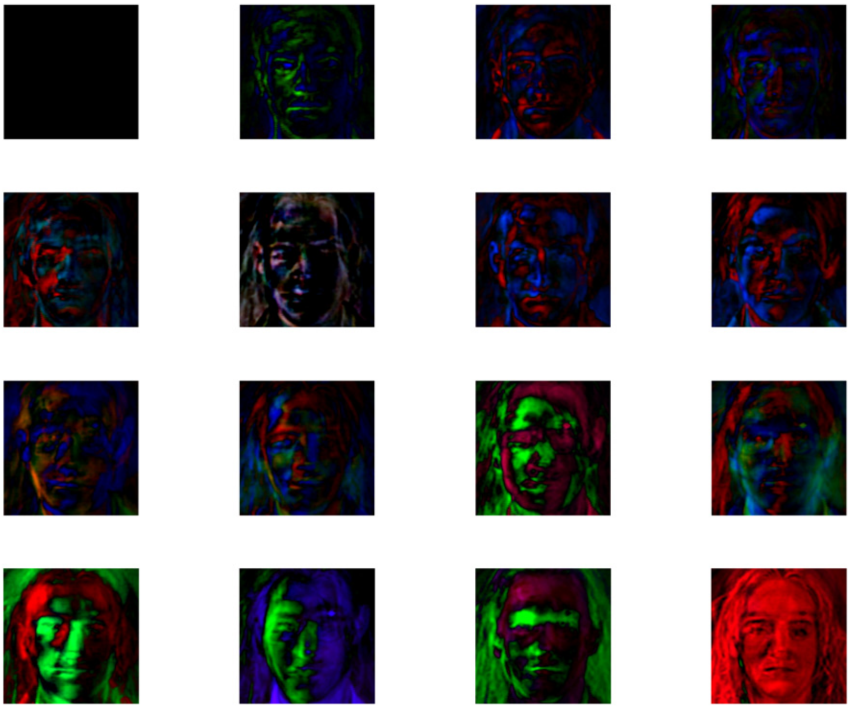

Fig. 4. Eigenfaces in color of 16 faces from Faces95.

5. Conclusions

In this paper we present a structure-preserving method to solve the (right) eigenvalue problems of Hermitian quaternionic linear operators on quaternionic vector spaces. For Hermitian quaternionic linear operators, we find that their real counterparts have two structures, i.e., symmetry and *JRS*-symmetry, and these structures are unchanged under orthogonal *JRS*-symplectic equivalence transformations. So we construct a structure-preserving method for the tridiagonalization of the real counterpart, and make it possible that the $4n$-dimensional problem is reduced to an n-dimensional problem. Indeed, we compute a real symmetric tridiagonal matrix which is similar to the original Hermitian quaternionic linear operator. This is the core of our method. There are at least two advantages of the new structure-preserving tridiagonalization algorithm.

- It costs only about a quarter of operations and one-eighth of the CPU time of the famous Householder tridiagonalization algorithm, and the obtained tridiagonal matrices are still symmetric and *JRS*-symmetric.
- Compared with the order *tridiagonalize* of Quaternion toolbox for Matlab, it can compute the real symmetric tridiagonal matrix by and only by real computations, and the principal advantage over the order *tridiagonalize* of Quaternion toolbox is a four-fold reduction in CPU time.

Numerical examples show that our algorithms are reliable, and that the larger the dimension of the problem the more obvious is the advantage of the structure-preserving method.

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