



# Cox risk model with variable premium rate and stochastic return on investment



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## ARTICLE INFO

### Article history:

Received 28 November 2012

Received in revised form 8 April 2013

### Keywords:

Cox risk model

Optimal investment

Expected discounted penalty function

Variable premium rate

## ABSTRACT

This paper studies the ruin probability for a Cox risk model with intensity depending on premiums and stochastic investment returns, and the model proposed in this paper allows the dependence between premiums and claims. When the surplus is invested in the bond market with constant interest force, coupled integral equations for the Gerber–Shiu expected discounted penalty function (GS function) are derived; together with the initial value and Laplace transformation of the GS function, we provide a numerical procedure for obtaining the GS function. When the surplus can be invested in risky asset driven by a drifted Brownian motion, we focus on finding a minimal upper bound of ruin probability and find that optimal piecewise constant policy yields the minimal upper bound. It turns out that the optimal piecewise constant policy is asymptotically optimal when initial surplus tends to infinity.

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## 1. Introduction

Ruin theory models with force of interest or stochastic investment return have received considerable attention in past two or three decades. For results on the ruin theory under models with constant interest force see, for example, [1–11] and references therein. In [12], an expected discounted penalty function is introduced, and it is called the Gerber–Shiu function (or GS function). This has been studied by many authors in the literature. One popular method to study the ruin probability or the GS function is to analyze the integral–differential equation satisfied by the ruin probability or the GS function, and another group of literature is on bounds estimation or asymptotic behavior of ruin probability. Most of the studies assume that the premium income rate is a fixed constant. Some work on variable premium rate models can be found in [13–15]. This paper focuses on the Cox risk model with variable premium rate specified by a function of the Cox process intensity, and thus the model allows the dependence between premium incomes and claims. Since more premium income means more customers, more claims probably will occur. So the model is reasonable. The first part of this paper devotes to the GS function when the model receives constant interest force. Coupled integral equations satisfied by the GS function are obtained. Together with the initial value of the GS function, we can derive the expression for the GS function.

The second part of the paper focuses on optimal investment policy when the model has stochastic investment return. In a model with constant interest force, if the claim sizes have exponential moments (i.e. the “light tailed claims”), the ruin probability decreases exponentially as the initial surplus increases. However, when there is a stochastic investment return, the situation can be different. Frovola et al. [16], Gjessing and Paulsen [17], Kalashnikov and Norberg [18] investigated the

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problem under the assumption that all the surpluses are invested in the risky market; it has been shown that even if the claims are “light tailed claims”, the ruin probability decreases only in the order of a negative power of the initial surplus. This somehow indicates that investing the surplus into the risky market cannot be optimal. Naturally, one interesting problem is: if an insurer has the opportunity to invest in the risky asset, what is the optimal investment policy if the insurer wants to minimize the ruin probability? In particular, can the insurer do better than keeping the surplus in the bond? Browne [19] considered this problem for the drifted Brownian motion risk model and found that the optimal policy is to invest constant amount in the risky asset, independent of the surplus of the wealth process. In this case ruin probability has a closed form expression and is much smaller than the one without any investment in risky asset. However, in most cases, it is not easy to obtain an explicit solution for the optimal policy. Alternatively, there are some papers focused on finding optimal policy minimizing the upper bound of ruin probability (c.f. [20,21]). They found that the optimal policy to minimize the upper bound of ruin probability is a kind of constant policy, and they proved that such constant policy is asymptotic optimal when the initial surplus tends to infinity. Motivated by the work of Gaier et al. [20], the second part of the paper aims to find optimal investment policy minimizing the upper bound of ruin probability and proves its asymptotic optimality. Results obtained in this part can be regarded as an extension of Gaier et al. [20] to the case of the Cox risk model with dependence between the premiums and claims.

This paper is organized as follows. Section 2 provides an introduction to the model and the problem formulation. In Section 3, a coupled system of integral equations satisfied by the GS function is obtained and the initial value of the GS function is derived. Section 4 investigates the optimal investment policy for minimizing the upper bound of ruin probability and proves that the optimal constant investment policy is asymptotically optimal when initial surplus tends to infinity.

## 2. Model and problem formulation

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space. The surplus process of an insurer is specified by

$$X_t = u + \int_0^t c(\lambda_s) ds - \sum_{i=1}^{N_t} Y_i, \quad (2.1)$$

where  $u > 0$  is the initial surplus,  $\{N_t, t \geq 0\}$ , denotes the number of claims arrived up to time  $t$ , is a Cox process with intensity process  $\{\lambda_t, t \geq 0\}$ .  $\{Y_i, i \geq 1\}$  are i.i.d. random variables with common distribution function  $F(x)$  and  $F(0) = 0$ .  $\{\lambda_t, t \geq 0\}$  is a positive-valued, continuous time Markov chain with state space  $E = \{\alpha_i, i = 1, 2, \dots, n\}$  and generator  $\mathbf{Q} = (q_{ij})_{n \times n}$ .  $c(\cdot)$  is a continuous, positive valued function defined on  $\mathbb{R}^+$ . Define  $\tau_1$  the first time that the process  $\{\lambda_t, t \geq 0\}$  leaves the initial state, i.e.  $\tau_1 = \inf\{t : t > 0, \lambda_t \neq \lambda_0\}$ . By the classical results on continuous time Markov chain, if  $q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ , then we have the following results.

**Lemma 1.** Suppose that  $\lambda_0 = \alpha_i$ , then for any  $\alpha_i \in E$ , the following properties holds:

$$\mathbb{P}(\tau_1 > t) = e^{-q_i t}; \quad (2.2)$$

$$\mathbb{P}(\tau_1 \leq t, \lambda_{\tau_1} = \alpha_j) = (1 - e^{-q_i t}) \frac{q_{ij}}{q_i}; \quad (2.3)$$

$$\mathbb{P}(\lambda_{\tau_1} = \alpha_j) = \frac{q_{ij}}{q_i}. \quad (2.4)$$

The proof of Lemma 1 can be found in [22]. Let  $\mathcal{F}_t^\lambda = \sigma\{\lambda_s, 0 \leq s \leq t\}$ ,  $\mathcal{F}_t^X = \sigma\{X_s, 0 \leq s \leq t\}$  and  $\mathcal{F}_t = \sigma\{(\lambda_s, X_s), 0 \leq s \leq t\}$ . In this paper we shall use Lemma 2.19 in [22], and we cite it here.

**Lemma 2.** (i)  $N_t$  has independent increments with respect to  $\mathcal{F}_\infty^\lambda$ ;  
(ii)  $N_t - N_s$  is a Poisson distribution with mean  $\int_s^t \lambda_r dr$  with respect to  $\mathcal{F}_\infty^\lambda$ .

One common assumption in the insurance risk model is the “positive safety loading” condition, which guarantees that the expected net income of the insurer is positive. Assume that the process  $\{\lambda_t, t > 0\}$  is stationary with initial distribution  $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ . Then the following condition guarantees that “positive safety loading” holds.

$$\mathbb{E}c(\lambda_t) = \mathbb{E}c(\lambda_0) > \mathbb{E}\lambda_0 \mathbb{E}Y := p\mathbb{E}Y. \quad (2.5)$$

Note that  $q_i < \infty$  and  $\lambda_t$  is a finite-state Markov chain; it follows from the standard results on the stochastic process (c.f. [23]) that  $\mathbb{E}|\lambda_t - \lambda_0|^2 \rightarrow 0$  ( $t \rightarrow 0+$ ) and consequently  $\mathbb{E}|c(\lambda_t) - c(\lambda_0)|^2 \rightarrow 0$  ( $t \rightarrow 0+$ ) and we also have  $\mathbb{E} \int_0^t c(\lambda_s) ds = \int_0^t \mathbb{E}c(\lambda_s) ds$  (c.f. Theorem 2.3, [23]). In fact, Eq. (2.5) ensures that for any  $t \geq 0$ , the expected total premium income is larger than the expected aggregate claims since

$$\begin{aligned} \mathbb{E} \int_0^t c(\lambda_s) ds &= \int_0^t \mathbb{E}c(\lambda_s) ds = t\mathbb{E}c(\lambda_0) > \mathbb{E} \left[ \sum_{i=1}^{N(t)} Y_i \right] = \mathbb{E}Y \mathbb{E} \left[ \int_0^t \lambda_s ds \right] = \mathbb{E}Y \mathbb{E}\lambda_0 t \\ &\iff \mathbb{E}c(\lambda_t) = \mathbb{E}c(\lambda_0) > \mathbb{E}Y \mathbb{E}\lambda_0 := p\mathbb{E}Y. \end{aligned}$$

In particular, putting  $c(\lambda_t) = (1 + \rho)p\mathbb{E}Y$  with  $\rho > 0$ , our model reduces to the one considered in [22].

Let  $T_i(u) = \inf\{t : X_t < 0 | X_0 = u, \lambda_0 = \alpha_i\}$ , the ruin time of  $X_t$  with  $\lambda_0 = \alpha_i$ ,  $X_0 = u$ , and  $T(u) = \inf\{t : X_t < 0 | X_0 = u\}$  the ruin time of process (2.1), with the convention that  $\inf \emptyset = \infty$ . Denote the ultimate ruin probability with initial surplus  $u$  and initial intensity state  $\alpha_i$  by  $\psi(u, i)$ , i.e.

$$\psi(u, \alpha_i) = \mathbb{P}\{T_i(u) < \infty\} = \mathbb{P}\{T(u) < \infty | X_0 = u, \lambda_0 = \alpha_i\}, \quad (2.6)$$

the ruin probability with initial surplus  $u$  by  $\psi(u)$ , i.e.

$$\psi(u) = \mathbb{P}\{T(u) < \infty\} = \mathbb{P}\left\{\inf_t X_t < 0 | X_0 = u\right\} = \sum_{i=1}^n \psi(u, i) \pi_i, \quad (2.7)$$

and the probability that ruin occurs before or on the  $n_{th}$  claim by

$$\psi_n(u) = \mathbb{P}\left\{T(u) \leq L_n | X_0 = u\right\}, \quad (2.8)$$

where  $L_n$  denotes the  $n$ th claim time. Besides the ruin probability, other important ruin quantities in ruin theory include the Laplace transform of the ruin time, denoted by  $\mathbb{E}[e^{-\alpha T}]$ ; the surplus immediately before ruin, denoted by  $X_{T-}$ ; the deficit at ruin, denoted by  $|X_T|$ , etc. are also important. A unified approach to study these ruin quantities is the GS function which is defined as

$$\phi_{i,\beta}(u) = \mathbb{E}\left[\omega(X_{T(u)-}, |X_{T(u)}|) e^{-\beta T(u)} 1_{\{T(u) < \infty\}} | \lambda_0 = \alpha_i\right], \quad (2.9)$$

where  $\omega(x, y)$ ,  $x \geq 0$ ,  $y \geq 0$  is a nonnegative function such that  $\phi_{i,\beta}(u)$  exists. In this paper  $\omega(x, y)$  is assumed to be bounded, i.e.  $\sup_{x,y} \omega(x, y) = M < \infty$ ,  $x \geq 0$ ,  $y \geq 0$ ,  $M$  is a positive constant. The following boundary conditions are trivial.

$$\phi_{i,\beta}(\infty) = \psi(\infty, \alpha_i) = \psi(\infty) = 0. \quad (2.10)$$

### 3. Gerber–Shiu expected discounted penalty function

This section focuses on the case that the insurer would like to invest all the surpluses to the bond market with force of interest  $\delta$ . Then, the dynamic of the surplus process is specified as

$$dX_t = X_t \delta dt + c(\lambda_t) dt - dZ_t, \quad (3.1)$$

where  $Z_t = \sum_{i=1}^{N_t} Y_i$  denotes aggregate claims up to time  $t$ . Eq. (3.1) implies that

$$e^{-\delta t} dX_t - \delta e^{-\delta t} X_t dt = d(e^{-\delta t} X_t) = e^{-\delta t} c(\lambda_t) dt - e^{-\delta t} dZ_t.$$

Replace  $t$  with  $r$  in Eq. (3.1) and integrate both sides w.r.t.  $r$  from 0 to  $t$ ; it follows that

$$X_t = e^{\delta t} \left( u + \int_0^t e^{-\delta r} c(\lambda_r) dr - \int_0^t e^{-\delta r} dZ_r \right). \quad (3.2)$$

**Theorem 3.1.** Vector  $(\phi_{1,\beta}(u), \phi_{2,\beta}(u), \dots, \phi_{d,\beta}(u))$  is the solutions to the following matrix equation

$$\begin{aligned} \Phi_\beta(u) &= \int_0^u \mathbf{K}(u, t) \Phi_\beta(t) dt + \mathbf{B}(u) \Phi_\beta(0) - \mathbf{C}(u) \int_0^u m(t) dt - \int_0^u \mathbf{T}(u) \Phi_\beta(t) dt \\ &= \int_0^u [\mathbf{K}(u, t) - \mathbf{T}(u)] \Phi_\beta(t) dt + \mathbf{B}(u) \Phi_\beta(0) - \mathbf{C}(u) \int_0^u m(t) dt, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \Phi_\beta(u) &:= (\phi_{1,\beta}(u), \phi_{2,\beta}(u), \dots, \phi_{d,\beta}(u))^T, \\ \mathbf{K}(u, t) &:= \text{diag} \left( \frac{\alpha_1(1 - F(u - t)) + \beta + \delta}{\delta u + c(\alpha_1)}, \dots, \frac{\alpha_d(1 - F(u - t)) + \beta + \delta}{\delta u + c(\alpha_d)} \right), \\ \mathbf{B}(u) &:= \text{diag} \left( \frac{c(\alpha_1)}{\delta u + c(\alpha_1)}, \frac{c(\alpha_2)}{\delta u + c(\alpha_2)}, \dots, \frac{c(\alpha_d)}{\delta u + c(\alpha_d)} \right), \\ \mathbf{C}(u) &:= \text{diag} \left( \frac{\alpha_1}{\delta u + c(\alpha_1)}, \frac{\alpha_2}{\delta u + c(\alpha_2)}, \dots, \frac{\alpha_d}{\delta u + c(\alpha_d)} \right), \end{aligned} \quad (3.4)$$

$\mathbf{T}(u) = (t_{ij}(u))_{n \times n}$  a matrix with  $t_{ij}(u) = \frac{q_{ij}}{\delta u + c(\alpha_i)}$ ,  $i, j = 1, 2, \dots, d$  and  $m(t)$  denotes  $\int_t^\infty \omega(t, y - t) dF(y)$ .

**Proof.** Suppose that  $(X_0, \lambda_0) = (u, \alpha_i)$ . Inspired by the “differential argument” applied in [3], consider a very short time interval  $[0, \Delta t]$ ; there are four cases:

(i) no claim arrives and  $\lambda_t$  does not jump in  $[0, \Delta t]$ , then

$$X_{\Delta t} = ue^{\delta \Delta t} + e^{\delta \Delta t} \int_0^{\Delta t} e^{-\delta r} c(\lambda_r) dr = ue^{\delta \Delta t} + c(\alpha_i) \frac{e^{\delta \Delta t} - 1}{\delta} \approx ue^{\delta \Delta t} + c(\alpha_i) \Delta t$$

with probability  $(1 - q_i \Delta t)(1 - \alpha_i \Delta t) + o(\Delta t)$ . Note that when  $\Delta t$  is very small,  $e^{\delta \Delta t} \approx (1 + \delta \Delta t)$ ;

(ii)  $\lambda_t$  does not jump but one claim occurs with arrival time  $\Delta s (< \Delta t)$ , then we have

$$X_{\Delta t} = ue^{\delta \Delta t} + \Delta t c(\alpha_i) - Y_1 e^{\delta(\Delta t - \Delta s)}$$

with probability  $(1 - q_i \Delta t)\alpha_i \Delta t + o(\Delta t)$ . Note that in this case we should further consider whether the claim causes ruin or not;

(iii)  $\lambda_t$  jumps but no claim occurs in the time interval  $[0, \Delta t]$ ; denote the jump time by  $\Delta h (< \Delta t)$ , then we have

$$X_{\Delta t} = ue^{\delta \Delta t} + \Delta h c(\alpha_i) + c(\alpha_j)(\Delta t - \Delta h)$$

with probability  $q_i \Delta t \frac{q_{ij}}{q_i} (1 - \alpha_i \Delta t) + o(\Delta t)$ ;

(iv) other cases happen with probability  $o(\Delta t)$ .

By the Markov property of process  $(X_t, \lambda_t)$  we have

$$\begin{aligned} \phi_{i,\beta}(u) &= \mathbb{E} \left[ \mathbb{E} \left[ \omega(X_{T(u)-}, |X_{T(u)}|) 1_{T(u) < \infty} \middle| X_{\Delta t}, \lambda_{\Delta t} \right] \middle| X_0 = u, \lambda_0 = \alpha_i \right] \\ &= (1 - q_i \Delta t)(1 - \alpha_i \Delta t) e^{-\beta \Delta t} \phi_{i,\beta}([u(1 + \delta \Delta t) + c(\alpha_i) \Delta t]) \\ &\quad + (1 - q_i \Delta t) \alpha_i \Delta t e^{-\beta \Delta t} \left[ \int_{ue^{\delta \Delta t} + c(\alpha_i) \Delta t}^{\infty} \omega(ue^{\delta \Delta t} + c(\alpha_i) \Delta t, y - ue^{\delta \Delta t} - c(\alpha_i) \Delta t) dF(y) \right. \\ &\quad \left. + \int_0^{ue^{\delta \Delta t} + c(\alpha_i) \Delta t} \phi_{i,\beta}(ue^{\delta \Delta t} + c(\alpha_i) \Delta t - y) dF(y) \right] \\ &\quad + e^{-\beta \Delta t} \sum_{i \neq j} \frac{q_{ij}}{q_i} q_i \Delta t (1 - \alpha_i \Delta t) \phi_{j,\beta}(ue^{\delta \Delta t} + c(\alpha_i) \Delta h + c(\alpha_j)(\Delta t - \Delta h)) + o(\Delta t). \end{aligned} \quad (3.5)$$

Rearranging Eq. (3.5) yields

$$\begin{aligned} \phi_{i,\beta}(u) - \phi_{i,\beta}(u + [u\delta + c(\alpha_i)]\Delta t) &= -(q_i + \alpha_i + \beta) \phi_{i,\beta}([u(1 + \delta \Delta t) + c(\alpha_i) \Delta t]) \\ &\quad + (1 - q_i \Delta t) \alpha_i \Delta t e^{-\beta \Delta t} \left[ \int_{ue^{\delta \Delta t} + c(\alpha_i) \Delta t}^{\infty} \omega(ue^{\delta \Delta t} + c(\alpha_i) \Delta t, y - ue^{\delta \Delta t} - c(\alpha_i) \Delta t) dF(y) \right. \\ &\quad \left. + \int_0^{ue^{\delta \Delta t} + c(\alpha_i) \Delta t} \phi_{i,\beta}(ue^{\delta \Delta t} + c(\alpha_i) \Delta t - y) dF(y) \right] \\ &\quad + e^{-\beta \Delta t} \sum_{i \neq j} \frac{q_{ij}}{q_i} q_i \Delta t (1 - \alpha_i \Delta t) \phi_{j,\beta}(ue^{\delta \Delta t} + c(\alpha_i) \Delta h + c(\alpha_j)(\Delta t - \Delta h)) + o(\Delta t). \end{aligned} \quad (3.6)$$

Eq. (3.6) indicates that  $\phi_{i,\beta}(u)$  is continuous. Under the assumption that function  $\omega(x, y)$  is bounded, by the dominated convergence theorem, differentiating both sides of Eq. (3.6) with respect to  $\Delta t$  yields

$$\begin{aligned} \phi'_{i,\beta}(u)[u\delta + c(\alpha_i)] &= (\alpha_i + \beta + \delta) \phi_{i,\beta}(u) \\ &\quad - \alpha_i \left[ \int_u^{\infty} \omega(u, y - u) dF(y) + \int_0^u \phi_{i,\beta}(u - y) dF(y) \right] - \sum_{j=1}^d q_{ij} \phi_{j,\beta}(u). \end{aligned} \quad (3.7)$$

Replace argument  $u$  in the above equation by  $t$  and integrate both sides of Eq. (3.7) with respect to  $t$  from 0 to  $u$ , and note that

$$\begin{aligned} \int_0^u (\delta t + c(\alpha_i)) \phi'_{i,\beta}(t) dt &= \int_0^u (\delta t + c(\alpha_i)) d\phi_{i,\beta}(t) = (\delta t + c(\alpha_i)) \phi_{i,\beta}(t) \Big|_0^u - \int_0^u \delta \phi_{i,\beta}(t) dt \\ &= (\delta u + c(\alpha_i)) \phi_{i,\beta}(u) - c(\alpha_i) \phi_{i,\beta}(0) - \delta \int_0^u \phi_{i,\beta}(t) dt; \end{aligned}$$

denote  $\int_t^{\infty} \omega(t, y - t) dF(y)$  by  $m(t)$ , then we have

$$\phi_{i,\beta}(u) = \frac{\alpha_i + \beta + \delta}{\delta u + c(\alpha_i)} \int_0^u \phi_{i,\beta}(t) dt + \frac{c(\alpha_i)}{\delta u + c(\alpha_i)} \phi_{i,\beta}(0)$$

$$\begin{aligned}
& -\frac{\alpha_i}{\delta u + c(\alpha_i)} \int_0^u m(t)dt - \frac{\alpha_i}{\delta u + c(\alpha_i)} \int_0^u \int_0^t \phi_{i,\beta}(t-y)dF(y)dt \\
& - \sum_{j=1}^d \frac{q_{ij}}{\delta u + c(\alpha_i)} \int_0^u \phi_{j,\beta}(t)dt.
\end{aligned} \tag{3.8}$$

Since

$$\begin{aligned}
\int_0^u \int_0^t \phi_{i,\beta}(t-y)dF(y)dt &= \int_0^u \int_y^u \phi_{i,\beta}(t-y)dt dF(y) \\
&= \int_0^u \int_0^{u-y} \phi_{i,\beta}(t)dt dF(y) = F(y) \int_0^{u-y} \phi_{i,\beta}(t)dt \Big|_0^u - \int_0^u F(y) d \left( \int_0^{u-y} \phi_{i,\beta}(t)dt \right) \\
&= \int_0^u F(u-t) \phi_{i,\beta}(t)dt,
\end{aligned} \tag{3.9}$$

it follows that

$$\begin{aligned}
\phi_{i,\beta}(u) &= \int_0^u \frac{\alpha_i(1-F(u-t)) + \beta + \delta}{\delta u + c(\alpha_i)} \phi_{i,\beta}(t)dt + \frac{c(\alpha_i)}{\delta u + c(\alpha_i)} \phi_{i,\beta}(0) \\
&\quad - \frac{\alpha_i}{\delta u + c(\alpha_i)} \int_0^u m(t)dt - \sum_{j=1}^d \frac{q_{ij}}{\delta u + c(\alpha_i)} \int_0^u \phi_{j,\beta}(t)dt, \quad i = 1, 2, \dots, d.
\end{aligned} \tag{3.10}$$

For  $i = 1, 2, \dots, d$ , Eq. (3.10) composes a coupled system of integro equations and Eq. (3.3) is the matrix form. This completes the proof.  $\square$

Eq. (3.3) provides the way to obtain the value of  $\Phi_\beta(u)$  by the Piccard recursive method once the value of  $\Phi_\beta(0)$  is known. The rest of this section provides a result for  $\Phi_\beta(0)$  under some suitable conditions. To proceed our discussion, let

$$\begin{aligned}
K(r, \alpha_i) &= \mathbb{E} \left[ e^{-r(X_{L_1}-u)} | \lambda_0 = \alpha_i \right], \\
K(r) &= \mathbb{E} \left[ e^{-r(X_{L_1}-u)} \right] = \sum_{i=1}^d \pi_i \mathbb{E} \left[ e^{-r(X_{L_1}-u)} | \lambda_0 = \alpha_i \right] = \sum_{i=1}^d \pi_i K(r, \alpha_i).
\end{aligned} \tag{3.11}$$

The proof of Lemma 3.2 can be found in the Appendix.

**Lemma 3.2.** Suppose that  $R_i$  is a positive root of equation  $K(r, \alpha_i) = 1$ ,  $i = 1, 2, \dots, d$  and  $R$  is the positive root of equation  $K(r) = 1$ . Let  $\bar{R} := \min R_i$ . Then

$$\psi(u, \alpha_i) \leq \varrho e^{-\bar{R}u}, \quad i = 1, 2, \dots, d \tag{3.12}$$

$$\psi(u) \leq \varrho e^{-Ru}, \tag{3.13}$$

where  $\varrho$  is specified by  $\varrho^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{-Ru} dF(u)}{e^{Rt} \bar{F}(t)}$ . Naturally, since  $\omega(x, y)$  is nonnegative and bounded by  $M$ , we have

$$\phi_{i,\beta}(u) \leq \varrho M e^{-\bar{R}u}, \quad i = 1, 2, \dots, d. \tag{3.14}$$

**Theorem 3.3.** Suppose the conditions of Lemma 3.2 hold, we have

$$\Phi_\beta(0) = \left( \frac{\alpha_1}{c(\alpha_1)}, \frac{\alpha_2}{c(\alpha_2)}, \dots, \frac{\alpha_d}{c(\alpha_d)} \right)^T \Theta, \tag{3.15}$$

where  $\Theta = \lim_{u \rightarrow \infty} \int_0^u m(u-t)dt$ .

**Proof.** Revisit Eq. (3.10) with  $t$  replaced by  $u-t$ , by some mathematical manipulations, it follows that

$$\begin{aligned}
\phi_{i,\beta}(u) &= \int_0^u \frac{\alpha_i(1-F(t)) + \beta + \delta}{\delta u + c(\alpha_i)} \phi_{i,\beta}(u-t)dt - \frac{\alpha_i}{\delta u + c(\alpha_i)} \int_0^u m(u-t)dt \\
&\quad + \frac{c(\alpha_i)}{\delta u + c(\alpha_i)} \phi_{i,\beta}(0) - \sum_{j=1}^d \frac{q_{ij}}{\delta u + c(\alpha_i)} \int_0^u \phi_{j,\beta}(u-t)dt, \quad i = 1, 2, \dots, d.
\end{aligned} \tag{3.16}$$

Multiply  $\delta u + c(\alpha_i)$  on both sides of Eq. (3.16); note that Eq. (3.14) guarantees that

$$\lim_{u \rightarrow \infty} \phi_{i,\beta}(u)(\delta u + c(\alpha_i)) = 0, \quad i = 1, 2, \dots, d. \quad (3.17)$$

Using Eq. (3.17), together with boundary condition equation (2.10) and the dominated convergence theorem, letting  $u \rightarrow \infty$  on both sides of Eq. (3.16) yields

$$\phi_{i,\beta}(0) = \frac{\alpha_i}{c(\alpha_i)} \lim_{u \rightarrow \infty} \int_0^u m(u-t)dt. \quad (3.18)$$

Summarizing the previous discussion, we complete the proof.  $\square$

**Remark 1.** Eq. (3.17) plays a key role in Theorem 3.3, one can easily see that  $\phi_{i,\beta}(u) = o(u^{-(1+\gamma)})$  for some  $\gamma > 0$  sufficiently justifies Eq. (3.17), thus conditions in Lemma 3.2 seem too strong. However, Lemma 3.2 can also serve as an exponential bound estimation for the GS function, which is a classical research topic in risk theory.  $\square$

**Remark 2.** Taking Laplace transform on both sides of Eq. (3.7) yields

$$c(\alpha_i)s\tilde{\phi}_{i,\beta}(s) - \alpha_i\Theta - \delta \frac{d\tilde{\phi}_{i,\beta}(s)}{ds} = (\alpha_i + \beta)\tilde{\phi}_{i,\beta}(s) - \alpha_i\tilde{m}(s) - \alpha_i\tilde{F}(s)\tilde{\phi}_{i,\beta}(s) - \sum_{j=1}^d q_{ij}\tilde{\phi}_{j,\beta}(s), \quad (3.19)$$

where

$$\begin{aligned} \tilde{\phi}_{i,\beta}(s) &= \int_0^\infty e^{-su} \phi_{i,\beta}(u) du, \quad i = 1, 2, \dots, d; \\ \tilde{m}(s) &= \int_0^\infty e^{-su} m(u) du; \\ \tilde{F}(s) &= \int_0^\infty e^{-su} dF(u). \end{aligned}$$

Eq. (3.19) can be rewritten in the matrix form:

$$\delta \frac{d\tilde{\phi}_\beta(s)}{ds} = \mathbf{D}_\beta(s)\tilde{\phi}_\beta(s) - \mathbf{H}_1\Phi_\beta(0) + \mathbf{H}_2\tilde{\mathbf{M}}(s), \quad (3.20)$$

where  $\tilde{\phi}_\beta(s) = (\tilde{\phi}_{1,\beta}(s), \dots, \tilde{\phi}_{d,\beta}(s))^T$  and  $\mathbf{D}_\beta(s)$  is a  $d \times d$  matrix of the form

$$\mathbf{D}_\beta(s) = [(\mathbf{H}_1s + \mathbf{Q} - \beta\mathbf{I}) - \mathbf{H}_2 + \mathbf{H}_2\tilde{\mathbf{F}}(s)] \quad (3.21)$$

with

$$\begin{aligned} \mathbf{I} &= \text{diag}(1, 1, \dots, 1) \\ \mathbf{H}_1 &= \text{diag}(c(\alpha_1), c(\alpha_2), \dots, c(\alpha_d)); \\ \mathbf{H}_2 &= \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_d), \\ \tilde{\mathbf{M}}(s) &= (\tilde{m}(s), \tilde{m}(s), \dots, \tilde{m}(s))^T. \end{aligned} \quad (3.22)$$

Eq. (3.20) is a first order, nonlinear matrix ODE. To our knowledge, there is no close form solution to Eq. (3.20). Using the method of inverting Laplace transformation, Eq. (3.3) provides a numerical method for solving Eq. (3.20).  $\square$

**Example 1.** Consider the case that  $\lambda_t \equiv \alpha$  and  $c(\lambda_t) = (1 + \rho)\alpha\mathbb{E}Y$  with  $\rho > 0$ , then our model reduces to the compound Poisson risk model. If  $\omega(x, y) \equiv 1$  and  $\beta = 0$  then  $\psi_{i,\beta}(0)$  is the ruin probability for the classical risk model with positive safety loading  $\rho$ . By Eq. (3.18), we have

$$\psi(0) = \frac{\alpha}{(1 + \rho)\alpha\mathbb{E}Y} \lim_{u \rightarrow \infty} \int_0^u \int_x^\infty dF(t)dx = \frac{1}{1 + \rho}. \quad (3.23)$$

This is a classical result for the compound Poisson risk model (c.f. [22]).

#### 4. Minimizing the upper bound of ruin probability

Motivated by Gaier et al. [20], this section focuses on finding investment policy that minimizes the upper bound of ruin probability, and we also prove its asymptotic optimality. Thus, the purpose of this section is to investigate whether there

are constants  $R_i^*$  and  $C(\alpha_i)$  such that

$$\hat{\psi}(u, \alpha_i) \leq C(\alpha_i)e^{-R_i^*u}, \quad (4.1)$$

where  $\hat{\psi}(u, \alpha_i)$  is the minimum ruin probability over all admissible investment policies and it is also known as the value function in control theory. Of course, there is always a possibility not to invest anything in the risky asset at all, resulting in an exponential bound for the ruin probability  $\hat{\psi}(u, \alpha_i)$ , which is the so-called Lundberg upper bound for the Cox risk model without investment (c.f. [22]). Our purpose is to find the tightest upper bound for the minimum ruin probability, that is to say we want to find the optimal (i.e. the largest) coefficient  $R_i^*$  such that (4.1) holds.

To proceed our discussion, we assume that there are two kinds of assets available for investors in the financial market: a risk-free asset and risky asset, and their dynamics are specified respectively by

$$\begin{aligned} dr(t) &= \delta r(t)dt, \\ \frac{dP_t}{P_t} &= \mu dt + \sigma dB_t, \end{aligned} \quad (4.2)$$

where  $\{B_t, t \geq 0\}$  is a standard Brownian Motion,  $\delta, \mu$  and  $\sigma$  are positive constants.  $P = \{P_t, t \geq 0\}$  and  $\lambda = \{\lambda_t, t \geq 0\}$  are mutually independent. Due to the non-arbitrage assumption of financial market, it is assumed that  $\mu > \delta > 0$ .

Denote by  $\{A_t\}$  the amount invested in the risky asset at time  $t$  and  $X_t^A$  the wealth process with policy  $\{A_t, t \geq 0\}$ .  $X_t^A - A_t$  is the amount invested in bond. Denote by  $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$  the smallest filtration satisfying the usual condition such that the process  $\{(\lambda_t, P_t), t \geq 0\}$  is measurable. Assume that strategies  $\{A_t, t \geq 0\}$  are predictable w.r.t.  $\mathcal{F}_t$  and the insurers are allowed to invest more than its current wealth in risky asset. This means that the value of an admissible policy at time  $t$  may depend on the history of the process  $(X_t^A, \lambda_t, P_t)$  up to time  $t$ , but it may not depend on the size of a claim occurring at time  $t$ . Thus the admissible set is

$$\mathcal{A} = \left\{ A = (A_t)_{t \geq 0} : A \text{ is predictable and } \mathbb{P} \left[ \int_0^t A_s^2 ds < \infty \right] \text{ for all } t \in [0, \infty) \right\}.$$

Fleming and Soner [24] state that when the state process of a controlled system is Markovian, then a Markov optimal control is also a general optimal control. Note that  $(X_t^A, \lambda_t)$  is a controlled Markov vector process; thus it is sufficient to consider the Markovian control here, i.e.  $A_t$  takes the form of

$$A_t = A(X_{t-}^A, \lambda_t), \quad (4.3)$$

where  $A(\cdot, \cdot)$  is the deterministic of investment policy  $A_t$ .

**Remark 3.** The dynamic of  $X_t^A$  is

$$dX_t^A = c(\lambda_t)dt - dZ_t + A_t(\mu - \delta)dt + X_t^A\delta dt + A_t\sigma dB_t, \quad (4.4)$$

which implies that

$$e^{-\delta t} dX_t^A - \delta e^{-\delta t} X_t^A dt = e^{-\delta t} [c(\lambda_t)dt - dZ_t + A_t(\mu - \delta)dt + A_t\sigma dB_t]$$

and thus

$$d(e^{-\delta t} X_t^A) = e^{-\delta t} [c(\lambda_t)dt - dZ_t + A_t(\mu - \delta)dt + A_t\sigma dB_t] =: e^{-\delta t} d\tilde{X}_t^A \quad (4.5)$$

$e^{-\delta t} X_t^A$  is the discounted process of  $X_t^A$  and thus has the same ruin probability of  $X_t^A$ . Consequently,  $\tilde{X}_t^A$  has the same ruin probability with  $X_t^A$ , for mathematical convenience, we only study the optimal policy for process  $\tilde{X}_t^A$ .  $\square$

Denote  $\tilde{\mu} = \mu - \delta > 0$ , then

$$d\tilde{X}_t^A = [\tilde{\mu} A_t + c(\lambda_t)]dt + A_t\sigma dB_t - dZ_t. \quad (4.6)$$

Denote the time of ruin with initial surplus  $u$  and policy  $A$  by

$$T(u, A(\cdot, \cdot)) = \inf \left\{ t \geq 0 : \tilde{X}_t^{A(\cdot, \cdot)} < 0 | \tilde{X}_0^{A(\cdot, \cdot)} = u \right\} \quad (4.7)$$

and ruin probability by  $\psi^{A(\cdot, \cdot)}(u, \alpha_i) = \mathbb{P}(T(u, A(\cdot, \cdot)) < \infty | \tilde{X}_0^{A(\cdot, \cdot)} = u, \lambda_0 = \alpha_i)$ . The value function is

$$\hat{\psi}(u, \alpha_i) = \inf_{A(\cdot, \cdot) \in \mathcal{A}} \psi^{A(\cdot, \cdot)}(u, i). \quad (4.8)$$

Denote by  $\mathcal{A}_c$  the piecewise constant control policy and the value of  $A_t$  only depend on the value of the intensity process  $\lambda_t$ , i.e.

$$\mathcal{A}_c = \{A \in \mathcal{A}, A_t = A(\lambda_t), t \geq 0\}. \quad (4.9)$$

It is obvious that  $\mathcal{A}_c \subset \mathcal{A}$ . The idea of this section is to find an optimal policy in  $\mathcal{A}_c$ . Then we prove that the optimal policy in  $\mathcal{A}_c$  is the limits of the true optimal policy in  $\mathcal{A}$  when  $u \rightarrow \infty$ . To distinguish two different type investment strategies, denote by  $A(\cdot)$  the piecewise constant policy and by  $A(\cdot, \cdot)$  the general policy. Suppose that function  $V(x, l)$  belongs to the domain of the infinitesimal operator of process  $(\tilde{X}_t^A, \lambda_t)$ , then for all  $A(\cdot) \in \mathcal{A}_c$ ,

$$\begin{aligned} \mathcal{L}^{A(\cdot)} V(u, \alpha_i) &= c(\alpha_i) V_x(u, \alpha_i) + \alpha_i \mathbb{E}[V(u - Y, \alpha_i) - v(u, \alpha_i)] + A(\alpha_i) \tilde{\mu} V_x(u, \alpha_i) \\ &\quad + \frac{1}{2} A(\alpha_i)^2 \sigma^2 V_{xx}(u, \alpha_i) + \sum_{j=1}^d q_{ij} V(u, \alpha_j), \quad i = 1, 2, 3, \dots, d, \end{aligned} \quad (4.10)$$

where  $V_x, V_{xx}$  denote the first and second partial derivatives of  $V(u, \alpha_i)$  with respect to  $u$ . The following boundary condition is natural,

$$V(+\infty, \alpha_i) = 0, \quad i = 1, 2, \dots, d. \quad (4.11)$$

The Dynkin Theorem (see [25]) claims that  $M(t) = V(\tilde{X}_t^A, \lambda_t)$  is a martingale for any  $V$  such that

$$\mathcal{L}^{A(\cdot)} V = 0. \quad (4.12)$$

Since the main purpose of this paper is to find an optimal exponential upper bound for ruin probability and corresponding optimal investment piecewise constant policy, motivated by Grandell [22] (Prop. 52 of Chapt. 4), we restrict ourself to function  $V$  with the form of

$$V(u, \alpha_i) = g(\alpha_i) e^{-ru}, \quad i = 1, 2, \dots, d. \quad (4.13)$$

**Theorem 4.1.** Fix  $A > 0$ , for any  $i = 1, 2, \dots, d$ , if there exist  $g : R_+ \mapsto R_+$  and  $R_i(A) > 0, i = 1, 2, \dots, d$  such that

$$\alpha_i [\mathbb{E} e^{rY} - 1] + \frac{1}{2} A^2 \sigma^2 r^2 g(\alpha_i) - r[c(\alpha_i)g(\alpha_i) + A\tilde{\mu}g(\alpha_i)] + \sum_{j=1}^d q_{ij}g(\alpha_j) \equiv 0, \quad (4.14)$$

then,

$$\psi^A(u, \alpha_i) \leq \frac{g(\alpha_i) e^{-R_i(A)u}}{\mathbb{E}[g(\lambda_{T(u,A)}) 1_{\{T(u,A) < \infty\}} | \lambda_0 = \alpha_i]}. \quad (4.15)$$

**Proof.** Plugging (4.13) into (4.12), it is easy to see that

$$\mathcal{L}^A V(u, \alpha_i) = e^{-ru} \left[ \alpha_i [\mathbb{E} e^{rY} - 1] + \frac{1}{2} A^2 \sigma^2 r^2 g(\alpha_i) - r[c(\alpha_i)g(\alpha_i) + A\tilde{\mu}g(\alpha_i)] + \sum_{j=1}^d q_{ij}g(\alpha_j) \right],$$

which shows that Eq. (4.14) is equivalent to (4.12). Therefore

$$M(t, R_i(A), A) := g(\lambda_t) e^{-R_i(A)\tilde{X}_t^A} \quad (4.16)$$

is a  $\mathcal{F}$ -martingale. By the optional sampling theorem, we have

$$\begin{aligned} \mathbb{E}[g(\lambda_0) e^{-R_i(A)u}] &= \mathbb{E}[M(0, R_i(A), A)] = \mathbb{E}[M(T(u, A) \wedge n, R_i(A), A)] \\ &\geq \mathbb{E}[M(T(u, A) \wedge n, R_i(A), A) 1_{\{T(u, A) \leq n\}}] \\ &= \mathbb{E}[M(T(u, A) \wedge n, R_i(A), A) | T(u, A) \leq n] \mathbb{P}(T(u, A) \leq n). \end{aligned} \quad (4.17)$$

Thus,

$$\psi_n^A(u, \alpha_i) = \mathbb{P}(T(u, A) \leq n | \lambda_0 = \alpha_i) \leq \frac{g(\alpha_i) e^{-R_i(A)u}}{\mathbb{E}[M(T(u, A) \wedge n, R_i(A), A) | T(u, A) \leq n]}. \quad (4.18)$$

Let  $n \rightarrow \infty$ , note that  $\tilde{X}_{T(u,A)}^A < 0$  and thus  $e^{-R_i(A)\tilde{X}_{T(u,A)}^A} > 1$ , we have

$$\begin{aligned} \psi^A(u, \alpha_i) &= \mathbb{P}(T(u, A) < \infty | \lambda_0 = \alpha_i) \leq \frac{g(\lambda_0) e^{-R_i(A)u}}{\mathbb{E}[M(T(u, A), R_i(A), A) | T(u, A) < \infty]} \\ &< \frac{g(\alpha_i) e^{-R_i(A)u}}{\mathbb{E}[g(\lambda_{T(u,A)}) 1_{\{T(u,A) < \infty\}} | \lambda_0 = \alpha_i]}. \end{aligned} \quad (4.19)$$

This completes the proof.



Let

$$C(\alpha_i, A) := \frac{g(\alpha_i)}{\mathbb{E}[g(\lambda_{T(u,A)}) \mathbf{1}_{\{T(u,A) < \infty\}} | \lambda_0 = \alpha_i]},$$

$$C(\alpha_i) := \max_{A \in \mathcal{A}_e} C(\alpha_i, A).$$

Since  $\hat{\psi}(u, \alpha_i) = \inf_{A(\cdot, \cdot) \in \mathcal{A}} \psi^{A(\cdot, \cdot)}(u, i)$ , we have

$$\hat{\psi}(u, \alpha_i) \leq C(\alpha_i, A) e^{-R_i(A)u} \leq C(\alpha_i) e^{-R_i(A)u}. \quad (4.20)$$

The purpose of this section is to find the “tightest” upper bound for  $\hat{\psi}(u, \alpha_i)$ . One should note that the coefficient  $R_i(A)$  depends on the value of  $A$  and current state of the intensity process  $\lambda_t$ . To obtain the tightest upper bound, it is sufficient to find the maximum of  $R_i(A)$  over all  $A$ . Denote by  $R_i^*$  the maximum of  $R_i(A)$  and  $A_i^*$  is the maximizer of  $R_i^*$ . Then we have

$$\hat{\psi}(u, \alpha_i) \leq \psi^{A_i^*}(u, \alpha_i) \leq C(\alpha_i) e^{-R_i^* u}. \quad (4.21)$$

Following this procedure, we can determine a sequence of investment policies which minimize the upper bound of ruin probability and the policies vary w.r.t. the state of  $\lambda_t$ . Lemma 4.2 defines the relationship between  $A_i^*$  and  $R_i^*$  and provides the method to find the expressions of  $A_i^*$  and  $R_i^*$ .

**Lemma 4.2.** For any fixed  $A$ , if  $\sum_{j=1}^d q_{ij} g(\alpha_j) < 0$ , then there always exists a positive  $R_i(A)$  such that Eq. (4.14) holds and only  $A_i^* = \frac{\tilde{\mu}}{\sigma^2 R_i^*}$  minimizes the left hand side of Eq. (4.14), which results in a maximum  $R_i^*$ .

**Proof.** Let

$$h(r, A) := \alpha_i [\mathbb{E} e^{rY} - 1] + \frac{1}{2} A^2 \sigma^2 r^2 g(\alpha_i) - r [c(\alpha_i) g(\alpha_i) + A \tilde{\mu} g(\alpha_i)] + \sum_{j=1}^d q_{ij} g(\alpha_j)$$

$$h_1(r) := \alpha_i [\mathbb{E} e^{rY} - 1] - rc(\alpha_i) g(\alpha_i)$$

$$h_2(r) := - \left[ \frac{1}{2} A^2 \sigma^2 r^2 g(\alpha_i) - A \tilde{\mu} r g(\alpha_i) + \sum_{j=1}^d q_{ij} g(\alpha_j) \right]. \quad (4.22)$$

With the assumption, for any fixed  $A$ ,  $h(0, A) = \sum_{j=1}^d q_{ij} g(\alpha_j) < 0$  and  $\frac{\partial^2 h(r, A)}{\partial r^2} = \alpha_i \mathbb{E}[Y^2 e^{rY}] + A^2 \sigma^2 g(\alpha_i) > 0$ . Thus there must exist a unique positive  $R_i(A)$  such that Eq. (4.14) holds. It is easy to see that  $A = \frac{\tilde{\mu}}{\sigma^2 r}$  is the maximizer of  $h_2(r)$  for all  $r$ . Note that for any fixed  $A$ , root of Eq. (4.14) is the intersection of  $h_1(r)$  and  $h_2(r)$ . Since  $h_1(0) < 0$  and  $\frac{d^2 h_1(r)}{dr^2} > 0$ , it follows that  $A = \frac{\tilde{\mu}}{\sigma^2 r}$  yields the maximum  $r$  satisfying Eq. (4.14). This root is dependent on the current state of intensity process  $\lambda_t$  and denote it by  $R_i^*$ . Consequently, the optimal investment constant policy is determined by  $A_i^* = \frac{\tilde{\mu}}{\sigma^2 R_i^*}$ . This completes the proof.  $\square$

Given that current state of  $\lambda_t$  is  $\alpha_i$ , by Lemma 3.2 we know that  $R_i^*$  is the solution to the following equation.

$$\alpha_i [\mathbb{E} e^{rY} - 1] - rc(\alpha_i) g(\alpha_i) - \frac{\tilde{\mu}}{2\sigma^2} g(\alpha_i) + \sum_{j=1}^d q_{ij} g(\alpha_j). \quad (4.23)$$

The following theorem summarizes previous discussions.

**Theorem 4.3.** The optimal piecewise constant policy for minimizing the upper bound of ruin probability is specified as

$$A^*(\lambda_t) = A_i^*, \quad \text{given that } \lambda_t = \alpha_i, \quad (4.24)$$

where  $A_i^* = \frac{\tilde{\mu}}{\sigma R_i^*}$  and  $R_i^*$  is determined by Eq. (4.23).

**Remark 4.** Denote by  $f(\alpha_i, r, A(\cdot, \cdot))$  the left hand side of Eq. (4.14) by replacing  $A$  with  $A(\cdot, \cdot)$ , then

$$f(\alpha_i, R_i^*, A(\cdot, \cdot)) = f(\alpha_i, R_i^*, A_i^*) - (A(\cdot, \cdot) - A_i^*) R_i^* \tilde{\mu} g(\alpha_i) + \frac{1}{2} [(A(\cdot, \cdot)^2 - A_i^{*2})] \sigma^2 R_i^{*2} g(\alpha_i)$$

$$= f(\alpha_i, R_i^*, A_i^*) - (A(\cdot, \cdot) - A_i^*) R_i^* \tilde{\mu} g(\alpha_i)$$

$$+ \frac{1}{2} [(A(\cdot, \cdot) - A_i^*)^2 + 2A_i^* (A(\cdot, \cdot) - A_i^*)] \sigma^2 R_i^{*2} g(\alpha_i). \quad (4.25)$$

Note that  $A_i^* = \frac{\tilde{\mu}}{\sigma^2 R_i^*}$  and  $f(\alpha_i, R_i^*, A_i^*) = 0$ , Eq. (4.25) can be reformulated as

$$f(\alpha_i, R_i^*, A(\cdot, \cdot)) = f(\alpha_i, R_i^*, A_i^*) + \frac{1}{2}(A(\cdot, \cdot) - A_i^*)^2 = \frac{1}{2}(A(\cdot, \cdot) - A_i^*)^2 > 0. \quad (4.26)$$

This means that

$$M(t, R_i^*, A(\cdot, \cdot)) := g(\lambda_t) e^{-R_i^* X_t^A} \quad (4.27)$$

is a submartingale for any investment policy  $A(\cdot, \lambda_t) \neq A^*(\lambda_t)$  and we cannot have  $\psi_i(u, A(\cdot, \cdot)) \leq C(\alpha_i) e^{-R_i^* A_i^*}$ . This indicates that Eq. (4.19) only holds for the piecewise constant policy  $A^*(\lambda_t)$  and thus the optimal investment policy in  $A^*(\cdot, \cdot) \in \mathcal{A}$  can be approximated by optimal piecewise constant policies  $A^*(\cdot) \in \mathcal{A}_c$  when the initial value  $u$  tends to infinity. However, the statements are not strict in mathematics. The rest of this section gives the proof of such approximation when the claims have uniform exponential moment in tail distribution.  $\square$

**Definition 4.4** (c.f. Gaier et al. [20]). We say that  $\xi$  has a uniform exponential moment in the tail distribution for  $r$ , if

$$\sup_{y \geq 0} \mathbb{E} \left[ e^{-r(y-\xi)} \middle| \xi > y \right] < \infty. \quad (4.28)$$

The proofs of the following two Lemmas are similar to that of Theorem 4 and Lemma 5 of Gaier et al. [20] and we state it without proof.

**Lemma 4.5.** Assume that  $Y$  has a uniform exponential moment in the tail distribution for  $R_i^*$ ; then for each  $A(\cdot, \cdot) \in \mathcal{A}$ , the process  $M(t, R_i^*, A(\cdot, \cdot))$  is a uniformly integrable submartingale.

**Lemma 4.6.** Assume that  $Y$  has a uniform exponential moment in the tail distribution for  $R_i^*$ ; then for each  $A(\cdot, \cdot) \in \mathcal{A}$  and  $u > 0$ , the stopping process

$$\tilde{X}_t^{A(\cdot, \cdot)} := X_{t \wedge T(u, A(\cdot, \cdot))}^{A(\cdot, \cdot)} \quad (4.29)$$

converges almost surely on  $\{T(u, A(\cdot, \cdot)) = \infty\}$  to  $\infty$  when  $t \rightarrow \infty$ . In other words, either ruin occurs, or the insurer becomes infinitely rich. As a result, we know that

$$\tilde{M}(t, R_i^*, A(\cdot, \cdot)) := M(t \wedge T(u, A), R_i^*, A(\cdot, \cdot)) \quad (4.30)$$

converges to 0 as  $t \rightarrow \infty$  on the set  $\{T(u, A(\cdot, \cdot)) = \infty\}$ .

**Theorem 4.7.** Assume that  $Y$  has a uniform exponential moment in the tail distribution for  $R_i^*$ ; then for each  $A(\cdot, \cdot) \in \mathcal{A}$ ,  $\lambda_0 = \alpha_i$  and  $R_i^*$  we have

$$\begin{aligned} \psi^{A(\cdot, \cdot)}(u, \alpha_i) &= \mathbb{P}(T(u, A(\cdot, \cdot)) < \infty | \lambda_0 = \alpha_i) \\ &\geq \frac{g(\alpha_i) e^{-R_i^* u}}{\mathbb{E}[M(T(u, A(\cdot, \cdot)), R_i^*, A(\cdot, \cdot)) | T(u, A(\cdot, \cdot)) < \infty]} \\ &\geq \underline{C} e^{-R_i^* u}, \end{aligned} \quad (4.31)$$

where

$$\underline{C} = \frac{1}{\sup_{y \geq 0} \mathbb{E}[e^{-R_i^*(y-Y)} | Y > y]} > 0. \quad (4.32)$$

In particular, we have

$$\hat{\psi}(u, \alpha_i) \geq \underline{C} e^{-R_i^* u}. \quad (4.33)$$

**Proof.** Using a similar argument to that in Eq. (4.19), we have

$$\begin{aligned} g(\alpha_i) e^{-R_i^* u} &= \tilde{M}(0, R_i^*, A) \leq \mathbb{E}[\tilde{M}(T(u, A), R_i^*, A)] \\ &= \mathbb{E}[\tilde{M}(T(u, A), R_i^*, A) | T(u, A) < \infty] \mathbb{P}(T(u, A) < \infty) \\ &\quad + \mathbb{E}[\tilde{M}(T(u, A), R_i^*, A) | T(u, A) = \infty] \mathbb{P}(T(u, A) = \infty) \\ &= \mathbb{E}[\tilde{M}(T(u, A), R_i^*, A) | T(u, A) < \infty] \mathbb{P}(T(u, A) < \infty). \end{aligned}$$

Note that investment cannot cause ruin (c.f. [26]); thus ruin can only be caused by claim. Suppose that  $X_{T(u,A)-}^A = y > 0$ , then

$$\mathbb{E}[\tilde{M}(T(u, A), R_i^*, A) | T(u, A) < \infty] \leq \sup_{y \geq 0} \mathbb{E}[e^{-R_i^*(y-Y)} | Y > y]. \quad (4.34)$$

This completes the proof.  $\square$

**Remark 5.** What can we say from Eq. (4.21) and Theorem 4.7? One can find that the Lundberg upper bound (4.21) does not hold for exponent  $R_i^*$  when  $A(\cdot, \cdot) \neq A^*$ , note that  $R_i^*$  is defined as the supremum of  $R_i(A)$ , Eq. (4.31) indicates that  $R_i^*$  is the maximal adjustment coefficient (i.e. the minimal upper bound for ruin probability) we can obtain over all  $A(\cdot, \cdot) \in \mathcal{A}$ . The optimal piecewise constant policy corresponding to adjustment coefficient  $R_i^*$  is  $A^* = \frac{\mu}{\sigma^2 R_i^*}$ .  $\square$

**Lemma 4.8.** Assume that  $Y$  has a uniform exponential moment in the tail distribution for  $R_i^*$ , let  $A(\cdot, \cdot)$  be the determining function of Markovian control process  $A_t$ . If there exist  $\epsilon > 0$  and  $u_\epsilon \geq 0$  such that

$$|A(u, \alpha_i) - A_i^*| \geq \epsilon \quad \text{for } u \geq u_\epsilon, \quad (4.35)$$

then, there exist  $r_\epsilon < R_i^*$  and  $A_\epsilon > 0$  such that

$$\psi^{A(\cdot, \cdot)}(u, \alpha_i) \geq A_\epsilon e^{-r_\epsilon u}. \quad (4.36)$$

**Theorem 4.9.** Let  $A^*(\cdot, \cdot)$  be the determining function of optimal Markov control policy of problem (4.8), then we have

$$\lim_{u \rightarrow \infty} A^*(u, \alpha_i) = A_i^*, \quad \text{for all } i = 1, 2, \dots, d. \quad (4.37)$$

**Proof.** Assume that  $\lim_{u \rightarrow \infty} A^*(u, \alpha_i) \neq A_i^*$ , then there exists  $\epsilon, u_\epsilon > 0$  such that

$$|A^*(u, \alpha_i) - A_i^*| \geq \epsilon \quad \text{for } u \geq u_\epsilon. \quad (4.38)$$

Therefore, by Lemma 4.8 we have

$$\hat{\psi}(u, \alpha_i) = \inf_{A(\cdot, \cdot) \in \mathcal{A}} \psi^{A(\cdot, \cdot)}(u, \alpha_i) \geq A_\epsilon e^{-r_\epsilon u} \quad \text{for some } r_\epsilon < R_i^*, \quad (4.39)$$

which yields that

$$\lim_{u \rightarrow \infty} \frac{V(u, \alpha_i)}{e^{-R_i^* u}} = \infty, \quad (4.40)$$

which is a contradiction to the fact that

$$\hat{\psi}(u, \alpha_i) \leq \inf_{A \in \mathcal{A}_e} \psi^A(u, \alpha_i) = \psi^{A_i^*}(u, \alpha_i) \leq C(\alpha_i) e^{-R_i^* u}. \quad (4.41)$$

**Remark 6.** One should note that the optimal piecewise constant policy is for the “discounted” risk process (that is the force of interest is  $\delta = 0$ ). Otherwise, the result is slightly different. If  $\delta \neq 0$ , by simple calculation, it is easy to see that the optimal investment policy at time  $t$  is given by  $e^{\delta t} A^*(\lambda_t)$ , where  $A^*(\lambda_t)$  is specified by Eq. (4.24).

**Remark 7.** What is the message of our result from practical point of view? When the initial surplus of an insurer is very large, for the optimal investment problem, minimizing ruin probability is a very conservative approach, especially in the sense of asymptotical optimality. Another remarkable fact, which follows from our analysis, is that, by incorporating additional risks (investment return from risky asset) we can decrease the ruin probability. And such decrease is quite substantial and leads to a different exponential decay for the ruin probability. Thus, when an insurer tries to invest in risky asset, even under a very conservative risk measure (e.g. ruin probability), optimal policy is still important.  $\square$

## Acknowledgments

Lin Xu would like to acknowledge support by Natural Science Foundation of China (Grant No. 11126238, 11201006) and Humanities and Social Sciences Fund of Ministry of Education, China (Grant No. 12YJC910012). Hailiang Yang would like to acknowledge the Research Grants Council of the Hong Kong Special Administrative Region, China (Grant No. HKU 706611P). Rongming Wang would like to acknowledge support by National Natural Science Foundation of China (11231005), Doctoral

Program Foundation of the Ministry of Education of China (20110076110004), Program for New Century Excellent Talents in University (NCET-09-0356) and “the Fundamental Research Funds for the Central Universities”.

## Appendix

For notational simplicity, we only present the bound for ruin probability  $\psi(u)$  and the idea can be extended to  $\psi(u, \alpha_i)$  easily. Note that ruin only occurs when a claim arrives, thus we can consider the so-called “skeleton-process” of process (3.2) in studying ruin probability. Denote the “discounted skeleton risk process” of process (3.2) by

$$\begin{aligned} M_n &:= e^{-\delta L_n} X_{L_n} = e^{-\delta L_n} \left[ X_{L_{n-1}} e^{\delta(L_n - L_{n-1})} + \int_{L_{n-1}}^{L_n} e^{L_n - \delta r} c(\lambda_r) dr - Y_n \right] \\ &= M_{n-1} + e^{-\delta L_{n-1}} \left[ \int_{L_{n-1}}^{L_n} e^{-\delta(r - L_{n-1})} c(\lambda_r) dr - Y_n e^{-\delta(L_n - L_{n-1})} \right] \end{aligned} \quad (\text{A.1})$$

with the convention that  $L_0 = 0$ . Obviously,

$$\bar{F}(x) = \left( \frac{\int_t^\infty e^{Ry} dF(y)}{e^{Rt} \bar{F}(t)} \right)^{-1} e^{-Rx} \int_x^\infty e^{Ry} dF(y) \leq \varrho e^{-Rx} \int_x^\infty e^{Ry} dF(y), \quad (\text{A.2})$$

where  $\varrho^{-1} = \inf_{t \geq 0} \frac{\int_t^\infty e^{-Ru} dF(u)}{e^{Rt} \bar{F}(t)}$ . Consider whether the first claim causes ruin or not; we have the following recursive formula.

$$\begin{aligned} \psi_n(u) &= \mathbb{P}(T \leq L_n) = \mathbb{P}\left(\bigcup_{k=1}^n \{M_k < 0\} | M_0 = u\right) \\ &= \mathbb{P}\left(\bigcup_{k=1}^n M_k < 0 | M_1 < 0\right) \mathbb{P}(M_1 < 0) + \mathbb{P}\left(\bigcup_{k=1}^n \{M_k < 0\} | M_1 > 0\right) \mathbb{P}(M_1 > 0) \\ &= \mathbb{P}\left(Y_1 > ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr\right) \\ &\quad + \mathbb{E}\left[\mathbb{P}\left(\bigcup_{k=2}^n \{M_k < 0\} \mid M_1 = u + \int_0^{L_1} e^{-\delta r} c(\lambda_r) dr - Y_1 e^{-\delta L_1}\right)\right] \\ &\quad \times \mathbb{P}\left(Y_1 < ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr\right) \\ &= \mathbb{E}\left[\bar{F}\left(ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr\right)\right. \\ &\quad \left.+ \int_0^{ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr} \psi_{n-1}\left(ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr - y\right) dF(y)\right]. \end{aligned} \quad (\text{A.3})$$

For  $n = 1$ , by inequality (A.2), it follows that

$$\begin{aligned} \psi_1(u) &= \mathbb{E}\left[\bar{F}\left(ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr\right)\right] \\ &\leq \mathbb{E}\left[\varrho e^{-R\left[ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr\right]} \int_{ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr}^\infty e^{Ry} dF(y)\right] \end{aligned} \quad (\text{A.4})$$

$$\leq \varrho e^{-Ru} \mathbb{E} e^{-R\left[u(e^{\delta L_1} - 1) + \int_0^{L_1} e^{\delta(L_1 - r)} c(\lambda_r) dr - Y\right]} \quad (\text{A.5})$$

$$= \varrho e^{-Ru} \mathbb{E} \left[ e^{-R(X_{L_1} - X_0)} \right] = \varrho e^{-Ru}. \quad (\text{A.6})$$

By an inductive method, we suppose that for  $n = k$  and  $u > 0$

$$\psi_k(u) \leq \varrho e^{-Ru}, \quad (\text{A.7})$$

then for  $n = k + 1$ ,

$$\begin{aligned}
 \psi_{k+1}(u) &= \mathbb{E} \left[ \bar{F}(ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr) \right. \\
 &\quad \left. + \int_0^{ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr} \psi_k \left( ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr - y \right) dF(y) \right] \\
 &\leq \mathbb{E} \left[ \varrho e^{-R \left[ ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr \right]} \int_{ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr}^{\infty} e^{Ry} dF(y) \right. \\
 &\quad \left. + \int_0^{ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr} \varrho e^{-R \left( ue^{\delta L_1} + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr - y \right)} dF(y) \right] \\
 &= \varrho e^{-Ru} \mathbb{E} \left[ \exp \left\{ -R \left[ u(e^{\delta L_1} - 1) + \int_0^{L_1} e^{\delta(L_1-r)} c(\lambda_r) dr - Y \right] \right\} \right] \\
 &= \varrho e^{-Ru} \mathbb{E} \left[ e^{-R(X_1 - X_0)} \right] = \varrho e^{-Ru}. \tag{A.8}
 \end{aligned}$$

The second step of the previous equation comes from (A.4). Let  $k \rightarrow \infty$  and note that  $\lim_{k \rightarrow \infty} \psi_k(u) = \psi(u)$ , this completes the proof.  $\square$

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