



Matrix decomposition algorithms for arbitrary order C^0 tensor product finite element systems

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ABSTRACT

Matrix decomposition algorithms (MDAs) are fast direct methods for the solution of systems of linear algebraic equations which arise in the approximation of Poisson's equation on the unit square using various techniques such as finite difference, spline collocation and spectral methods. The attraction of MDAs is that they employ fast Fourier transforms and require $O(N^2 \log N)$ operations on an $N \times N$ uniform partition of the unit square. In this paper, MDAs are formulated for the solution of the finite element Galerkin equations arising when spaces of C^0 piecewise polynomials of degree $k \geq 3$ are employed. Results of numerical experiments exhibit the expected optimal global convergence rates and super-convergence phenomena.

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1. Introduction

In [1], a comprehensive survey is given of fast direct methods, called matrix decomposition algorithms (MDAs), for solving certain systems of linear algebraic equations of the form

$$(A_1 \otimes B_2 + B_1 \otimes A_2) \mathbf{u} = \mathbf{f}, \quad (1.1)$$

where A_1 and B_1 are $M_1 \times M_1$ matrices, A_2 and B_2 are $M_2 \times M_2$ and \otimes denotes the matrix tensor product. Such systems arise in various commonly used techniques, such as finite difference, spline collocation and spectral methods, for solving Poisson's equation

$$-\Delta u = f(x, y) \quad (x, y) \in \Omega, \quad (1.2)$$

where $\Omega = (0, 1) \times (0, 1)$, the unit square, with boundary $\partial\Omega$, and Δ denotes the Laplace operator. The attraction of MDAs is that they employ fast Fourier transforms and require $O(N^2 \log N)$ operations on an $N \times N$ uniform partition of the unit square.

In this paper, we consider (1.2) subject to the following boundary conditions (BCs): on the horizontal sides of $\partial\Omega$, homogeneous Dirichlet BCs

$$u(x, 0) = u(x, 1) = 0, \quad x \in (0, 1), \quad (1.3)$$

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and on the vertical sides of $\partial\Omega$, that is, for $y \in [0, 1]$, one of the following:

$$u(0, y) = u(1, y) = 0 \quad (\text{Dirichlet BCs}); \quad (1.4)$$

$$u_x(0, y) = u_x(1, y) = 0 \quad (\text{Neumann BCs}); \quad (1.5)$$

$$u(0, y) = u_x(1, y) = 0 \quad (\text{Dirichlet–Neumann BCs}); \quad (1.6)$$

$$u_x(0, y) = u(1, y) = 0 \quad (\text{Neumann–Dirichlet BCs}); \quad (1.7)$$

$$u(0, y) = u(1, y), \quad u_x(0, y) = u_x(1, y) \quad (\text{periodic BCs}). \quad (1.8)$$

For each choice of boundary conditions, we consider the solution of (1.2) using the finite element Galerkin (FEG) method with tensor products of C^0 piecewise polynomials of degree $k \geq 3$, and formulate an MDA for the solution of the resulting Galerkin equations. This work is a substantial generalization of the case $k = 2$ which is considered in [2]; see also [3].

As indicated in [1], there is very little in the literature on the use of MDAs in FEG methods. The case $k = 1$ is described in [1] for Poisson problems, and in [4], MDAs are developed for problems arising in fluid dynamics, linear elasticity and electromagnetics. For the case in which the FEG with piecewise Hermite (C^1) bicubics is used, MDAs are described in [5]. Kaufman and Warner [6,7] developed and implemented MDAs for the FEG method with B-splines of arbitrary order applied to more general elliptic problems such that the matrices A_1 and B_1 are symmetric and positive definite. In such an MDA, FFTs cannot be used and, as a consequence, the total cost of the algorithm is $O(N^3)$ on an $N \times N$ partition, which, however, can be nonuniform. Spectral Galerkin MDAs for Helmholtz problems using ultraspherical polynomials are presented in [8,9]. This approach is extended to spectral Galerkin MDAs for Helmholtz problems using Jacobi polynomials in [10,11]. Spectral–Galerkin approaches are also applied to Helmholtz Neumann problems in [12] and to fourth order problems in [10,13] using Jacobi polynomials, and to problems of order $2n$ in [14] using ultraspherical polynomials.

In the first step of our algorithm, we determine an orthogonal matrix Q such that $D^{A_1} = Q^T A_1 Q$ and $D^{B_1} = Q^T B_1 Q$ are block diagonal with M diagonal blocks, denoted by

$$D^{A_1} = \text{diag} \{D_1^{A_1}, D_2^{A_1}, \dots, D_M^{A_1}\} \quad \text{and} \quad D^{B_1} = \text{diag} \{D_1^{B_1}, D_2^{B_1}, \dots, D_M^{B_1}\},$$

where $D_i^{A_1}$ and $D_i^{B_1}$ are each of order k_i , where $\sum_{i=1}^M k_i = M_1$. Then the system (1.1) is transformed into the system

$$(D^{A_1} \otimes B_2 + D^{B_1} \otimes A_2)\mathbf{v} = \mathbf{g}, \quad (1.9)$$

where $\mathbf{g} = (Q^T \otimes I_{M_2})\mathbf{f}$ and I_{M_2} denotes the unit matrix of order M_2 , and $\mathbf{u} = (Q \otimes I_{M_2})\mathbf{v}$. Note that the matrix $D^{A_1} \otimes B_2 + D^{B_1} \otimes A_2$ is block diagonal, with diagonal blocks $D_i^{A_1} \otimes B_2 + D_i^{B_1} \otimes A_2$, $i = 1, 2, \dots, M$. To solve (1.9), we determine a diagonal matrix Λ_i and a nonsingular matrix Z_i such that

$$D_i^{A_1} Z_i = D_i^{B_1} Z_i \Lambda_i, \quad Z_i^T D_i^{B_1} Z_i = I_{k_i} \quad i = 1, \dots, M, \quad (1.10)$$

where I_{k_i} denotes the unit matrix of order k_i . From (1.10), it follows that

$$Z_i^T D_i^{A_1} Z_i = \Lambda_i, \quad i = 1, \dots, M. \quad (1.11)$$

Thus, to solve (1.1), we use the following MDA:

Matrix Decomposition Algorithm

- | | |
|---------------|--|
| (i) Compute | Λ_i and Z_i , $i = 1, \dots, M$, satisfying (1.10). |
| (ii) Compute | $\mathbf{g} = (Q^T \otimes I_{M_2})\mathbf{f}$. |
| (iii) Compute | $\mathbf{p}_i = (Z_i^T \otimes I_{M_2})\mathbf{g}_i$, $i = 1, \dots, M$. |
| (iv) Solve | $(\Lambda_i \otimes B_2 + I_{k_i} \otimes A_2)\mathbf{q}_i = \mathbf{p}_i$, $i = 1, \dots, M$. |
| (v) Compute | $\mathbf{v}_i = (Z_i \otimes I_{M_2})\mathbf{q}_i$, $i = 1, \dots, M$. |
| (vi) Compute | $\mathbf{u} = (Q \otimes I_{M_2})\mathbf{v}$. |

To determine the computational cost of the MDA, first note that k and k_i , $i = 1, \dots, M$, are independent of M , M_1 and M_2 , and secondly M is $\mathcal{O}(N)$, while M_1 and M_2 are $\mathcal{O}(kN)$. Then, using the QR algorithm, the cost of step (i) is $\mathcal{O}(k^3 N)$ operations. In the tensor product systems of the form (1.1) considered in this paper, A_1 , A_2 , B_1 and B_2 have an interesting block structure (cf., Fig. 1) which can be exploited so that step (iv) of the algorithm requires $\mathcal{O}(k^4 N^2)$ operations. The matrix–vector products in steps (iii) and (v) cost $\mathcal{O}(k^3 N^2)$ operations, while the matrix–vector products in steps (ii) and (vi) can be performed using fast Fourier transforms (FFTs) at a cost of $\mathcal{O}(k^2 N^2 \log N)$ operations. Thus the total cost of the algorithm is $\mathcal{O}(k^2 N^2 \log N + k^4 N^2)$ operations.

A brief outline of the remainder of this paper is as follows. In Section 2, we introduce the spaces of piecewise polynomials employed in the FEG method, and describe the related mass and stiffness matrices and the structure of the Galerkin equations. In Sections 3–6, we formulate MDAs for Dirichlet, Neumann, mixed and periodic boundary conditions, respectively. Results of numerical experiments for each choice of boundary conditions are presented in Section 7. These demonstrate the expected optimal global error estimates and superconvergence phenomena. Section 8 comprises concluding remarks including a brief description of future research projects.

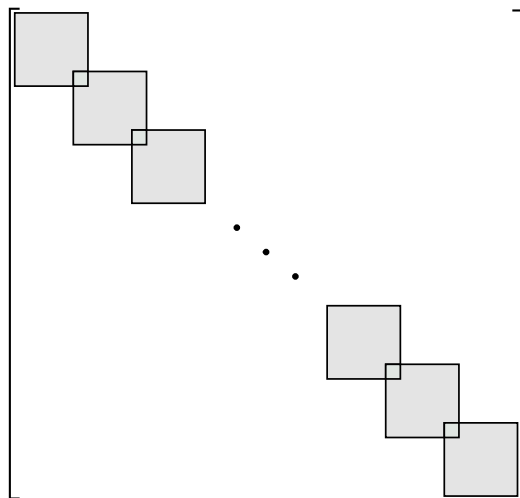


Fig. 1. The sparsity structure of the stiffness and mass matrices \mathcal{A} and \mathcal{B} .

2. The finite element Galerkin (FEG) method

Let $\{L_j\}_{j=0}^k$ denote the basis of standard Lagrange interpolation on the $k+1$ uniform mesh points $t_j = j/k, j = 0, 1, \dots, k$, on $[0, 1]$. Then

$$L_j(t) = \prod_{i=0, i \neq j}^k \frac{t - t_i}{t_j - t_i}$$

so that

$$L_j(t_i) = \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \quad (2.1)$$

Let $\{x_i\}_{i=0}^N$ be a uniform partition of $[0, 1]$ such that $x_i = ih, i = 0, \dots, N$, where N is a positive integer and $h = 1/N$ is the meshsize. We denote by $\bar{A} = [\bar{a}_{ij}]_{i,j=1}^{k+1}$ and $\bar{B} = [\bar{b}_{ij}]_{i,j=1}^{k+1}$ the $(k+1) \times (k+1)$ element stiffness and mass matrices, respectively, with

$$\bar{a}_{ij} = \frac{1}{h} \int_0^1 L'_{i-1}(t) L'_{j-1}(t) dt, \quad \bar{b}_{ij} = h \int_0^1 L_{i-1}(t) L_{j-1}(t) dt, \quad i, j = 1, 2, \dots, k+1. \quad (2.2)$$

We have the following lemma.

Lemma 2.1. *The matrices \bar{A} and \bar{B} are symmetric positive semi-definite and symmetric positive definite, respectively, and satisfy*

$$\bar{a}_{ij} = \bar{a}_{k+2-i, k+2-j}, \quad \bar{b}_{ij} = \bar{b}_{k+2-i, k+2-j}, \quad i, j = 1, \dots, k+1. \quad (2.3)$$

Proof. It follows from (2.2) that the matrices are symmetric; they are also positive semi-definite since they are Gram matrices. Since $\{L_j\}_{j=0}^k$ are linearly independent, it follows that the matrix \bar{B} is positive definite. From the basic properties of the basis $\{L_j\}_{j=0}^k$, it is easy to show that (2.3) holds.

From Lemma 2.1, it follows that there are at most $\nu(k)$ different elements in the $(k+1) \times (k+1)$ matrices \bar{A} and \bar{B} , where

$$\nu(k) = \begin{cases} \frac{k^2 + 4k + 3}{4}, & \text{if } k \text{ is odd,} \\ \frac{k^2 + 4k + 4}{4}, & \text{if } k \text{ is even.} \end{cases}$$

Let $\gamma_i, i = 1, \dots, \nu(k)$, denote these elements, and set $\gamma = [\gamma_1, \dots, \gamma_{\nu(k)}]$. Let $n_l = (k - l + 2)l$ with

$$l = \begin{cases} 1, \dots, \mu(k), & \text{if } k \text{ is odd,} \\ 1, \dots, \mu(k) + 1, & \text{if } k \text{ is even,} \end{cases}$$

and

$$\mu(k) = \begin{cases} \frac{k-1}{2}, & \text{if } k \text{ is odd,} \\ \frac{k-2}{2}, & \text{if } k \text{ is even.} \end{cases}$$

Then, when k is odd, the quantities $\gamma_i, i = 1, 2, \dots, v(k)$, in the matrices \bar{A} and \bar{B} appear as follows (for clarity, we only give the part on and above the diagonal and the anti-diagonal):

$$\begin{array}{cccccccccccc} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \cdots & \cdots & \cdots & \gamma_{k-1} & \gamma_k & \gamma_{n_1} \\ & \gamma_{n_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{n_2} & \\ & & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & & \ddots \\ & & & \gamma_{n_{\mu(k)-1}+1} & \cdots & \cdots & \gamma_{n_{\mu(k)}} & & & \\ & & & & \gamma_{v(k)-1} & \gamma_{v(k)} & & & & \end{array}$$

When k is even, we have

$$\begin{array}{cccccccccccc} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \cdots & \cdots & \gamma_{k-1} & \gamma_k & \gamma_{n_1} \\ & \gamma_{n_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{n_2} & \\ & & \ddots & \cdots & \cdots & \cdots & \cdots & & \ddots \\ & & & \gamma_{n_{\mu(k)}+1} & \cdots & \gamma_{n_{\mu(k)+1}} & & & \\ & & & & \gamma_{v(k)} & & & & \end{array}$$

For any integer $k \geq 3$, let \mathcal{S}_k be the space of C^0 piecewise polynomials on $[0, 1]$ defined by

$$\mathcal{S}_k = \{v \in C^0[0, 1] : v|_{[x_{i-1}, x_i]} \in P_k, i = 1, \dots, N\},$$

where P_k is the set of polynomials of degree $\leq k$, and let

$$\mathcal{S}_k^{\mathcal{D}} = \{v \in \mathcal{S}_k : v(0) = v(1) = 0\}, \quad \mathcal{S}_k^{\mathcal{DN}} = \{v \in \mathcal{S}_k : v(0) = 0\},$$

$$\mathcal{S}_k^{\mathcal{ND}} = \{v \in \mathcal{S}_k : v(1) = 0\}, \quad \mathcal{S}_k^{\mathcal{P}} = \{v \in \mathcal{S}_k : v(0) = v(1)\}.$$

Note that $\dim(\mathcal{S}_k) = kN + 1$, $\dim(\mathcal{S}_k^{\mathcal{D}}) = kN - 1$, and $\dim(\mathcal{S}_k^{\mathcal{DN}}) = \dim(\mathcal{S}_k^{\mathcal{ND}}) = \dim(\mathcal{S}_k^{\mathcal{P}}) = kN$.

We define a basis for $\mathcal{S}_k, \{\phi_\ell\}_{\ell=0}^{Nk}$, by

$$\begin{aligned} \phi_0(x) &= \begin{cases} L_0\left(\frac{x-x_0}{h}\right), & x_0 \leq x \leq x_1, \\ 0, & \text{otherwise,} \end{cases} \\ \phi_{ik}(x) &= \begin{cases} L_k\left(\frac{x-x_{i-1}}{h}\right), & x_{i-1} \leq x \leq x_i, \\ L_0\left(\frac{x-x_i}{h}\right), & x_i \leq x \leq x_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad i = 1, \dots, N-1, \\ \phi_{ik+j}(x) &= \begin{cases} L_j\left(\frac{x-x_i}{h}\right), & x_i \leq x \leq x_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad i = 0, \dots, N-1, j = 1, \dots, k-1, \\ \phi_{kN}(x) &= \begin{cases} L_k\left(\frac{x-x_{N-1}}{h}\right), & x_{N-1} \leq x \leq x_N, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

From (2.1), it follows that

$$\phi_i(jh/k) = \delta_{ij}, \quad i, j = 0, 1, \dots, kN.$$

We define the $(kN + 1) \times (kN + 1)$ matrices \mathcal{A} (the stiffness matrix) and \mathcal{B} (the mass matrix) by

$$\mathcal{A} = [a_{ij}]_{i,j=1}^{kN+1}, \quad a_{ij} = (\phi'_{i-1}, \phi'_{j-1}), \quad \mathcal{B} = [b_{ij}]_{i,j=1}^{kN+1}, \quad b_{ij} = (\phi_{i-1}, \phi_{j-1}),$$

where

$$(\phi, \psi) = \int_0^1 \phi(x) \psi(x) dx.$$

Table 1The values of α and β for various values of k .

| k | 2 | | 3 | | 4 | | 5 | |
|-----|----------------|-----------------|-------------------|-------------------|---------------------|--------------------|-------------------------|--------------------------|
| | α | β | α | β | α | β | α | β |
| 1 | $\frac{7}{3}$ | $\frac{2}{15}$ | $\frac{37}{10}$ | $\frac{8}{105}$ | $\frac{985}{189}$ | $\frac{146}{2835}$ | $\frac{62197}{9072}$ | $\frac{1907}{49896}$ |
| 2 | $-\frac{8}{3}$ | $\frac{1}{15}$ | $-\frac{189}{40}$ | $\frac{33}{560}$ | $-\frac{6848}{945}$ | $\frac{148}{2835}$ | $-\frac{378125}{36288}$ | $\frac{24775}{532224}$ |
| 3 | $\frac{1}{3}$ | $-\frac{1}{30}$ | $\frac{27}{20}$ | $-\frac{3}{140}$ | $\frac{1016}{315}$ | $-\frac{29}{945}$ | $\frac{115625}{18144}$ | $-\frac{9925}{266112}$ |
| 4 | $\frac{16}{3}$ | $\frac{8}{15}$ | $-\frac{13}{40}$ | $\frac{19}{1680}$ | $-\frac{1472}{945}$ | $\frac{4}{405}$ | $-\frac{78125}{18144}$ | $\frac{17125}{798336}$ |
| 5 | | | $\frac{54}{5}$ | $\frac{27}{70}$ | $\frac{347}{945}$ | $-\frac{29}{5670}$ | $\frac{34375}{18144}$ | $-\frac{1525}{266112}$ |
| 6 | | | $-\frac{297}{40}$ | $-\frac{27}{560}$ | $\frac{3328}{189}$ | $\frac{128}{405}$ | $-\frac{2059}{5184}$ | $\frac{493}{177408}$ |
| 7 | | | | | $-\frac{4736}{315}$ | $-\frac{64}{945}$ | $\frac{123125}{4536}$ | $\frac{111625}{399168}$ |
| 8 | | | | | $\frac{5888}{945}$ | $\frac{128}{2835}$ | $-\frac{495625}{18144}$ | $-\frac{24625}{266112}$ |
| 9 | | | | | $\frac{496}{21}$ | $\frac{104}{315}$ | $\frac{304375}{18144}$ | $\frac{2125}{29568}$ |
| 10 | | | | | | | $-\frac{41875}{5184}$ | $-\frac{62875}{1596672}$ |
| 11 | | | | | | | $\frac{198125}{4536}$ | $\frac{62375}{199584}$ |
| 12 | | | | | | | $-\frac{45625}{1296}$ | $-\frac{13625}{133056}$ |

3. Dirichlet boundary conditions

Let $\Gamma_{\mathcal{D}}(\gamma)$ be the matrix obtained by deleting the first and the last rows and columns of the matrix $\Gamma(\gamma)$. Then using the basis $\{\phi_i\}_{i=1}^{kN-1}$ for $\mathcal{S}_k^{\mathcal{D}}$, we have $A_1 = \Gamma_{\mathcal{D}}(\alpha/h)$ and $B_1 = \Gamma_{\mathcal{D}}(h\beta)$ in (1.1). Denote the discrete sine transform matrix [15] by

$$S_{\ell} = \left[\sqrt{\frac{2}{\ell+1}} \left(\sin \frac{ij\pi}{\ell+1} \right) \right]_{i,j=1}^{\ell}.$$

For convenience, in this section, we let

$$\theta_1 = pN + m, \quad \theta_2 = qN + n, \quad \theta_+ = \theta_1 + \theta_2, \quad \theta_- = \theta_1 - \theta_2.$$

The $(pN + m, qN + n)$ entry of the matrix

$$\Sigma_{\mathcal{D}}(\gamma) = S_{kN-1} \Gamma_{\mathcal{D}}(\gamma) S_{kN-1} \quad (3.1)$$

is

$$\sigma_{pN+m, qN+n}^{\mathcal{D}} = \frac{2}{kN} \sum_{i=1}^{v(k)} c_i^{\mathcal{D}}(p, q, m, n) \gamma_i,$$

where, for $l = 0, 1, 2, \dots, \mu(k)$,

$$\begin{aligned} c_{n_l+1}^{\mathcal{D}} &= \sum_{i=1}^N \left[\sin \frac{(ki-k+l)\theta_1\pi}{kN} \sin \frac{(ki-k+l)\theta_2\pi}{kN} + \sin \frac{(ki-l)\theta_1\pi}{kN} \sin \frac{(ki-l)\theta_2\pi}{kN} \right] \\ &= \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k)\theta_-\pi}{2kN} - \cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k)\theta_+\pi}{2kN} \right], \end{aligned} \quad (3.2)$$

$$\begin{aligned} c_{n_l+j}^{\mathcal{D}} &= \sum_{i=1}^N \left[\sin \frac{(ki-k+l)\theta_1\pi}{kN} \sin \frac{(ki-k+j+l-1)\theta_2\pi}{kN} \right. \\ &\quad + \sin \frac{(ki-k+j+l-1)\theta_1\pi}{kN} \sin \frac{(ki-k+l)\theta_2\pi}{kN} \\ &\quad + \sin \frac{(ki-j-l+1)\theta_1\pi}{kN} \sin \frac{(ki-l)\theta_2\pi}{kN} + \sin \frac{(ki-l)\theta_1\pi}{kN} \sin \frac{(ki-j-l+1)\theta_2\pi}{kN} \left. \right] \\ &= 2 \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k+j-1)\theta_-\pi}{2kN} \cos \frac{(j-1)\theta_+\pi}{2kN} \right. \\ &\quad \left. - \cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k+j-1)\theta_+\pi}{2kN} \cos \frac{(j-1)\theta_-\pi}{2kN} \right], \quad j = 2, 3, \dots, k-2l, \end{aligned} \quad (3.3)$$

$$\begin{aligned}
c_{n_{l+1}}^{\mathcal{D}} &= \sum_{i=1}^N \left[\sin \frac{(ki - k + l)\theta_1\pi}{kN} \sin \frac{(ki - l)\theta_2\pi}{kN} + \sin \frac{(ki - l)\theta_1\pi}{kN} \sin \frac{(ki - k + l)\theta_2\pi}{kN} \right] \\
&= \sum_{i=1}^N \left[\cos \frac{(2i - 1)\theta_-\pi}{2N} \cos \frac{(2l - k)\theta_+\pi}{2kN} - \cos \frac{(2i - 1)\theta_+\pi}{2N} \cos \frac{(2l - k)\theta_-\pi}{2kN} \right];
\end{aligned} \quad (3.4)$$

when k is even,

$$c_{v(k)}^{\mathcal{D}} = \sum_{i=1}^N \sin \frac{(2i - 1)\theta_1\pi}{2N} \sin \frac{(2i - 1)\theta_2\pi}{2N} = \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(2i - 1)\theta_-\pi}{2N} - \cos \frac{(2i - 1)\theta_+\pi}{2N} \right]; \quad (3.5)$$

see [Appendix](#) for details.

We classify the first $kN - 1$ positive integers into the following $N + 1$ sets:

$$\begin{cases} J_i^{\mathcal{D}} = \{i, 2N - i, 2N + i, 4N - i, 4N + i, \dots, k_i^{\mathcal{D}}\}, & i = 1, \dots, N - 1, \\ J_N^{\mathcal{D}} = \{N, 3N, 5N, 7N, 9N, \dots, k_N^{\mathcal{D}}\}, \\ J_{N+1}^{\mathcal{D}} = \{2N, 4N, 6N, 8N, 10N, \dots, k_{N+1}^{\mathcal{D}}\}, \end{cases}$$

with

$$\begin{cases} k_i^{\mathcal{D}} = (k - 1)N + i, & k_N^{\mathcal{D}} = (k - 2)N, & k_{N+1}^{\mathcal{D}} = (k - 1)N, & \text{if } k \text{ is odd,} \\ k_i^{\mathcal{D}} = kN - i, & k_N^{\mathcal{D}} = (k - 1)N, & k_{N+1}^{\mathcal{D}} = (k - 2)N, & \text{if } k \text{ is even.} \end{cases}$$

Remark 3.1. In this section, p, q, m, n are integers satisfying $1 \leq pN + m \leq kN - 1$ and $1 \leq qN + n \leq kN - 1$. They are uniquely determined by the $N + 1$ sets $J_i^{\mathcal{D}}, i = 1, 2, \dots, N + 1$. For example, let $N = 10, k = 5$. Then, $1 \leq 19 \leq 5N - 1$; we have $19 = 2N - 1$ or $19 = N + 9$. Obviously, 19 belongs to $J_1^{\mathcal{D}}$. By using the forms of the elements in $J_1^{\mathcal{D}}$, in which p must be even, p and m can be uniquely determined, i.e., $p = 2$ and $m = -1$.

We then have the following lemma, which follows from (3.2)–(3.5) on using the fact that, for any $j, l \in \mathbb{Z}$,

$$\sum_{i=1}^N \cos \frac{(2i - 1)j\pi}{2N} = \begin{cases} N, & j = 4lN, \\ -N, & j = 2N(2l - 1), \\ 0, & \text{otherwise.} \end{cases} \quad (3.6)$$

Lemma 3.1. Let $pN + m \in J_s^{\mathcal{D}}$ and $qN + n \in J_t^{\mathcal{D}}$. If $s \neq t$, then,

$$c_i^{\mathcal{D}} = 0, \quad i = 1, 2, \dots, v(k).$$

If $s = t$, then, for $l = 0, 1, \dots, \mu(k)$,

$$\begin{aligned}
c_{n_{l+1}}^{\mathcal{D}} &= \begin{cases} N \cos \frac{l(p - q)\pi}{k}, & m = n \neq 0, \\ -N \cos \frac{l(p + q)\pi}{k}, & m = -n \neq 0, \\ 2N \sin \frac{lp\pi}{k} \sin \frac{lq\pi}{k}, & m = n = 0, \end{cases} \\
c_{n_{l+j}}^{\mathcal{D}} &= \begin{cases} 2N \cos \frac{(2l + j - 1)(p - q)\pi}{2k} \cos \frac{(j - 1)[(p + q)N + 2m]\pi}{2kN}, & m = n \neq 0, \\ -2N \cos \frac{(2l + j - 1)(p + q)\pi}{2k} \cos \frac{(j - 1)[(p - q)N + 2m]\pi}{2kN}, & m = -n \neq 0, \\ 2N \left[\sin \frac{(l + j - 1)p\pi}{k} \sin \frac{lq\pi}{k} + \sin \frac{lp\pi}{k} \sin \frac{(l + j - 1)q\pi}{k} \right], & m = n = 0, \end{cases} \\
j &= 2, 3, \dots, k - 2l, \\
c_{n_{l+1}}^{\mathcal{D}} &= \begin{cases} N \cos \left[\frac{l((p + q)N + 2m)\pi}{kN} - \frac{m\pi}{N} \right], & m = n \neq 0, \\ -N \cos \left[\frac{l((p - q)N + 2m)\pi}{kN} - \frac{m\pi}{N} \right], & m = -n \neq 0, \\ (-1)^{p+1} 2N \sin \frac{lp\pi}{k} \sin \frac{lq\pi}{k}, & m = n = 0; \end{cases}
\end{aligned}$$

when k is even,

$$c_{v(k)}^{\mathcal{D}} = \begin{cases} (-1)^{\frac{p-q}{2}} \frac{N}{2}, & m = n \neq 0, \\ -(-1)^{\frac{p+q}{2}} \frac{N}{2}, & m = -n \neq 0, \\ \left[(-1)^{\frac{p-q}{2}} - (-1)^{\frac{p+q}{2}} \right] \frac{N}{2}, & m = n = 0. \end{cases}$$

The main theorem of this section is the following, the proof of which follows from Lemma 3.1.

Theorem 3.1. *There exists a permutation matrix P such that $P^T \Sigma_{\mathcal{D}} P = \tilde{\Sigma}_{\mathcal{D}}$, where $\Sigma_{\mathcal{D}}$ is given by (3.1) and $\tilde{\Sigma}_{\mathcal{D}} = \text{diag} \{D_i\}_{i=1}^{N+1}$, where D_i is $k_i \times k_i$ with*

$$k_i = \begin{cases} (k-1)/2, & i = 1, N+1, \\ k, & i = 2, \dots, N, \end{cases}$$

if k is odd, and

$$k_i = \begin{cases} k/2, & i = 1, \\ k, & i = 2, \dots, N, \\ k/2 - 1, & i = N+1, \end{cases}$$

if k is even. If $Q = S_{kN-1}P$, then $Q^T \Gamma_{\mathcal{D}} Q = \tilde{\Sigma}_{\mathcal{D}} = \text{diag} \{D_i\}_{i=1}^{N+1}$.

The sparsity patterns of the matrices $\Gamma_{\mathcal{D}}$, $\Sigma_{\mathcal{D}}$ and $\tilde{\Sigma}_{\mathcal{D}}$ are shown in Fig. 3, where nz denotes the number of nonzero elements in the matrices. Using the LAPACK driver routine DSYGV (version 3.3.1) [16], we determine Λ_i and Z_i satisfying (1.10), as required in step (i) of the MDA.

4. Neumann boundary conditions

Using the basis $\{\phi_i\}_{i=0}^{kN}$ for \mathcal{H}_k , we have $A_1 = \Gamma(\alpha/h)$ and $B_1 = \Gamma(h\beta)$ in (1.1). Denote the discrete cosine transform matrix [15] by

$$C_{\ell+1} = \sqrt{\frac{2}{\ell}} \left[w_j \cos \frac{(i-1)(j-1)\pi}{\ell} \right]_{i,j=1}^{\ell+1},$$

where

$$w_j = \begin{cases} 1/\sqrt{2}, & j = 1, \ell+1, \\ 1, & j = 2, \dots, \ell. \end{cases}$$

In this section,

$$\theta_1 = pN + m - 1, \quad \theta_2 = qN + n - 1, \quad \theta_+ = \theta_1 + \theta_2, \quad \theta_- = \theta_1 - \theta_2.$$

The $(pN + m, qN + n)$ entry of the matrix

$$\Sigma_{\mathcal{N}}(\gamma) = C_{kN+1}^T \Gamma(\gamma) C_{kN+1} \tag{4.1}$$

is

$$\sigma_{pN+m, qN+n}^{\mathcal{N}} = \frac{2w_{pN+m}w_{qN+n}}{kN} \sum_{i=1}^{v(k)} c_i^{\mathcal{N}}(p, q, m, n) \gamma_i,$$

where, for $l = 0, 1, 2, \dots, \mu(k)$,

$$\begin{aligned} c_{n_l+1}^{\mathcal{N}} &= \sum_{i=1}^N \left[\cos \frac{(ki-k+l)\theta_1\pi}{kN} \cos \frac{(ki-k+l)\theta_2\pi}{kN} + \cos \frac{(ki-l)\theta_1\pi}{kN} \cos \frac{(ki-l)\theta_2\pi}{kN} \right] \\ &= \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k)\theta_+\pi}{2kN} + \cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k)\theta_-\pi}{2kN} \right], \end{aligned}$$

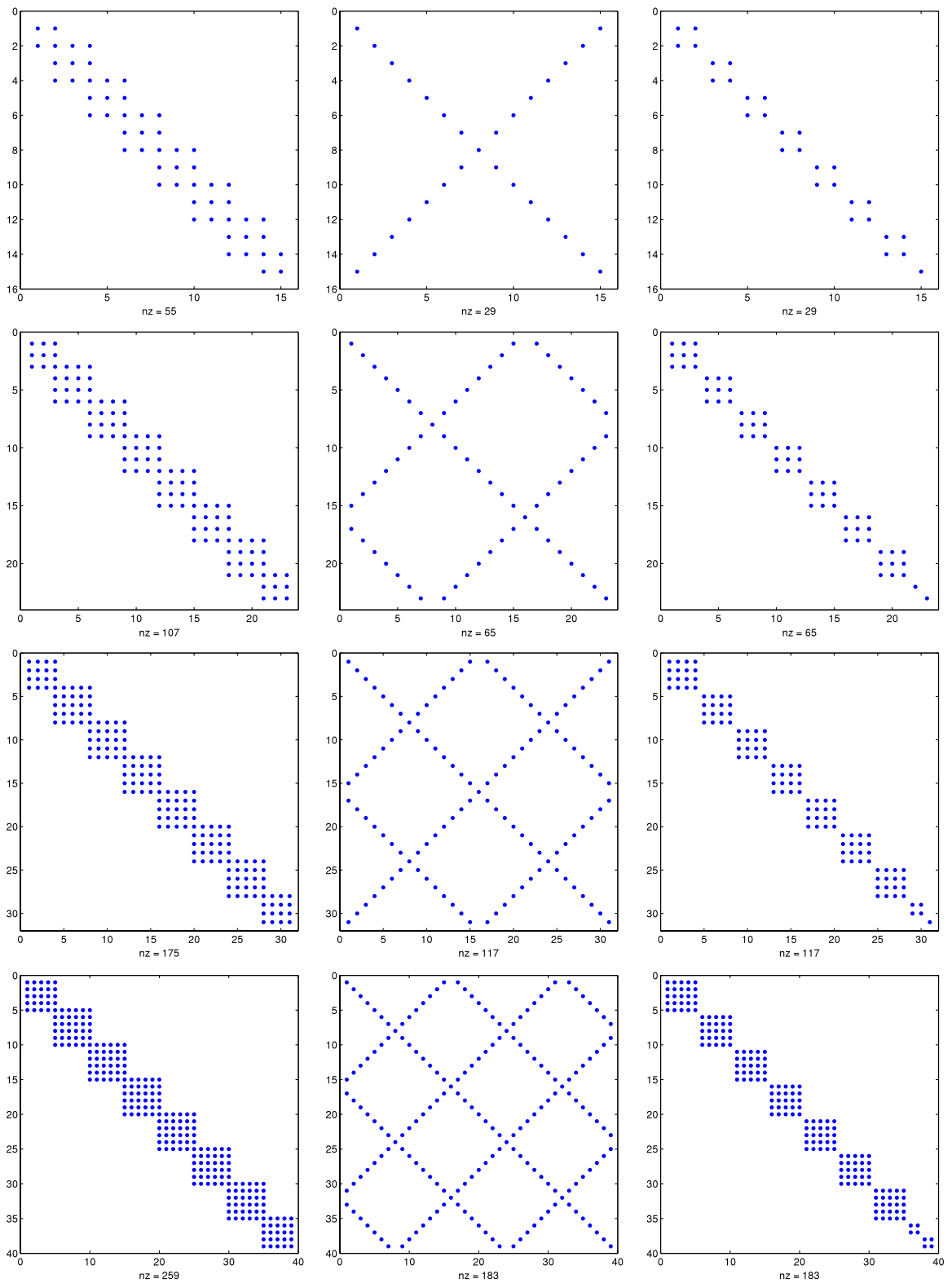


Fig. 3. Sparsity patterns of the matrices Γ_D (left), Σ_D (middle) and $\tilde{\Sigma}_D$ (right) for $N = 8$; from top to bottom: $k = 2, 3, 4, 5$.

$$\begin{aligned}
c_{n_l+j}^{\mathcal{N}} &= \sum_{i=1}^N \left[\cos \frac{(ki-k+l)\theta_1\pi}{kN} \cos \frac{(ki-k+j+l-1)\theta_2\pi}{kN} + \cos \frac{(ki-k+j+l-1)\theta_1\pi}{kN} \cos \frac{(ki-k+l)\theta_2\pi}{kN} \right. \\
&\quad \left. + \cos \frac{(ki-j-l+1)\theta_1\pi}{kN} \cos \frac{(ki-l)\theta_2\pi}{kN} + \cos \frac{(ki-l)\theta_1\pi}{kN} \cos \frac{(ki-j-l+1)\theta_2\pi}{kN} \right], \\
&= 2 \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k+j-1)\theta_+\pi}{2kN} \cos \frac{(j-1)\theta_-\pi}{2kN} \right. \\
&\quad \left. + \cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k+j-1)\theta_-\pi}{2kN} \cos \frac{(j-1)\theta_+\pi}{2kN} \right], \quad j = 2, 3, \dots, k-2l, \\
c_{n_{l+1}}^{\mathcal{N}} &= \sum_{i=1}^N \left[\cos \frac{(ki-k+l)\theta_1\pi}{kN} \cos \frac{(ki-l)\theta_2\pi}{kN} + \cos \frac{(ki-l)\theta_1\pi}{kN} \cos \frac{(ki-k+l)\theta_2\pi}{kN} \right] \\
&= \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k)\theta_-\pi}{2kN} + \cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k)\theta_+\pi}{2kN} \right];
\end{aligned}$$

when k is even,

$$c_{v(k)}^{\mathcal{N}} = \sum_{i=1}^N \cos \frac{(i-\frac{1}{2})\theta_1\pi}{N} \cos \frac{(i-\frac{1}{2})\theta_2\pi}{N} = \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_+\pi}{2N} + \cos \frac{(2i-1)\theta_-\pi}{2N} \right];$$

cf., [Appendix](#).

We classify the first $kN+1$ positive integers into the following $N+1$ sets:

$$\begin{cases} J_1^{\mathcal{N}} = \{1, 2N+1, 4N+1, 6N+1, 8N+1, \dots, k_1^{\mathcal{N}}\}, \\ J_i^{\mathcal{N}} = \{i, 2N+2-i, 2N+i, 4N+2-i, 4N+i, \dots, k_i^{\mathcal{N}}\}, \quad i = 2, \dots, N, \\ J_{N+1}^{\mathcal{N}} = \{N+1, 3N+1, 5N+1, 7N+1, 9N+1, \dots, k_{N+1}^{\mathcal{N}}\}, \end{cases}$$

with

$$k_i^{\mathcal{N}} = \begin{cases} (k-1)N+i, & \text{if } k \text{ is odd,} \\ kN+2-i, & \text{if } k \text{ is even.} \end{cases}$$

Remark 4.1. In this section, p, q, m, n are integers satisfying $1 \leq pN+m \leq kN+1$ and $1 \leq qN+n \leq kN+1$. They are uniquely determined by the $N+1$ sets $J_i^{\mathcal{N}}, i = 1, 2, \dots, N+1$. See [Remark 3.1](#).

Using (3.6), we have the following lemma.

Lemma 4.1. Let $pN+m \in J_s^{\mathcal{N}}$ and $qN+n \in J_t^{\mathcal{N}}$. If $s \neq t$, then,

$$c_i^{\mathcal{N}} = 0, \quad i = 1, 2, \dots, v(k).$$

If $s = t$, then, for $l = 0, 1, \dots, \mu(k)$,

$$\begin{aligned}
c_{n_l+1}^{\mathcal{N}} &= \begin{cases} N \cos \frac{l(p-q)\pi}{k}, & m = n \neq 1, \\ N \cos \frac{l(p+q)\pi}{k}, & m+n=2, m \neq n, \\ 2N \cos \frac{lp\pi}{k} \cos \frac{lq\pi}{k}, & m = n = 1, \end{cases} \\
c_{n_l+j}^{\mathcal{N}} &= \begin{cases} 2N \cos \frac{(2l+j-1)(p-q)\pi}{2k} \cos \frac{(j-1)[(p+q)N+2m-2]\pi}{2kN}, & m = n \neq 1, \\ 2N \cos \frac{(2l+j-1)(p+q)\pi}{2k} \cos \frac{(j-1)[(p-q)N+2m-2]\pi}{2kN}, & m+n=2, \\ & m \neq n, \\ 2N \left[\cos \frac{(l+j-1)p\pi}{k} \cos \frac{lq\pi}{k} + \cos \frac{lp\pi}{k} \cos \frac{(l+j-1)q\pi}{k} \right], & m = n = 1, \end{cases} \\
&\quad j = 2, 3, \dots, k-2l, \\
c_{n_{l+1}}^{\mathcal{N}} &= \begin{cases} N \cos \left[\frac{l((p+q)N+2m-2)\pi}{kN} - \frac{(m-1)\pi}{N} \right], & m = n \neq 1, \\ N \cos \left[\frac{l((p-q)N+2m-2)\pi}{kN} - \frac{(m-1)\pi}{N} \right], & m+n=2, \\ & m \neq n, \\ (-1)^p 2N \cos \frac{lp\pi}{k} \cos \frac{lq\pi}{k}, & m = n = 1; \end{cases}
\end{aligned}$$

when k is even,

$$c_{v(k)}^{\mathcal{N}} = \begin{cases} (-1)^{\frac{p-q}{2}} \frac{N}{2}, & m = n \neq 1, \\ (-1)^{\frac{p+q}{2}} \frac{N}{2}, & m + n = 2, m \neq n, \\ \left[(-1)^{\frac{p+q}{2}} + (-1)^{\frac{p-q}{2}} \right] \frac{N}{2}, & m = n = 1. \end{cases}$$

The following theorem corresponds to [Theorem 3.1](#).

Theorem 4.1. *There exists a permutation matrix P such that $P^T \Sigma_{\mathcal{N}} P = \tilde{\Sigma}_{\mathcal{N}}$, where $\Sigma_{\mathcal{N}}$ is given by (4.1) and $\tilde{\Sigma}_{\mathcal{N}} = \text{diag} \{D_i\}_{i=1}^{N+1}$, where D_i is $k_i \times k_i$ with*

$$k_i = \begin{cases} (k+1)/2, & i = 1, N+1, \\ k, & i = 2, \dots, N, \end{cases}$$

if k is odd, and

$$k_i = \begin{cases} k/2 + 1, & i = 1, \\ k, & i = 2, \dots, N, \\ k/2, & i = N+1, \end{cases}$$

if k is even. If $Q = C_{kN+1} P$, then $Q^T \Gamma_{\mathcal{N}} Q = \tilde{\Sigma}_{\mathcal{N}} = \text{diag} \{D_i\}_{i=1}^{N+1}$.

The sparsity patterns of the matrices $\Gamma_{\mathcal{N}}$, $\Sigma_{\mathcal{N}}$ and $\tilde{\Sigma}_{\mathcal{N}}$ are shown in [Fig. 4](#).

5. Mixed boundary conditions

We only consider the Dirichlet–Neumann case; the Neumann–Dirichlet case can be obtained by the same approach as in Section 5 of [\[2\]](#).

Let $\Gamma_{\mathcal{M}}(\gamma)$ be the matrix obtained by deleting the first row and column of the matrix $\Gamma(\gamma)$. Then using the basis $\{\phi_i\}_{i=1}^{kN}$ for $\mathcal{S}_k^{\mathcal{D}, \mathcal{N}}$, we have $A_1 = \Gamma_{\mathcal{M}}(\alpha/h)$ and $B_1 = \Gamma_{\mathcal{M}}(h\beta)$ in (1.1). Denote the discrete sine transform II matrix [\[15\]](#) by

$$\tilde{S}_{\ell} = \sqrt{\frac{2}{\ell}} \left[\sin \frac{i(2j-1)\pi}{2\ell} \right]_{i,j=1}^{\ell}.$$

In this section,

$$\theta_1 = 2(pN + m) - 1, \quad \theta_2 = 2(qN + n) - 1, \quad \theta_+ = \theta_1 + \theta_2, \quad \theta_- = \theta_1 - \theta_2.$$

The $(pN + m, qN + n)$ entry of the matrix

$$\Sigma_{\mathcal{M}}(\gamma) = \tilde{S}_{kN}^T \Gamma_{\mathcal{M}}(\gamma) \tilde{S}_{kN} \tag{5.1}$$

is

$$\sigma_{pN+m, qN+n}^{\mathcal{M}} = \frac{2}{kN} \sum_{i=1}^{v(k)} c_i^{\mathcal{M}}(p, q, m, n) \gamma_i,$$

where, for $l = 0, 1, 2, \dots, \mu(k)$,

$$\begin{aligned} c_{n_l+1}^{\mathcal{M}} &= \sum_{i=1}^N \left[\sin \frac{(ki-k+l)\theta_1\pi}{2kN} \sin \frac{(ki-k+l)\theta_2\pi}{2kN} + \sin \frac{(ki-l)\theta_1\pi}{2kN} \sin \frac{(ki-l)\theta_2\pi}{2kN} \right] \\ &= \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{4N} \cos \frac{(2l-k)\theta_-\pi}{4kN} - \cos \frac{(2i-1)\theta_+\pi}{4N} \cos \frac{(2l-k)\theta_+\pi}{4kN} \right], \\ c_{n_l+j}^{\mathcal{M}} &= \sum_{i=1}^N \left[\sin \frac{(ki-k+l)\theta_1\pi}{2kN} \sin \frac{(ki-k+j+l-1)\theta_2\pi}{2kN} + \sin \frac{(ki-k+j+l-1)\theta_1\pi}{2kN} \sin \frac{(ki-k+l)\theta_2\pi}{2kN} \right. \\ &\quad \left. + \sin \frac{(ki-j-l+1)\theta_1\pi}{2kN} \sin \frac{(ki-l)\theta_2\pi}{2kN} + \sin \frac{(ki-l)\theta_1\pi}{2kN} \sin \frac{(ki-j-l+1)\theta_2\pi}{2kN} \right], \\ &= 2 \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{4N} \cos \frac{(2l-k+j-1)\theta_-\pi}{4kN} \cos \frac{(j-1)\theta_+\pi}{4kN} \right. \\ &\quad \left. - \cos \frac{(2i-1)\theta_+\pi}{4N} \cos \frac{(2l-k+j-1)\theta_+\pi}{4kN} \cos \frac{(j-1)\theta_-\pi}{4kN} \right], \quad j = 2, 3, \dots, k-2l, \end{aligned}$$

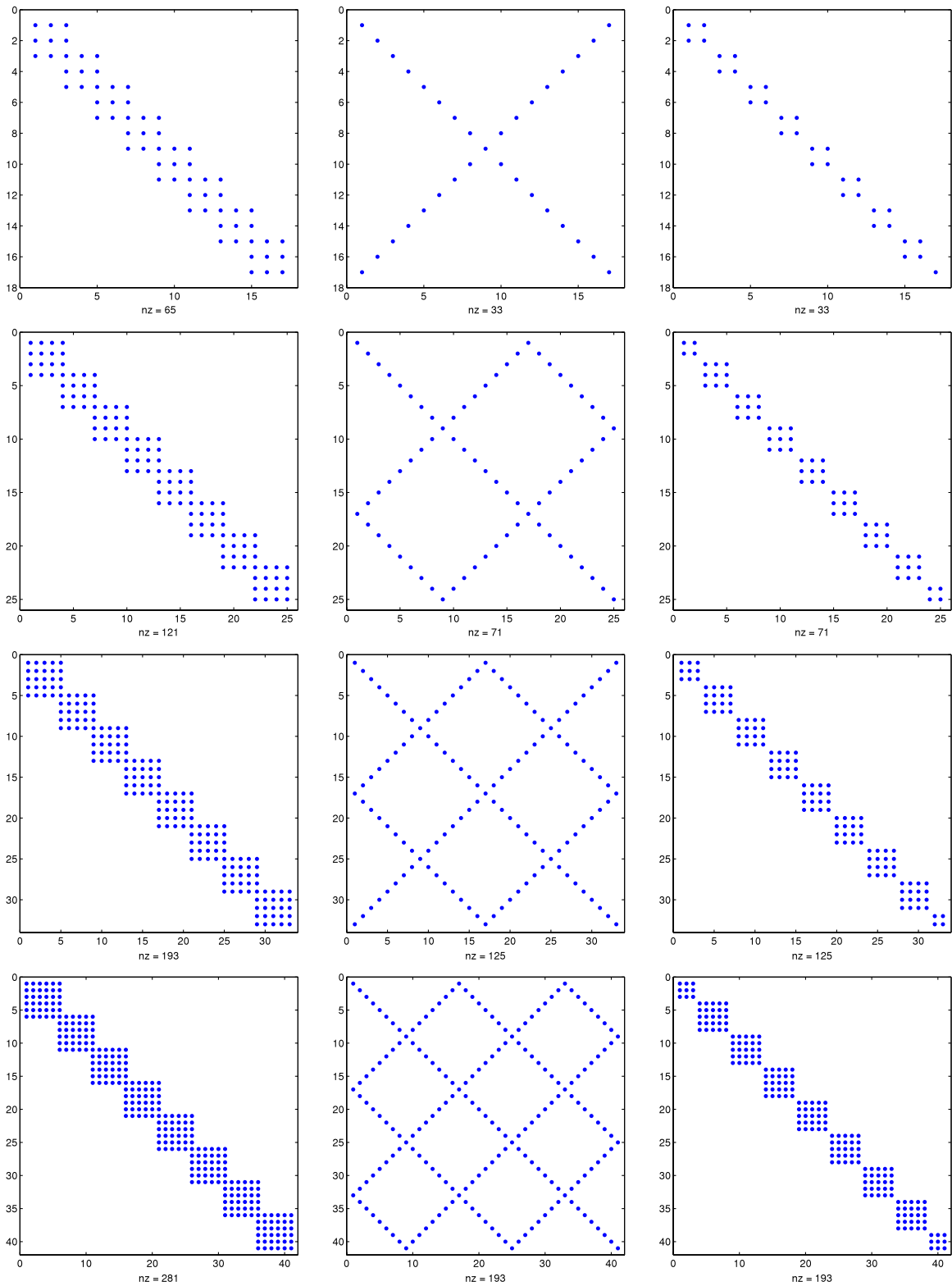


Fig. 4. Sparsity patterns of the matrices Γ_N (left), Σ_N (middle) and $\tilde{\Sigma}_N$ (right) for $N=8$; from top to bottom: $k=2, 3, 4, 5$.

$$\begin{aligned} c_{n_l+1}^{\mathcal{M}} &= \sum_{i=1}^N \left[\sin \frac{(ki-k+l)\theta_1\pi}{2kN} \sin \frac{(ki-l)\theta_2\pi}{2kN} + \sin \frac{(ki-l)\theta_1\pi}{2kN} \sin \frac{(ki-k+l)\theta_2\pi}{2kN} \right] \\ &= \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{4N} \cos \frac{(2l-k)\theta_+\pi}{4kN} - \cos \frac{(2i-1)\theta_+\pi}{4N} \cos \frac{(2l-k)\theta_-\pi}{4kN} \right]; \end{aligned}$$

when k is even,

$$c_{v(k)}^{\mathcal{M}} = \sum_{i=1}^N \sin \frac{(i-\frac{1}{2})\theta_1\pi}{2N} \sin \frac{(i-\frac{1}{2})\theta_2\pi}{2N} = \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{4N} - \cos \frac{(2i-1)\theta_+\pi}{4N} \right];$$

cf., [Appendix](#).

We classify the first kN positive integers into the following N sets:

$$J_i^{\mathcal{M}} = \{i, 2N+1-i, 2N+i, 4N+1-i, 4N+i, \dots, k_i^{\mathcal{M}}\}, \quad i = 1, \dots, N,$$

with

$$k_i^{\mathcal{M}} = \begin{cases} (k-1)N+i, & \text{if } k \text{ is odd,} \\ kN+1-i, & \text{if } k \text{ is even.} \end{cases}$$

Remark 5.1. In this section, p, q, m, n are integers satisfying $1 \leq pN+m \leq kN$ and $1 \leq qN+n \leq kN$. They are uniquely determined by the N sets $J_i^{\mathcal{M}}, i = 1, 2, \dots, N$. See [Remark 3.1](#).

Using (3.6), we have the following lemma.

Lemma 5.1. Let $pN+m \in J_s^{\mathcal{M}}$ and $qN+n \in J_t^{\mathcal{M}}$. If $s \neq t$, then,

$$c_i^{\mathcal{M}} = 0, \quad i = 1, 2, \dots, v(k).$$

If $s = t$, then, for $l = 0, 1, \dots, \mu(k)$,

$$\begin{aligned} c_{n_l+1}^{\mathcal{M}} &= \begin{cases} N \cos \frac{l(p-q)\pi}{k}, & m = n, \\ -N \cos \frac{l(p+q)\pi}{k}, & m+n=1, \end{cases} \\ c_{n_l+j}^{\mathcal{M}} &= \begin{cases} 2N \cos \frac{(2l+j-1)(p-q)\pi}{2k} \cos \frac{(j-1)[(p+q)N+2m-1]\pi}{2kN}, & m = n, \\ -2N \cos \frac{(2l+j-1)(p+q)\pi}{2k} \cos \frac{(j-1)[(p-q)N+2m-1]\pi}{2kN}, & m+n=1, \end{cases} \\ j &= 2, 3, \dots, k-2l, \\ c_{n_l+1}^{\mathcal{M}} &= \begin{cases} N \cos \left[\frac{l((p+q)N+2m-1)\pi}{kN} - \frac{(2m-1)\pi}{2N} \right], & m = n, \\ -N \cos \left[\frac{l((p-q)N+2m-1)\pi}{kN} - \frac{(2m-1)\pi}{2N} \right], & m+n=1, \end{cases} \end{aligned}$$

when k is even,

$$c_{v(k)}^{\mathcal{M}} = \begin{cases} (-1)^{\frac{p-q}{2}} \frac{N}{2}, & m = n, \\ -(-1)^{\frac{p+q}{2}} \frac{N}{2}, & m+n=1. \end{cases}$$

The main theorem of this section is the following.

Theorem 5.1. There exists a permutation matrix P such that $P^T \Sigma_{\mathcal{M}} P = \tilde{\Sigma}_{\mathcal{M}}$, where $\Sigma_{\mathcal{M}}$ is given by (5.1) and $\tilde{\Sigma}_{\mathcal{M}} = \text{diag} \{D_i\}_{i=1}^N$, where each D_i is $k \times k$. If $Q = C_{kN+1} P$, then $Q^T \Gamma_{\mathcal{M}} Q = \tilde{\Sigma}_{\mathcal{M}} = \text{diag} \{D_i\}_{i=1}^N$.

The sparsity patterns of the matrices $\Gamma_{\mathcal{M}}$, $\Sigma_{\mathcal{M}}$ and $\tilde{\Sigma}_{\mathcal{M}}$ are shown in [Fig. 5](#).

6. Periodic boundary conditions

Let

$$\Gamma_{\mathcal{P}}(\gamma) = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix},$$

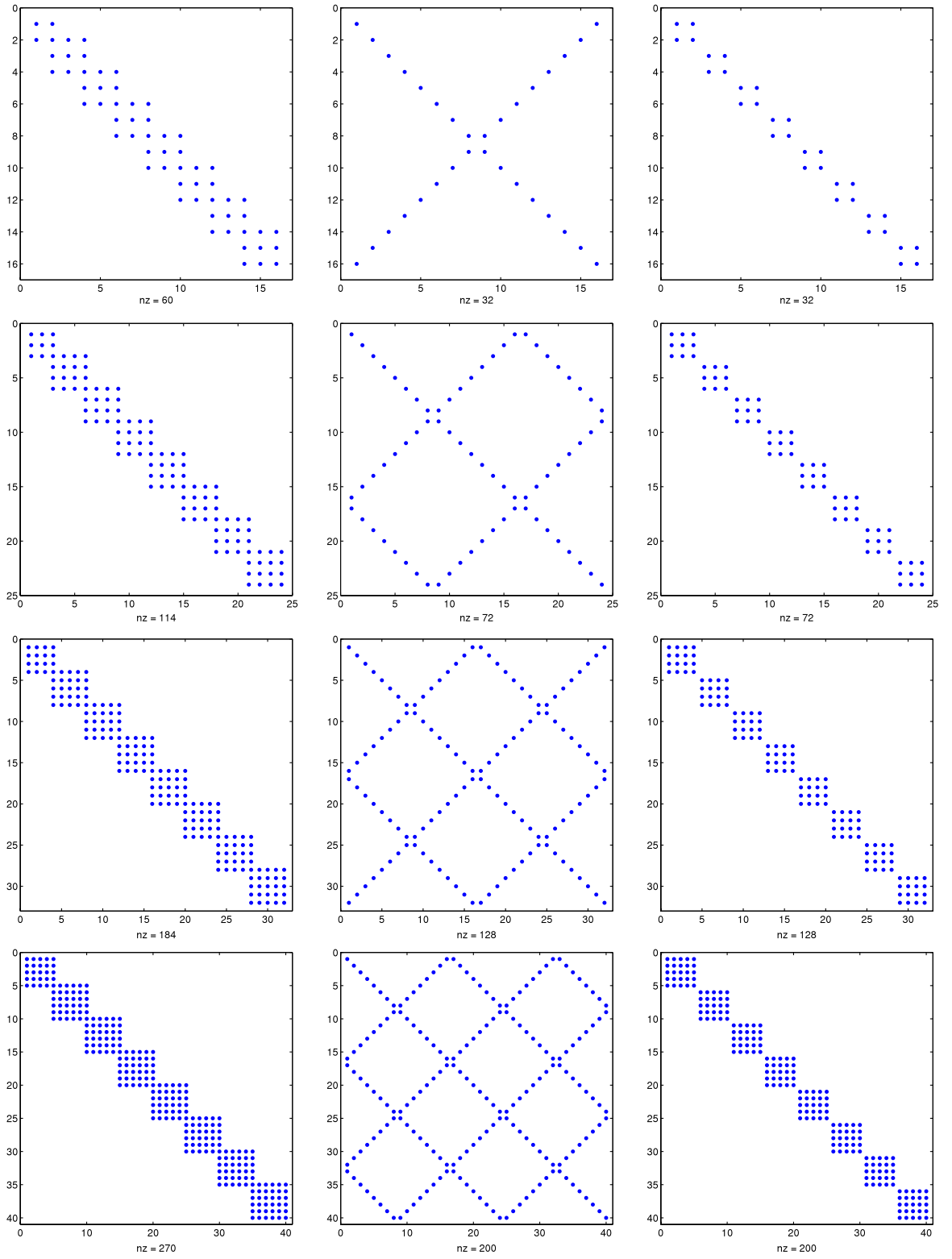


Fig. 5. Sparsity patterns of the matrices $\Gamma_{\mathcal{M}}$ (left), $\Sigma_{\mathcal{M}}$ (middle) and $\tilde{\Sigma}_{\mathcal{M}}$ (right) for $N = 8$; from top to bottom: $k = 2, 3, 4, 5$.

where

$$\Gamma_{11} = R(\gamma_{k+1}, 2\gamma_1, \gamma_{k+1}),$$

$$\Gamma_{12} = \begin{bmatrix} R(\gamma_k, \gamma_2, 0) & R(\gamma_{k-1}, \gamma_3, 0) & \cdots & R(\gamma_2, \gamma_k, 0) \end{bmatrix},$$

$$\Gamma_{21} = \begin{bmatrix} R(0, \gamma_2, \gamma_k) \\ R(0, \gamma_3, \gamma_{k-1}) \\ \vdots \\ R(0, \gamma_k, \gamma_2) \end{bmatrix}, \quad \Gamma_{22} = E(\gamma) \otimes I_N,$$

where

$$R(a, b, c) = \begin{bmatrix} b & c & & a \\ a & b & c & \\ & \ddots & \ddots & \ddots \\ & & a & b & c \\ c & & & a & b \end{bmatrix},$$

and the matrix $E(\gamma)$ is obtained by deleting the first and the last rows and columns of the element stiffness matrix \bar{A} when $\gamma = \alpha/h$ or the element mass matrix \bar{B} when $\gamma = h\beta$. Then using the basis functions

$$\{\phi_0 + \phi_{kN}, \phi_k, \dots, \phi_{kN-k}, \phi_1, \phi_{k+1}, \dots, \phi_{kN-k+1}, \dots, \phi_{k-1}, \phi_{2k-1}, \dots, \phi_{kN-1}\}$$

for $\mathcal{S}_k^{\mathcal{P}}$, we have $A_1 = \Gamma_{\mathcal{P}}(\alpha/h)$ and $B_1 = \Gamma_{\mathcal{P}}(h\beta)$ in (1.1). Let F_N denote the Fourier transformation, that is,

$$F_N = \sqrt{\frac{1}{N}} \left[e^{-\frac{(i-1)(j-1)2\pi\iota}{N}} \right]_{i,j=1}^N,$$

where $\iota = \sqrt{-1}$, and $\mathcal{F}_{kN} = I_k \otimes F_N$. By the basic properties of circulant matrices, we have

$$\Sigma_{\mathcal{P}} = \mathcal{F}_{kN}^H \Gamma_{\mathcal{P}}(\gamma) \mathcal{F}_{kN} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & \Gamma_{22} \end{bmatrix},$$

where

$$D_{11} = \text{diag}(d_{1,1}, \dots, d_{1,N}),$$

$$D_{12} = [\text{diag}(d_{2,1}, \dots, d_{2,N}) \quad \cdots \quad \text{diag}(d_{k,1}, \dots, d_{k,N})],$$

$$D_{21} = D_{12}^H,$$

and, for $i = 1, \dots, N, j = 2, \dots, k$,

$$d_{1,i} = 2\gamma_1 + 2\gamma_{k+1} \cos \frac{2(i-1)\pi}{N}, \quad d_{j,i} = \gamma_j + \gamma_{k+2-j} e^{\frac{(i-1)2\pi\iota}{N}}.$$

Obviously, $\Sigma_{\mathcal{P}}$ can be reordered to become a block diagonal matrix, denoted by $\tilde{\Sigma}_{\mathcal{P}}$, with N diagonal blocks of order k . The sparsity patterns of the matrices $\Gamma_{\mathcal{P}}$, $\Sigma_{\mathcal{P}}$ and $\tilde{\Sigma}_{\mathcal{P}}$ are shown in Fig. 6.

7. Numerical results

As in [2], we solved (1.2) with $f(x, y)$ corresponding to the exact solution

$$u(x, y) = \begin{cases} e^{xy}(x-x^2)(y-y^2) & \text{for (1.4),} \\ e^{xy}(x-x^2)^2(y-y^2) & \text{for (1.5) and (1.6),} \\ [1 + \sin(2\pi x)]e^y(y-y^2) & \text{for (1.8).} \end{cases}$$

In Tables 2–5, we present errors and the corresponding convergence rates when $k = 3, 4, 5$ in the norm defined by

$$\|u - u^h\|_{l^\infty} = \max_{0 \leq i, j \leq Nk} |u(i\hat{h}, j\hat{h}) - u^h(i\hat{h}, j\hat{h})|,$$

where $\hat{h} = h/k$, and the L^2 and H^1 norms, for the various boundary conditions. Convergence rates in the various norms are determined using the formula

$$\text{Rate} = \frac{\log(e_{N/2}/e_N)}{\log 2},$$

where e_N is the error corresponding to the $N \times N$ partition of Ω . As expected, the convergence rates for the l^∞ , L^2 and H^1 norms are $k+1$, $k+1$ and k , respectively.

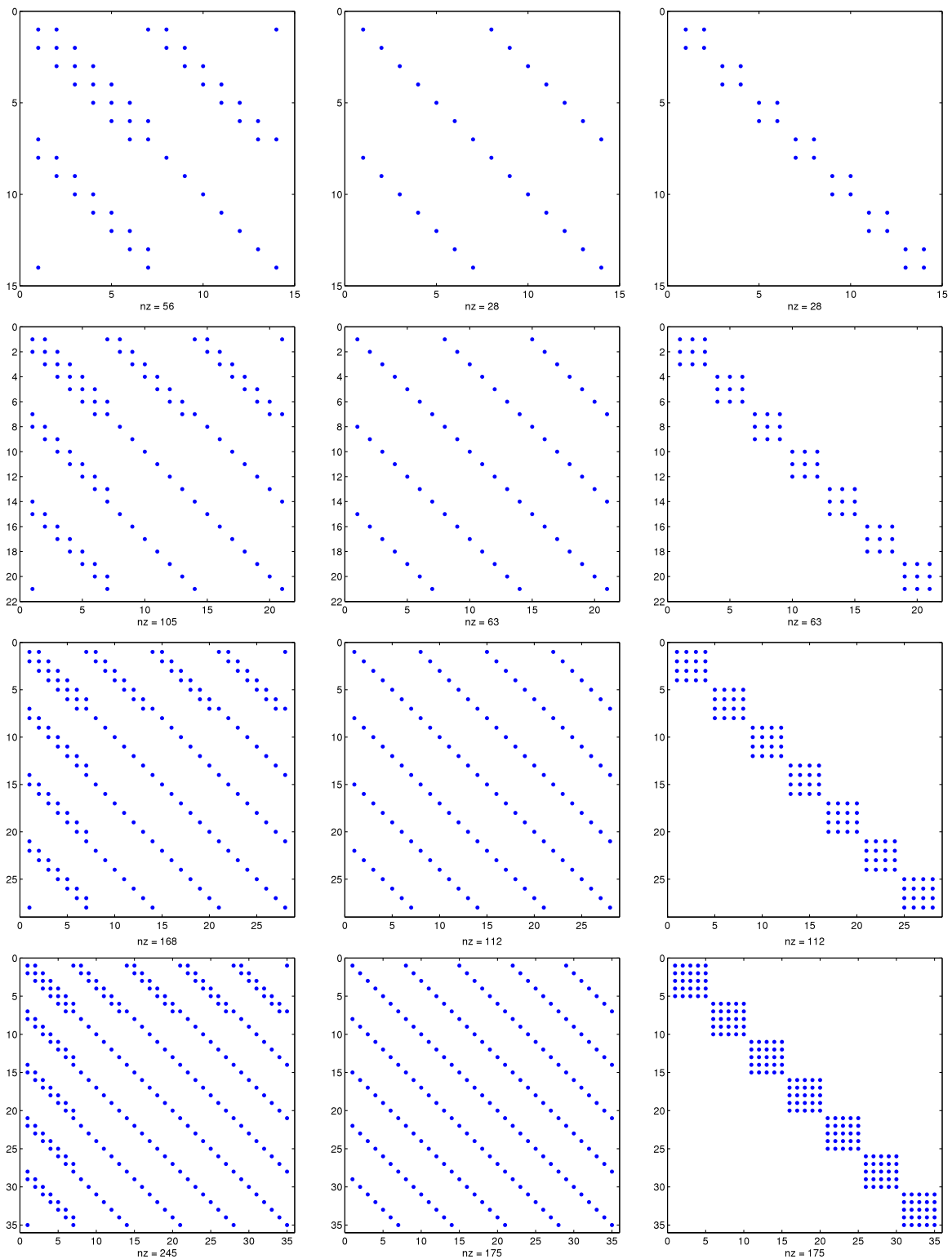


Fig. 6. Sparsity patterns of the matrices $\Gamma_{\mathcal{P}}$ (left), $\Sigma_{\mathcal{P}}$ (middle) and $\tilde{\Sigma}_{\mathcal{P}}$ (right) for $N = 7$; from top to bottom: $k = 2, 3, 4, 5$.

Let \mathcal{L} and \mathcal{G} denote the sets of the $(k + 1)$ th Lobatto points and the k th Gauss points in all elements, respectively. We define the following norms on Lobatto points and Gauss points:

$$\|u - u^h\|_{l^\infty, \mathcal{L}} = \max_{(x,y) \in \mathcal{L}} |u(x, y) - u^h(x, y)|,$$

Table 2Errors and convergence rates for the Dirichlet BC in l^∞ , L^2 and H^1 norms.

| k | N | $\ u - u^h\ _{l^\infty}$ | | $\ u - u^h\ _{L^2}$ | | $\ u - u^h\ _{H^1}$ | |
|-----|-----|--------------------------|-------|---------------------|--------|---------------------|-------|
| | | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 4.063e-06 | – | 1.705e-06 | – | 8.296e-05 | – |
| | 8 | 2.665e-07 | 3.930 | 1.088e-07 | 3.971 | 1.038e-05 | 2.999 |
| | 16 | 1.711e-08 | 3.961 | 6.837e-09 | 3.991 | 1.297e-06 | 3.000 |
| | 32 | 1.070e-09 | 3.999 | 4.280e-10 | 3.998 | 1.621e-07 | 3.000 |
| 4 | 4 | 1.247e-07 | – | 2.446e-08 | – | 1.582e-06 | – |
| | 8 | 3.765e-09 | 5.050 | 7.756e-10 | 4.979 | 9.899e-08 | 3.998 |
| | 16 | 1.223e-10 | 4.944 | 2.433e-11 | 4.994 | 6.189e-09 | 4.000 |
| | 32 | 3.861e-12 | 4.985 | 7.612e-13 | 4.998 | 3.868e-10 | 4.000 |
| 5 | 4 | 1.857e-09 | – | 2.841e-10 | – | 2.296e-08 | – |
| | 8 | 3.137e-11 | 5.887 | 4.495e-12 | 5.982 | 7.185e-10 | 4.998 |
| | 16 | 4.859e-13 | 6.013 | 7.218e-14 | 5.961 | 2.246e-11 | 4.999 |
| | 32 | 2.546e-13 | 0.932 | 1.288e-13 | –0.835 | 9.162e-13 | 4.616 |

Table 3Errors and convergence rates for the Neumann BC in l^∞ , L^2 and H^1 norms.

| k | N | $\ u - u^h\ _{l^\infty}$ | | $\ u - u^h\ _{L^2}$ | | $\ u - u^h\ _{H^1}$ | |
|-----|-----|--------------------------|-------|---------------------|-------|---------------------|-------|
| | | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 1.664e-05 | – | 8.153e-06 | – | 3.911e-04 | – |
| | 8 | 1.192e-06 | 3.803 | 5.228e-07 | 3.963 | 4.967e-05 | 2.977 |
| | 16 | 7.946e-08 | 3.907 | 3.289e-08 | 3.990 | 6.233e-06 | 2.994 |
| | 32 | 5.119e-09 | 3.956 | 2.059e-09 | 3.998 | 7.799e-07 | 2.999 |
| 4 | 4 | 8.944e-07 | – | 2.418e-07 | – | 1.550e-05 | – |
| | 8 | 3.141e-08 | 4.832 | 7.683e-09 | 4.976 | 9.781e-07 | 3.986 |
| | 16 | 1.040e-09 | 4.916 | 2.411e-10 | 4.994 | 6.128e-08 | 3.996 |
| | 32 | 3.341e-11 | 4.960 | 7.543e-12 | 4.998 | 3.832e-09 | 3.999 |
| 5 | 4 | 2.256e-08 | – | 4.670e-09 | – | 3.747e-07 | – |
| | 8 | 3.918e-10 | 5.848 | 7.396e-11 | 5.981 | 1.180e-08 | 4.989 |
| | 16 | 6.441e-12 | 5.927 | 1.160e-12 | 5.995 | 3.694e-10 | 4.997 |
| | 32 | 1.276e-13 | 5.657 | 3.424e-14 | 5.082 | 1.155e-11 | 4.999 |

Table 4Errors and convergence rates for the D–N mixed BC in l^∞ , L^2 and H^1 norms.

| k | N | $\ u - u^h\ _{l^\infty}$ | | $\ u - u^h\ _{L^2}$ | | $\ u - u^h\ _{H^1}$ | |
|-----|-----|--------------------------|-------|---------------------|-------|---------------------|-------|
| | | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 1.663e-05 | – | 8.153e-06 | – | 3.911e-04 | – |
| | 8 | 1.192e-06 | 3.803 | 5.228e-07 | 3.963 | 4.967e-05 | 2.977 |
| | 16 | 7.946e-08 | 3.907 | 3.289e-08 | 3.990 | 6.233e-06 | 2.994 |
| | 32 | 5.119e-09 | 3.956 | 2.059e-09 | 3.998 | 7.799e-07 | 2.999 |
| 4 | 4 | 8.944e-07 | – | 2.418e-07 | – | 1.550e-05 | – |
| | 8 | 3.141e-08 | 4.832 | 7.683e-09 | 4.976 | 9.781e-07 | 3.986 |
| | 16 | 1.040e-09 | 4.916 | 2.411e-10 | 4.994 | 6.128e-08 | 3.996 |
| | 32 | 3.342e-11 | 4.960 | 7.543e-12 | 4.998 | 3.832e-09 | 3.999 |
| 5 | 4 | 2.256e-08 | – | 4.670e-09 | – | 3.747e-07 | – |
| | 8 | 3.918e-10 | 5.848 | 7.396e-11 | 5.981 | 1.180e-08 | 4.989 |
| | 16 | 6.446e-12 | 5.925 | 1.160e-12 | 5.995 | 3.694e-10 | 4.997 |
| | 32 | 1.400e-13 | 5.525 | 4.410e-14 | 4.717 | 1.155e-11 | 4.999 |

$$\|u - u^h\|_{l^2, \mathcal{L}} = \left(h^2 \sum_{(x,y) \in \mathcal{L}} |u(x,y) - u^h(x,y)|^2 \right)^{1/2},$$

$$\|\nabla(u - u^h)\|_{l^\infty, \mathcal{G}} = \max_{(x,y) \in \mathcal{G}} \{ \max(|u_x(x,y) - u_x^h(x,y)|, |u_y(x,y) - u_y^h(x,y)|) \},$$

$$\|\nabla(u - u^h)\|_{l^2, \mathcal{G}} = \left(h^2 \sum_{(x,y) \in \mathcal{G}} (|u_x(x,y) - u_x^h(x,y)|^2 + |u_y(x,y) - u_y^h(x,y)|^2) \right)^{1/2}.$$

In Tables 6–9, we present errors and the corresponding convergence rates at the Lobatto and Gauss points when $k = 3, 4, 5$ in the norms defined above, for the various boundary conditions. Expected superconvergence phenomena [17] are observed,

Table 5Errors and convergence rates for the periodic BC in l^∞ , L^2 and H^1 norms.

| k | N | $\ u - u^h\ _{l^\infty}$ | | $\ u - u^h\ _{L^2}$ | | $\ u - u^h\ _{H^1}$ | |
|-----|-----|--------------------------|-------|---------------------|-------|---------------------|-------|
| | | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 4.320e-04 | – | 3.446e-04 | – | 1.658e-02 | – |
| | 8 | 3.228e-05 | 3.742 | 2.219e-05 | 3.957 | 2.109e-03 | 2.974 |
| | 16 | 2.111e-06 | 3.935 | 1.397e-06 | 3.989 | 2.648e-04 | 2.994 |
| | 32 | 1.335e-07 | 3.983 | 8.751e-08 | 3.997 | 3.314e-05 | 2.998 |
| 4 | 4 | 5.394e-05 | – | 2.561e-05 | – | 1.639e-03 | – |
| | 8 | 2.143e-06 | 4.654 | 8.179e-07 | 4.968 | 1.041e-04 | 3.977 |
| | 16 | 7.075e-08 | 4.921 | 2.570e-08 | 4.992 | 6.531e-06 | 3.994 |
| | 32 | 2.239e-09 | 4.982 | 8.042e-10 | 4.998 | 4.086e-07 | 3.999 |
| 5 | 4 | 4.145e-06 | – | 1.621e-06 | – | 1.296e-04 | – |
| | 8 | 8.336e-08 | 5.636 | 2.578e-08 | 5.974 | 4.109e-06 | 4.979 |
| | 16 | 1.380e-09 | 5.917 | 4.046e-10 | 5.994 | 1.289e-07 | 4.995 |
| | 32 | 2.257e-11 | 5.934 | 6.376e-12 | 5.988 | 4.031e-09 | 4.999 |

Table 6

Errors and superconvergence at Lobatto and Gauss points for the Dirichlet BC.

| k | N | $\ u - u^h\ _{l^\infty, \mathcal{L}}$ | | $\ u - u^h\ _{L^2, \mathcal{L}}$ | | $\ \nabla(u - u^h)\ _{l^\infty, \mathcal{G}}$ | | $\ \nabla(u - u^h)\ _{L^2, \mathcal{G}}$ | |
|-----|-----|---------------------------------------|--------|----------------------------------|--------|---|-------|--|-------|
| | | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 4.525e-07 | – | 3.410e-07 | – | 9.930e-05 | – | 4.734e-05 | – |
| | 8 | 1.375e-08 | 5.040 | 7.218e-09 | 5.562 | 7.464e-06 | 3.734 | 2.926e-06 | 4.016 |
| | 16 | 2.995e-10 | 5.521 | 1.593e-10 | 5.502 | 5.091e-07 | 3.874 | 1.821e-07 | 4.006 |
| | 32 | 6.511e-12 | 5.524 | 4.079e-12 | 5.288 | 3.642e-08 | 3.805 | 1.137e-08 | 4.001 |
| 4 | 4 | 6.136e-09 | – | 3.939e-09 | – | 1.888e-06 | – | 1.145e-06 | – |
| | 8 | 8.786e-11 | 6.126 | 4.528e-11 | 6.443 | 6.711e-08 | 4.814 | 3.408e-08 | 5.070 |
| | 16 | 1.047e-12 | 6.391 | 5.472e-13 | 6.371 | 2.443e-09 | 4.780 | 1.045e-09 | 5.027 |
| | 32 | 1.209e-14 | 6.437 | 2.765e-14 | 4.307 | 8.719e-11 | 4.808 | 3.250e-11 | 5.007 |
| 5 | 4 | 6.398e-11 | – | 4.318e-11 | – | 2.699e-08 | – | 2.013e-08 | – |
| | 8 | 3.512e-13 | 7.509 | 2.557e-13 | 7.400 | 4.648e-10 | 5.860 | 2.933e-10 | 6.101 |
| | 16 | 3.145e-14 | 3.481 | 9.349e-14 | 1.452 | 8.988e-12 | 5.693 | 4.470e-12 | 6.036 |
| | 32 | 2.541e-13 | –3.014 | 7.726e-13 | –3.047 | 9.594e-13 | 3.228 | 2.867e-12 | 0.641 |

Table 7

Errors and superconvergence at Lobatto and Gauss points for the Neumann BC.

| k | N | $\ u - u^h\ _{l^\infty, \mathcal{L}}$ | | $\ u - u^h\ _{L^2, \mathcal{L}}$ | | $\ \nabla(u - u^h)\ _{l^\infty, \mathcal{G}}$ | | $\ \nabla(u - u^h)\ _{L^2, \mathcal{G}}$ | |
|-----|-----|---------------------------------------|--------|----------------------------------|--------|---|-------|--|-------|
| | | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 1.492e-06 | – | 1.446e-06 | – | 2.623e-04 | – | 1.601e-04 | – |
| | 8 | 4.651e-08 | 5.004 | 4.035e-08 | 5.163 | 2.163e-05 | 3.600 | 1.021e-05 | 3.971 |
| | 16 | 1.423e-09 | 5.031 | 1.193e-09 | 5.080 | 1.563e-06 | 3.791 | 6.414e-07 | 3.993 |
| | 32 | 4.345e-11 | 5.033 | 3.662e-11 | 5.026 | 1.052e-07 | 3.893 | 4.014e-08 | 3.998 |
| 4 | 4 | 4.022e-08 | – | 3.431e-08 | – | 1.087e-05 | – | 7.768e-06 | – |
| | 8 | 5.336e-10 | 6.236 | 4.970e-10 | 6.109 | 4.245e-07 | 4.679 | 2.401e-07 | 5.016 |
| | 16 | 7.607e-12 | 6.132 | 7.444e-12 | 6.061 | 1.481e-08 | 4.841 | 7.471e-09 | 5.006 |
| | 32 | 1.214e-13 | 5.970 | 1.148e-13 | 6.019 | 4.890e-10 | 4.921 | 2.332e-10 | 5.002 |
| 5 | 4 | 5.617e-10 | – | 6.161e-10 | – | 2.899e-07 | – | 2.343e-07 | – |
| | 8 | 4.223e-12 | 7.056 | 4.450e-12 | 7.113 | 5.523e-09 | 5.714 | 3.526e-09 | 6.055 |
| | 16 | 3.360e-14 | 6.973 | 5.600e-14 | 6.312 | 9.537e-11 | 5.856 | 5.435e-11 | 6.020 |
| | 32 | 4.641e-14 | –0.466 | 1.742e-13 | –1.637 | 1.575e-12 | 5.920 | 9.454e-13 | 5.845 |

namely superconvergence of order $k + 2$ in the solution at the Lobatto points, in particular at the mesh points $\{(ih, jh)\}_{i,j=0}^N$ which are Lobatto points, and of order $k + 1$ in the first derivatives at the Gauss points.

In the quadratic case ($k = 2$) discussed in [2], integrals of the form (2.6) were approximated by interpolating the function $f(x, y)$ and then integrating exactly. We observed that this approach degrades the superconvergence when k is odd. As a consequence, these integrals were approximated using Gaussian quadrature with $k + 1$ points.

In Table 10, we present timings for the Dirichlet case. From these, it can be seen that, as predicted, for a fixed k , the CPU time is $O(N^2)$, and for fixed N , it is $O(k^2)$.

It should be noted that the systems of equations arising in step (iv) of the MDA have the sparsity pattern shown in Fig. 1. Thus they can be solved efficiently without fill-in as described in [2].

Table 8

Errors and superconvergence at Lobatto and Gauss points for the D–N mixed BC.

| k | N | $\ u - u^h\ _{L^\infty, \mathcal{L}}$ | | $\ u - u^h\ _{L^2, \mathcal{L}}$ | | $\ \nabla(u - u^h)\ _{L^\infty, \mathcal{G}}$ | | $\ \nabla(u - u^h)\ _{L^2, \mathcal{G}}$ | |
|-----|-----|---------------------------------------|--------|----------------------------------|--------|---|-------|--|-------|
| | | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 1.500e–06 | – | 1.438e–06 | – | 2.622e–04 | – | 1.601e–04 | – |
| | 8 | 4.661e–08 | 5.008 | 4.028e–08 | 5.158 | 2.163e–05 | 3.600 | 1.021e–05 | 3.971 |
| | 16 | 1.424e–09 | 5.032 | 1.193e–09 | 5.078 | 1.563e–06 | 3.791 | 6.415e–07 | 3.993 |
| | 32 | 4.347e–11 | 5.034 | 3.661e–11 | 5.026 | 1.052e–07 | 3.893 | 4.014e–08 | 3.998 |
| 4 | 4 | 4.023e–08 | – | 3.433e–08 | – | 1.087e–05 | – | 7.768e–06 | – |
| | 8 | 5.336e–10 | 6.236 | 4.971e–10 | 6.110 | 4.245e–07 | 4.679 | 2.401e–07 | 5.016 |
| | 16 | 7.608e–12 | 6.132 | 7.444e–12 | 6.061 | 1.481e–08 | 4.841 | 7.471e–09 | 5.006 |
| | 32 | 1.172e–13 | 6.021 | 1.173e–13 | 5.988 | 4.890e–10 | 4.921 | 2.332e–10 | 5.002 |
| 5 | 4 | 5.617e–10 | – | 6.161e–10 | – | 2.899e–07 | – | 2.343e–07 | – |
| | 8 | 4.222e–12 | 7.056 | 4.450e–12 | 7.113 | 5.523e–09 | 5.714 | 3.526e–09 | 6.055 |
| | 16 | 3.865e–14 | 6.771 | 6.614e–14 | 6.072 | 9.533e–11 | 5.856 | 5.435e–11 | 6.020 |
| | 32 | 7.088e–14 | –0.875 | 2.412e–13 | –1.866 | 1.636e–12 | 5.865 | 1.099e–12 | 5.628 |

Table 9

Errors and superconvergence at Lobatto and Gauss points for the periodic BC.

| k | N | $\ u - u^h\ _{L^\infty, \mathcal{L}}$ | | $\ u - u^h\ _{L^2, \mathcal{L}}$ | | $\ \nabla(u - u^h)\ _{L^\infty, \mathcal{G}}$ | | $\ \nabla(u - u^h)\ _{L^2, \mathcal{G}}$ | |
|-----|-----|---------------------------------------|-------|----------------------------------|-------|---|-------|--|-------|
| | | Error | Rate | Error | Rate | Error | Rate | Error | Rate |
| 3 | 4 | 7.645e–05 | – | 1.257e–04 | – | 5.764e–03 | – | 7.658e–03 | – |
| | 8 | 2.686e–06 | 4.831 | 3.972e–06 | 4.984 | 4.763e–04 | 3.597 | 4.888e–04 | 3.969 |
| | 16 | 8.699e–08 | 4.949 | 1.241e–07 | 5.000 | 3.225e–05 | 3.884 | 3.072e–05 | 3.992 |
| | 32 | 2.740e–09 | 4.989 | 3.878e–09 | 5.000 | 2.069e–06 | 3.962 | 1.923e–06 | 3.998 |
| 4 | 4 | 4.671e–06 | – | 1.010e–05 | – | 4.269e–04 | – | 7.807e–04 | – |
| | 8 | 9.736e–08 | 5.584 | 1.617e–07 | 5.966 | 1.777e–05 | 4.586 | 2.473e–05 | 4.981 |
| | 16 | 1.627e–09 | 5.903 | 2.540e–09 | 5.992 | 6.024e–07 | 4.883 | 7.751e–07 | 4.995 |
| | 32 | 2.659e–11 | 5.936 | 3.997e–11 | 5.990 | 1.934e–08 | 4.961 | 2.424e–08 | 4.999 |
| 5 | 4 | 2.560e–07 | – | 6.839e–07 | – | 2.816e–05 | – | 6.282e–05 | – |
| | 8 | 2.563e–09 | 6.642 | 5.453e–09 | 6.971 | 5.887e–07 | 5.580 | 9.904e–07 | 5.987 |
| | 16 | 2.140e–11 | 6.904 | 4.281e–11 | 6.993 | 9.927e–09 | 5.890 | 1.551e–08 | 5.997 |
| | 32 | 1.455e–12 | 3.878 | 4.615e–12 | 3.213 | 1.639e–10 | 5.921 | 2.432e–10 | 5.995 |

Table 10

CPU time (sec.) for solving the Dirichlet case.

| k | N | | | | |
|-----|------|------|------|-------|--------|
| | 4 | 8 | 16 | 32 | 64 |
| 3 | 0.18 | 0.63 | 2.51 | 10.21 | 41.45 |
| 4 | 0.27 | 1.13 | 4.61 | 18.82 | 76.13 |
| 5 | 0.45 | 1.79 | 7.37 | 30.28 | 121.51 |

8. Concluding remarks

In this paper, we present matrix decomposition algorithms for arbitrary order C^0 tensor product finite element approximation of Poisson's equation on a square subject to standard boundary conditions. Several extensions of the methods described in this paper can be easily formulated; see the discussion in [2].

The extension to the biharmonic Dirichlet problems of the form,

$$\begin{cases} \Delta^2 u(x, y) = f(x, y), & \text{in } \Omega = (0, 1) \times (0, 1), \\ u(x, y) = 0, \quad \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\partial/\partial n$ denotes the outward normal on the boundary $\partial\Omega$, is in progress.

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Appendix

Eqs. (3.2)–(3.4) are derived by repeated use of the trigonometric identities

$$\sin u \sin v = \frac{1}{2} [\cos(u - v) - \cos(u + v)], \quad (\text{A.1})$$

and

$$\cos(u - v) + \cos(u + v) = 2 \cos u \cos v,$$

and the judicious combination of terms. Thus, to derive (3.2), we have

$$\begin{aligned} c_{n_l+j}^{\mathcal{D}} &= \sum_{i=1}^N \left[\sin \frac{(ki - k + l)\theta_1\pi}{kN} \sin \frac{(ki - k + l)\theta_2\pi}{kN} + \sin \frac{(ki - l)\theta_1\pi}{kN} \sin \frac{(ki - l)\theta_2\pi}{kN} \right] \\ &= \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(ki - k + l)\theta_-\pi}{kN} - \cos \frac{(ki - k + l)\theta_+\pi}{kN} + \cos \frac{(ki - l)\theta_-\pi}{kN} - \cos \frac{(ki - l)\theta_+\pi}{kN} \right] \\ &= \sum_{i=1}^N \left[\cos \frac{(2i - 1)\theta_-\pi}{2N} \cos \frac{(2l - k)\theta_-\pi}{2kN} - \cos \frac{(2i - 1)\theta_+\pi}{2N} \cos \frac{(2l - k)\theta_+\pi}{2kN} \right]. \end{aligned}$$

For (3.3),

$$\begin{aligned} c_{n_l+j}^{\mathcal{D}} &= \sum_{i=1}^N \left[\sin \frac{(ki - k + l)\theta_1\pi}{kN} \sin \frac{(ki - k + j + l - 1)\theta_2\pi}{kN} + \sin \frac{(ki - k + j + l - 1)\theta_1\pi}{kN} \sin \frac{(ki - k + l)\theta_2\pi}{kN} \right. \\ &\quad \left. + \sin \frac{(ki - j - l + 1)\theta_1\pi}{kN} \sin \frac{(ki - l)\theta_2\pi}{kN} + \sin \frac{(ki - l)\theta_1\pi}{kN} \sin \frac{(ki - j - l + 1)\theta_2\pi}{kN} \right] \\ &= \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(ki - k + l)\theta_-\pi - (j - 1)\theta_2\pi}{kN} - \cos \frac{(ki - k + l)\theta_+\pi + (j - 1)\theta_2\pi}{kN} \right. \\ &\quad \left. + \cos \frac{(ki - k + l)\theta_-\pi + (j - 1)\theta_1\pi}{kN} - \cos \frac{(ki - k + l)\theta_+\pi + (j - 1)\theta_1\pi}{kN} \right. \\ &\quad \left. + \cos \frac{(ki - l)\theta_-\pi - (j - 1)\theta_1\pi}{kN} - \cos \frac{(ki - l)\theta_+\pi - (j - 1)\theta_1\pi}{kN} \right. \\ &\quad \left. + \cos \frac{(ki - l)\theta_-\pi + (j - 1)\theta_2\pi}{kN} - \cos \frac{(ki - l)\theta_+\pi - (j - 1)\theta_2\pi}{kN} \right] \\ &= \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(ki - k + l)\theta_-\pi - (j - 1)\theta_2\pi}{kN} + \cos \frac{(ki - l)\theta_-\pi + (j - 1)\theta_2\pi}{kN} \right. \\ &\quad \left. - \cos \frac{(ki - k + l)\theta_+\pi + (j - 1)\theta_2\pi}{kN} - \cos \frac{(ki - l)\theta_+\pi - (j - 1)\theta_2\pi}{kN} \right. \\ &\quad \left. + \cos \frac{(ki - k + l)\theta_-\pi + (j - 1)\theta_1\pi}{kN} + \cos \frac{(ki - l)\theta_-\pi - (j - 1)\theta_1\pi}{kN} \right. \\ &\quad \left. - \cos \frac{(ki - k + l)\theta_+\pi + (j - 1)\theta_1\pi}{kN} - \cos \frac{(ki - l)\theta_+\pi - (j - 1)\theta_1\pi}{kN} \right] \\ &= \sum_{i=1}^N \left[\cos \frac{(2i - 1)\theta_-\pi}{2N} \cos \frac{(-k + 2l)\theta_-\pi - 2(j - 1)\theta_2\pi}{2kN} \right. \\ &\quad \left. - \cos \frac{(2i - 1)\theta_+\pi}{2N} \cos \frac{(-k + 2l)\theta_+\pi + 2(j - 1)\theta_2\pi}{2kN} \right. \\ &\quad \left. + \cos \frac{(2i - 1)\theta_-\pi}{2N} \cos \frac{(-k + 2l)\theta_-\pi + 2(j - 1)\theta_1\pi}{2kN} \right. \\ &\quad \left. - \cos \frac{(2i - 1)\theta_+\pi}{2N} \cos \frac{(-k + 2l)\theta_+\pi + 2(j - 1)\theta_1\pi}{2kN} \right] \end{aligned}$$

$$= 2 \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k+j-1)\theta_-\pi}{2kN} \cos \frac{(j-1)\theta_+\pi}{2kN} \right. \\ \left. - \cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k+j-1)\theta_+\pi}{2kN} \cos \frac{(j-1)\theta_-\pi}{2kN} \right], \quad j = 2, 3, \dots, k-2l,$$

as required. Finally, for (3.4)

$$c_{n_{l+1}}^{\mathcal{D}} = \sum_{i=1}^N \left[\sin \frac{(ki-k+l)\theta_1\pi}{kN} \sin \frac{(ki-l)\theta_2\pi}{kN} + \sin \frac{(ki-l)\theta_1\pi}{kN} \sin \frac{(ki-k+l)\theta_2\pi}{kN} \right] \\ = \frac{1}{2} \sum_{i=1}^N \left[\cos \frac{(ki-k+l)\theta_1\pi - (ki-l)\theta_2\pi}{kN} - \cos \frac{(ki-k+l)\theta_1\pi + (ki-l)\theta_2\pi}{kN} \right. \\ \left. + \cos \frac{(ki-l)\theta_1\pi - (ki-k+l)\theta_2\pi}{kN} - \cos \frac{(ki-l)\theta_1\pi + (ki-k+l)\theta_2\pi}{kN} \right] \\ = \sum_{i=1}^N \left[\cos \frac{(2i-1)\theta_-\pi}{2N} \cos \frac{(2l-k)\theta_+\pi}{2kN} - \cos \frac{(2i-1)\theta_+\pi}{2N} \cos \frac{(2l-k)\theta_-\pi}{2kN} \right].$$

Note that the derivation of (3.5) is straightforward, requiring only one application of (A.1).

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