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Extending the convergence domain of the Secant and Moser method in Banach Space

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ABSTRACT

We present a new semilocal convergence analysis for the Secant and the Moser method in order to approximate a solution of an equation in a Banach space setting. Using the method of recurrent relations and weaker sufficient convergence criteria than in earlier studies such as Amat et al. (2014), Hernández and Rubio (2007), Hernández and Rubio (1999) and Hernández and Rubio (2002) we increase the convergence domain of these methods. The advantages are also obtained under less computational cost than in Amat et al. (2014), Hernández and Rubio (2007), Hernández and Rubio (1999) and Hernández and Rubio (2002). Numerical examples where the older convergence criteria are not satisfied but the new convergence criteria are satisfied are also provided in this study.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

A large number of problems in applied mathematics and engineering are solved by finding the solutions of certain equations [1–8]. Except in special cases, the most commonly used solution methods are iterative. In fact, starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. The study about the convergence matter of iterative procedures is usually based on two types: semilocal and local convergence analysis. The semilocal convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative methods; while the local one is, based on the information around a solution to find estimates on the radii of convergence balls. There is a plethora on local and semilocal convergence results on iterative methods can be found in [1–20].

Newton's method given by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad \text{for each } n = 0, 1, 2, \ldots, \tag{1.2}$$

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where x_0 is an initial point, is undoubtedly the most famous quadratically convergent iterative method for approximating x^* [1,6,8,11]. To avoid the computation of the first derivative at each step, the Secant method is used as an efficient alternative to Newton's method, which is written as follows [1–20]:

$$x_{n+1} = x_n - L_n^{-1} F(x_n), \quad L_n = [x_{n-1}, x_n; F], \text{ for each } n = 0, 1, 2, \ldots, \tag{1.3}$$

where $x_{-1}, x_0 \in D$, are initial points, $[x, y; F] \in L(X, Y)$, the space of bounded linear operators from X to Y is a divided difference of order one for the operator F at the points $x, y \in \Omega$ satisfying $[x, y; F](x - y) = F(x) - F(y)$ for each $x, y \in \Omega$ with $x \neq y$. If F is Fréchet differentiable then $F'(x) = [x, x; F]$.

The algorithm of the Secant method requires the solution of the system of equations say in the case $X = Y = R^m$ given by

$$L_n(x_{n+1} - x_n) = F(x_n), \quad \text{for each } n = 0, 1, 2, \ldots. \tag{1.4}$$

This iterative process has convergence order $\frac{1+\sqrt{5}}{2}$. In order to improve the convergence, Moser method [15] defined by

$$\begin{aligned} x_{n+1} &= x_n - B_n F(x_n) \\ B_{n+1} &= 2B_n - B_n L_{n+1} B_n, \end{aligned} \tag{1.5}$$

where $x_0 \in D$ is an initial point and $B_0 \in L(X, Y)$ is a given operator was introduced. Recently, a comparison was given in [11] between the Secant method (1.3) and the Moser method (1.5) in the semilocal case. The convergence criteria and the convergence domains between the two methods were also compared. In the present, study, using the method of recurrent relations, we obtain new weaker convergence criteria [11,15–18] for both methods under less computational cost of the constants involved. In particular, the advantages of our approach are: weaker sufficient convergence criteria; weaker Lipschitz conditions; tighter error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ for each $n = 0, 1, 2, \ldots$ and an at least as precise information on the location of the solution x^*.

The rest of the paper is organized as follows. Section 2 contains the convergence of both methods. The numerical examples where older convergence criteria do not hold but our convergence criteria hold can be found in Section 3.

In the rest of the paper we denote by $U(x, r)$, $\bar{U}(x, r)$ the open and closed balls in X with center $x \in D$ and of radius $r > 0$, respectively.

2. Semilocal convergence

In this section we shall study the semilocal convergence of the Moser method (1.5) for $S = (F, x_{-1}, x_0, B_0, L_0)$ belonging to the class $\mathcal{C} = C(l_0, l, \alpha, \mu, \beta, \delta)$ defined as follows:

Definition 2.1. Let $l_0 > 0$, $l > 0$, $\alpha \geq 0$, $\mu \geq 0$, $\beta > 0$ and $\delta \geq 0$ be given constants. Set

$$\begin{aligned} &\eta = \beta\mu, \qquad d_0 = \delta + \beta(l_0\alpha + l\eta), \qquad c_0 = \beta\eta(l_0 + l d_0(1 + d_0))\eta, \\ &d_1 = d_0^2 + c_0 \quad \text{and} \quad c_1 = c_0 d_0 (1 + d_0)(1 + d_1). \end{aligned} \tag{2.1}$$

Suppose that

$$d_0^2 < d_0 - c_0 \quad \text{and} \quad c_1 \leq c_0. \tag{2.2}$$

We say that S belongs to the class $\mathcal{C}$ if:

($\mathcal{C}_1$) F is a nonlinear operator defined on a convex subset D of a Banach space X and with values in a Banach space Y.

($\mathcal{C}_2$) x_0 and x_{-1} are two points belonging to the interior $\dot{D}$ of D and satisfying the inequations

$$\|x_0 - x_{-1}\| \leq \alpha$$

and

$$\|F(x_0)\| \leq \mu.$$

($\mathcal{C}_3$) $B_0 \in L(X, Y)$ and $L_0 \in L(X, Y)$ satisfy

$$\|B_0\| \leq \beta \quad \text{and} \quad \|I - L_0 B_0\| \leq \delta.$$

($\mathcal{C}_4$) F is Fréchet differentiable on D and there exist a divided difference of order one $[x, y; F] : D \times D \to L(X, Y)$ such that

$$\|[x, y; F] - [u, v; F]\| \leq l_0\|x - u\| + l\|y - v\| \quad \text{for each } x, y, u, v \in D.$$

($\mathcal{C}_5$) $\bar{U}(x_0, R) \subseteq D$, where $R = \frac{\eta}{1-\Delta}$ and $\Delta = d_0(1 + d_0) < 1$.

Using these conditions and notation we can state the main semilocal convergence result for the Moser method (1.5).

Theorem 2.2. *If $S \in \mathcal{C}$, then, sequence $\{x_n\}$ generated by the Moser method (1.5) is well defined, remains in $\bar{U}(x_0, R)$ for each $n = 0, 1, 2, \ldots$ and converges with R-order of convergence at least $\frac{1+\sqrt{5}}{2}$ to a solution x^* of equation $F(x) = 0$ such that*

$$\|x_n - x^*\| \leq \frac{\Delta^n}{1-\Delta}\eta\gamma^{\delta_{n-1}} \quad \text{for each } n = 0, 1, 2, \ldots,$$

where

$$\begin{aligned}
&\gamma_1 = \frac{d_1}{d_0}, \qquad \gamma_2 = \frac{c_1}{c_0}, \quad \gamma = \max\{\gamma_1, \gamma_2\} \in (0, 1), \\
&\delta_{-1} = 0, \qquad \delta_0 = 0, \qquad \delta_{n-1} = s_1 + s_2 + \cdots + s_{n-1} \quad \text{for each } n = 2, 3, \ldots \\
&s_n = \alpha_{n+2} - 1, \qquad s_1 + s_2 + \cdots + s_n = \alpha_{n+4} - (n+3) \quad \text{for each } n = 1, 2, \ldots.
\end{aligned} \tag{2.3}$$

Moreover, the sequence $\{B_n\}$ converges to B^*, the bounded right inverse of $F'(x^*) = [x^*, x^*; F]$.

Remark 2.3. Before presenting the proof of Theorem 2.1, let us compare the new results with the corresponding ones in [11]. They supposed in [11] instead of $(\mathcal{C}_4)$ the condition:

$(\mathcal{C}_4)'$ $\|[x, y; F] - [u, v; F]\| \leq K(\|x - u\| + \|y - v\|)$.

Notice that $(\mathcal{C}_4)'$ implies condition $(\mathcal{C}_4)$ if we set $l_0 = l = K$ but not necessarily vice versa unless if $K = \max\{l, l_0\}$. However, in general

$$l_0 \leq K \tag{2.4}$$

and

$$l \leq K \tag{2.5}$$

hold. Set

$$\bar{d_0} = \delta + \beta K(\alpha + \eta), \qquad \bar{c_0} = \beta K\eta(1 + \bar{d_0} + \bar{d_0}^2), \qquad \bar{d_1} = \bar{d_0}^2 + \bar{c_0}, \qquad \bar{c_1} = \bar{c_0}\bar{d_0}(1 + \bar{d_0})(1 + \bar{d_1}). \tag{2.6}$$

The sufficient convergence criteria in [11] corresponding to (2.2) are given by

$$\bar{d_0}^2 < \bar{d_0} - \bar{c_0} \quad \text{and} \quad \bar{c_1} \leq \bar{c_0}. \tag{2.7}$$

But if strict inequality holds in condition (2.4) or (2.5), then by (2.1) and (2.6) we get that

$$d_0 < \bar{d_0}, \qquad c_0 < \bar{c_0}, \qquad d_1 < \bar{d_1} \quad \text{and} \quad c_1 < \bar{c_1}. \tag{2.8}$$

Hence, it follows from (2.2) and (2.7) that

$$(2.7) \Rightarrow (2.2) \tag{2.9}$$

but not necessarily vice versa. That is the new conditions are weaker. Moreover, new sequences $\{d_n\}$ and $\{c_n\}$ (involved in the proof of Theorem 2.2) defined by

$$d_n = d_{n-1}^2 + c_{n-1}, \qquad c_n = c_{n-1}d_{n-1}(1 - d_{n-1})(1 + d_n) \quad \text{for each } n = 1, 2, \ldots$$

are tighter than the old sequences $\{\bar{d}_n\}$ and $\{\bar{c}_n\}$ defined by

$$\bar{d}_n = \bar{d}_{n-1}^2 + \bar{c}_{n-1}, \qquad \bar{c}_n = \bar{c}_{n-1}\bar{d}_{n-1}(1 - \bar{d}_{n-1})(1 + \bar{d}_n).$$

That is by a dimple inductive argument, if $l_0 < K$ or $l < K$, we have that

$$d_n < \bar{d}_n \quad \text{and} \quad c_n < \bar{c}_n \quad \text{for each } n = 0, 1, \ldots.$$

The convergence radius in [11] is given by

$$\bar{R} = \frac{\eta}{1 - \bar{\Delta}}, \quad \bar{\Delta} = \bar{d_0}(1 + \bar{d_0}).$$

It follows from (2.8) that $R < \bar{R}$ for $l_0 < K$ or $l < K$.

Hence, the information on the location of the solution is at least as precise. Notice that in practice the computation of constant K involves the computation of constants l_0 and l. Hence, the new advantages are obtained under the same computational cost for the constants involved as before.

Proof of Theorem 2.2. With the above changes in the definition of c_0, d_0, c_1 and d_1 the proofs are analogous to the corresponding ones in [11]. To avoid repetitions we state the results and only present the proof of Lemma 2.4 which differs from the proof of Lemma 3.2 in [11].

Lemma 2.4. *If the sequences $\{d_n\}$ and $\{c_n\}$ are decreasing and $x_n \in D$ for each $n = 1, 2, \ldots,$ then*

(I_n) $\|F(x_n)\| \le d_{n-1}\|F(x_{n-1})\|$,
(II_n) $\|B(x_n)\| \le (1 + d_{n-1})\|B_{n-1}\|$,
(III_n) $\|x_{n+1} - x_n\| \le d_{n-1}(1 - d_{n-1})\|B_{n-1}\|\,\|F(x_{n-1})\|$,
(IV_n) $\|I - L_{n+1}B_n\| \le d_n$,
(V_n) $\|L_{n+2} - L_{n+1}\|\,\|B_{n+1}\| \le c_n$

and

$$\|B_{n+1} - B_n\| \le d_n\|B_n\|.$$

Proof. We shall show that the preceding assertions hold for $n = 1$. It follows from (1.5), (2.1) and $(\mathcal{C}_3)$ that

$$\|x_1 - x_0\| = \|B_0F(x_0)\| \le \|B_0\|\,\|F(x_0)\| = \beta\mu = \eta.$$

By hypothesis $x_1 \in D$. That is

$$L_1 = [x_0, x_1; F] \quad \text{and} \quad B_1 = 2B_0 - B_0[x_0, x_1; F]B_0$$

exist. Using Moser method (1.5) for $n = 0$, we get in turn that

$$\begin{aligned} F(x_1) &= F(x_0) - [x_0, x_1; F](x_0 - x_1) \\ &= F(x_0) - [x_0, x_1; F]F(x_0) \\ &= (I - [x_0, x_1; F]B_0)F(x_0) \\ &= (I - L_1B_0)F(x_0). \end{aligned} \tag{2.10}$$

We also have that

$$\|I - L_1B_0\| \le \|I - L_0B_0\| + \|L_1 - L_0\|\,\|B_0\|$$

and by $(\mathcal{C}_2)$–$(\mathcal{C}_4)$ and (2.1)

$$\|L_1 - L_0\|\,\|B_0\| \le (l_0\|x_0 - x_{-1}\| + l\|x_1 - x_0\|)\|B_0\| \le (l_0\alpha + l\eta)\beta$$

and

$$\|I - L_1B_0\| \le \delta + \beta(l_0\alpha + l\eta) = d_0.$$

That is

$$\|F(x_1)\| \le d_0\|F(x_0)\|.$$

We also have by the second substep in Moser method (1.5) for $n = 0$ that

$$\|B_1\| = \|2B_0 - B_0L_1B_0\| \le (1 + \|I - L_1B_0\|)\|B_0\| \le (1 + d_0)\|B_0\|.$$

In view of the first substep of Moser method (1.5) for $n = 1$, we have

$$\|x_2 - x_1\| = \|B_1F(x_1)\| \le d_0(1 + d_0)\|B_0\|\,\|F(x_0)\| \le d_0(1 + d_0)\eta\delta = d_0(1 + d_0)\eta.$$

Consequently, we get

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le [1 + d_0(1 + d_0)]\|B_0\|\,\|F(x_0)\| \le [1 + d_0(1 + d_0)]\eta.$$

By hypothesis $x_2 \in D$. Hence, L_2 exists. Moreover, we have that

$$\begin{aligned} \|L_1 - L_2\|\,\|B_1\| &\le (l_0\|x_1 - x_0\| + \|x_2 - x_1\|)\|B_1\| \\ &\le (l_0\eta + ld_0(1 + d_0)\eta)(1 + d_0)\|B_0\| \\ &\le (l_0\eta + ld_0(1 + d_0)\eta)(1 + d_0)\eta\beta = c_0. \end{aligned}$$

Notice that

$$\|I - L_2B_1\| \le \|I - L_1B_0\|^2 + \|L_1 - L_2\|\,\|B_1\| \le d_0^2 + c_0 = d_1.$$

Hence, we showed the assertions for $n = 1$. The rest of the proof follows by straightforward mathematical induction on the integer n. □

Lemma 2.5. *Suppose that (2.2) holds. Then, the following assertions hold*

(i) $d_0(1 + d_0) < 1$,
(ii) *sequences $\{c_n\}$ and $\{d_n\}$ are decreasing*

(iii) $d_n < \gamma^{\alpha_n} d_{n-1}$ *and* $c_n < \gamma^{\beta_n} c_{n-1}$ *for each* $n = 1, 2, \ldots$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are the Fibonacci sequences defined by*

$$\alpha_1 = \alpha_2 = 1 \quad \text{and} \quad \alpha_{n+2} = \alpha_{n+1} + \alpha_n,$$
$$\beta_1 = 1, \qquad \beta_2 = 2 \quad \text{and} \quad \beta_{n+2} = \beta_{n+1} + \beta_n,$$

for each $n = 1, 2, \ldots$.

(iv) $d_n < \gamma^{s_n} d_0$ *and* $c_n = \gamma^{s_{n+1}} d_0$.

(v) $\alpha_n = \frac{l}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) > \frac{l}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ *for each* $n = 1, 2, \ldots$.

Proof. It follows exactly as the proof of Lemmas 3.3 and 4.1 in [11]. □

The proof of the rest of the assertions in Theorem 2.2 can be found in Theorems 3.4 and 4.2 in [11]. □

Next, we present the corresponding results to the ones in [11,16–18] for the Secant method (1.3) in an analogous way.

Consider the set $S^1 = (F, x_{-1}, x_0, L_0)$ belonging to the class $\mathcal{C}^1 = C(l_0, l, \alpha, \mu, \lambda)$ if:

Definition 2.6. Let $l_0 > 0, l > 0, \alpha \geq 0, \mu \geq 0$ and $\lambda > 0$ be given constants. Set

$$\rho = \lambda\mu, \qquad a_{-1} = \frac{\rho}{\alpha + \rho}, \qquad a_0 = \lambda(l_0\alpha + l\rho) \quad \text{and} \quad a_1 = \frac{a_0 a_{-1}}{(1 - a_0)^2}.$$

Suppose that

$$a_0 < (1 - a_1)^2 \quad \text{and} \quad a_{-1} < (1 - a_0)^2.$$

We say that S^1 belongs to the class $\mathcal{C}^1$ if $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_4)$ and

$(\mathcal{C}_5)'$ $\bar{U}(x_0, R_0) \subseteq D$,

where

$$R_0 = \frac{\eta}{1 - \Delta_0}, \qquad \Delta_0 = \frac{a_0}{1 - a_0} < 1$$

hold.

Then, as in our Theorem 2.2 (see also the proofs in [11,16–18]), we arrive at the following semilocal result for the Secant method (1.3).

Theorem 2.7. *If* $S^1 \in \mathcal{C}^1$, *then sequence* $\{x_n\}$ *generated by the Secant method (1.3) is well defined, remains in* $\bar{U}(x_0, R_0)$ *for each* $n = 0, 1, 2, \ldots$ *and converges with R-order of convergence at least* $\frac{1+\sqrt{5}}{2}$ *to a unique solution* x^* *of equation* $F(x) = 0$ *in* $\bar{U}(x_0, R_0)$.

Remark 2.8. A remark similar to Remark 2.3 can follow for the Secant method (1.3). Notice that in [11,16–18] the constants are defined (using $(\mathcal{C}_4)'$ instead of $(\mathcal{C}_4)$):

$$\bar{a}_{-1} = a_{-1}, \qquad \bar{a}_0 = \lambda K(\alpha + \rho), \qquad \bar{a}_1 = \frac{\bar{a}_0 \bar{a}_{-1}}{(1 - \bar{a}_0)^2}$$

and

$$\bar{R}_0 = \frac{\eta}{1 - \bar{\Delta}_0}, \qquad \bar{\Delta}_0 = \frac{\bar{a}_0}{1 - \bar{a}_0} < 1.$$

The sufficient convergence criteria are:

$$\bar{a}_0 \leq (1 - \bar{a}_1)^2 \quad \text{and} \quad \bar{a}_{-1} < (1 - \bar{a}_0)^2$$

and

$$\bar{U}(x_0, \bar{R}_0) \subseteq D.$$

Notice again that if $l_0 < K$ or $l < K$ we have:

$$a_0 < \bar{a}_0, \quad a_1 < \bar{a}_1 \quad \text{and} \quad R_0 < \bar{R}_0.$$

The rest of the comments are identical to the ones given in Remark 2.3.

3. Numerical examples

We present four numerical examples in which the old convergence criteria given in [11] are not satisfied but our new criteria are satisfied. The first two involve the Moser method (1.5) and the last two the Secant method (1.3).

Example 1 (*Moser Method*)**.** In the following example, we consider the real function

$$x^3 - 0.49 = 0. \tag{3.11}$$

We take the starting points $x_0 = 0.85,\ x_{-1} = 1$ and we consider the domain $\Omega = B(x_0, 0.5)$. In this case, we obtain

$$\mu = 0.124125,$$
$$\alpha = 0.15,$$
$$L_0 = 2.5725,$$
$$k = l = 6$$

and

$$l_0 = 3.$$

Choosing $B_0 = 0.4$, we obtain that $\delta = 0.029$ and $\eta = 0.04965$. Notice that the old hypothesis $d_0^2 < d_0 - c_0$ is not satisfied since

$$d_0^2 = 0.258227 > 0.190719 = d_0 - c_0$$

but with the new definitions of d_0 and c_0 given in Section 2 conditions of Theorem 2.2 are satisfied since

$$d_0^2 = 0.107689 < 0.320668 = d_0 - c_0,$$
$$c_1 = 0.00364164 < 0.00749232 = c_0$$

and

$$R = 0.212531 < 0.5.$$

So, Moser method starting from $x_0 \in B(x_0, 0.5)$ converges to the solution of (3.11) from Theorem 2.2.

Example 2 (*Moser Method*)**.** Consider the following nonlinear boundary value problem

$$\begin{cases} u'' = -u^3 - \dfrac{1}{4}\,u^2 \\ u(0) = 0, \qquad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)\,\left(u^3(t) + \frac{1}{4}\,u^2(t)\right)\,dt \tag{3.12}$$

where, $\mathcal{Q}$ is the Green function:

$$\mathcal{Q}(s,t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\le s\le 1} \int_0^1 |\mathcal{Q}(s,t)|\,dt = \frac{1}{8}.$$

Then problem (3.12) is in the form (1.1), where, F is defined as

$$[F(x)]\,(s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)\,\left(x^3(t) + \frac{1}{4}\,x^2(t)\right)\,dt.$$

The Fréchet derivative of the operator F is given by

$$[F'(x)y]\,(s) = y(s) - 3\int_0^1 \mathcal{Q}(s,t)x^2(t)y(t)dt - \frac{1}{2}\int_0^1 \mathcal{Q}(s,t)x(t)y(t)dt.$$

Choosing $x_0(s) = s$ and $R = 0.8$ we have that $\|F(x_0)\| \le 0.15625\ldots$. Define the divided difference defined by

$$[x, y; F] = \int_0^1 F'(\tau x + (1-\tau)y)d\tau.$$

Taking into account that

$$\begin{aligned}\|[\mathbf{x},\mathbf{y};F]-[\mathbf{u},\mathbf{v};F]\| &\le \int_0^1 \|F'(\tau\mathbf{x}+(1-\tau)\mathbf{y})-F'(\tau\mathbf{u}+(1-\tau)\mathbf{v})\|\,d\tau\\ &\le \frac{1}{8}\int_0^1 \Big(3\tau^2(\|x^2-v^2\|+\|y^2-u^2\|)+6\tau(1-\tau)\|xy-uv\|\\ &\quad+\frac{1}{2}\tau\|x-u\|+\frac{1}{2}(1-\tau)\|y-v\|\Big)\,d\tau\\ &\le \frac{1}{8}\left(\|\mathbf{x}^2-\mathbf{u}^2\|+\left(\|\mathbf{y}^2-\mathbf{v}^2\|\right)+(\|\mathbf{xy}-\mathbf{uv}\|)\right)+\frac{1}{32}\left(\|\mathbf{x}-\mathbf{u}\|+(\|\mathbf{y}-\mathbf{v}\|)\right).\end{aligned}$$

And it is easy to see that

$$k=l=\frac{1}{64}+\frac{1}{4}(2+2R)=0.915625$$

and

$$l_0=\frac{1}{64}+\frac{1}{4}(1+R)=0.465625.$$

Choosing $x_{-1}(s)=\frac{99s}{100}$ and $beta=1.67$ we find that

$$L_0=0.432213,$$
$$\delta=0.0517949,$$
$$\eta=0.260938$$

and

$$\alpha=0.01.$$

Notice that the old hypothesis $d_0^2<d_0-c_0$ is not satisfied since

$$d_0^2=0.217234>-0.518597=d_0-c_0$$

but with the new definitions of d_0 and c_0 given in Section 2 conditions of Theorem 2.2 are satisfied since

$$d_0^2=0.210285<0.334481=d_0-c_0,$$
$$c_1=0.110748<0.124088=c_0$$

and

$$R=0.787983<0.8.$$

So, Moser method starting from $x_0\in B(x_0,0.8)$ converges to the solution of (3.11) from Theorem 2.2.

Example 3 (*Secant Method*)**.** Let $X=Y=\mathcal{C}[0,1]$, the space of continuous functions defined in $[0,1]$ equipped with the max-norm. Let $\Omega=\{x\in\mathcal{C}[0,1];\ \|x\|\le R\}$, such that $R>1$ and F defined on Ω and given by

$$F(x)(s)=x(s)-f(s)-\frac{3}{2}\int_0^1 G(s,t)x(t)^3\,dt,\quad x\in C[0,1],\ s\in[0,1],$$

where $f\in\mathcal{C}[0,1]$ is a given function, λ is a real constant and the kernel G is the Green's function

$$G(s,t)=\begin{cases}(1-s)t, & t\le s,\\ s(1-t), & s\le t.\end{cases}$$

In this case, for each $x\in\Omega$, $F'(x)$ is a linear operator defined on Ω by the following expression:

$$[F'(x)(v)](s)=v(s)-\frac{9}{2}\int_0^1 G(s,t)x(t)^2v(t)\,dt,\quad v\in C[0,1],\ s\in[0,1].$$

If we choose $x_0(s)=f(s)=22$, we obtain

$$\|F(x_0)\|\le\frac{3}{16}.$$

On the other hand, for $x, y \in \Omega$ we have Choosing $R = 0.8$ we have that $\|F(x_0)\| \leq \frac{3}{8}$. Define the divided difference defined by

$$[x, y; F] = \int_0^1 F'(\tau x + (1-\tau)y)d\tau.$$

Taking into account that

$$\begin{aligned}\|[\mathbf{x}, \mathbf{y}; F] - [\mathbf{u}, \mathbf{v}; F]\| &\leq \int_0^1 \|F'(\tau \mathbf{x} + (1-\tau)\mathbf{y}) - F'(\tau \mathbf{u} + (1-\tau)\mathbf{v})\| \, d\tau \\ &\leq \frac{3}{2}\int_0^1 \left(3\tau^2(\|x^2 - v^2\| + \|y^2 - u^2\|) + 6\tau(1-\tau)\|xy - uv\|\right) d\tau \\ &\leq \frac{3}{2}\left(\|\mathbf{x}^2 - \mathbf{u}^2\| + \left(\|\mathbf{y}^2 - \mathbf{v}^2\|\right) + (\|\mathbf{xy} - \mathbf{uv}\|)\right).\end{aligned}$$

And it is easy to see that

$$k = l = 1.3125$$

and

$$l_0 = 0.65625.$$

Choosing $x_{-1}(s) = s$ we find that

$$\begin{aligned}L_0 &= 2.625, \\ \lambda &= 0.380952, \\ \rho &= 0.142857\end{aligned}$$

and

$$\alpha = 1.$$

Notice that the old hypothesis $a_0^2 < (1-a_1)^2$ is not satisfied since

$$a_0^2 = 0.571429 < 0.373457 = (1-a_1)^2$$

but with the new definitions of a_0 and a_1 given in Section 2 conditions of Theorem 2.7 are satisfied since

$$\begin{aligned}a_0^2 &= 0.321429 < 0.610352 = (1-a_1)^2, \\ a_{-1} &= 0.125 < 0.460459 = (1-a_0)^2\end{aligned}$$

and

$$R = 0.271429 < 0.75.$$

So, Secant method starting from $x_0 \in B(x_0, 0.8)$ converges to the solution of (3.11) from Theorem 2.7.

Example 4 (*Secant Method*)**.** Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ and let consider the real functions

$$F(x) = x^3 - A$$

where $A \in (1.2, 1.25)$ and we are going to apply secant-method to find the solution of $F(x) = 0$. We take the starting point $x_0 = 1$ we consider the domain $\Omega = B(x_0, 1)$ and we let x_{-1} free in order to find a relation between A and x_{-1} for which old criteria are not satisfied but new criteria are satisfied. In this case, we obtain

$$\mu = |(1-A)|, \qquad \lambda = \frac{1}{|1 + x_{-1} + x_{-1}^2|}, \qquad \rho = \frac{|(1-A)|}{|1 + x_{-1} + x_{-1}^2|},$$
$$l = 8 \quad \text{and} \quad l_0 = 4.$$

Taking all this data into account we obtain the following criteria:

(i) If $1.2 < A \leq 1.2338550182109815\ldots$ and $\varepsilon_1 < x_{-1} \leq \varepsilon_2$
(ii) if $1.2338550182109815\ldots < A < 1.25$ and $\varepsilon_1 < x_{-1} \leq \varepsilon_2$

where ε_1 is the smallest real root of

$$\begin{aligned} p_1(t) = {} & 1.42455 \times 10^6 - 3.34994 \times 10^6 A + 3.14976 \times 10^6 A^2 - 1.4807 \times 10^6 A^3 + 348160.A^4 - 32768.A^5 \\ & + \left(855178. - 1.61043 \times 10^6 A + 1.13216 \times 10^6 A^2 - 352256.A^3 + 40960.A^4\right) t \\ & + \left(1.50297 \times 10^6 - 2.73699 \times 10^6 A + 1.85312 \times 10^6 A^2 - 552960.A^3 + 61440.A^4\right) t^2 \\ & + \left(-1.80117 \times 10^6 + 3.69114 \times 10^6 A - 2.83661 \times 10^6 A^2 + 966656.A^3 - 122880.A^4\right) t^3 \\ & + \left(-97945. - 58672.A + 206208.A^2 - 116736.A^3 + 20480.A^4\right) t^4 \\ & + \left(-1.51615 \times 10^6 + 2.07942 \times 10^6 A - 954240.A^2 + 147456.A^3\right) t^5 \\ & + \left(1.61162 \times 10^6 - 2.54336 \times 10^6 A + 1.32115 \times 10^6 A^2 - 225280.A^3\right) t^6 \\ & + \left(-373132. + 585920.A - 312192.A^2 + 57344.A^3\right) t^7 \\ & + \left(822221. - 806184.A + 212352.A^2 - 6144.A^3\right) t^8 \\ & + \left(-777826. + 810176.A - 208000.A^2\right) t^9 \\ & + \left(296569. - 291584.A + 73408.A^2\right) t^{10} \\ & + \left(-241882. + 140480.A - 11648.A^2\right) t^{11} \\ & + \left(200237. - 105392.A + 960.A^2\right) t^{12} \\ & + (-87548. + 42368.A)t^{13} \\ & + (38114. - 9184.A)t^{14} \\ & + (-22244. + 1088.A)t^{15} \\ & + (9671. - 56.A)t^{16} \\ & - 2510.t^{17} \\ & + 375.t^{18} \\ & - 30.t^{19} \\ & + t^{20} \end{aligned}$$

and ε_2 is the smallest and the third smallest real root of

$$\begin{aligned} p_2(t) = {} & 4.35711 \times 10^6 - 1.52346 \times 10^7 A + 2.27881 \times 10^7 A^2 - 1.89065 \times 10^7 A^3 \\ & + 9.39821 \times 10^6 A^4 - 2.79962 \times 10^6 A^5 + 462848.A^6 - 32768.A^7 \\ & + \left(2.77159 \times 10^6 - 8.29674 \times 10^6 A + 1.03205 \times 10^7 A^2 - 6.82733 \times 10^6 A^3 \right. \\ & \left. + 2.53299 \times 10^6 A^4 - 499712.A^5 + 40960.A^6\right) t \\ & + \left(4.90296 \times 10^6 - 1.43386 \times 10^7 A + 1.74015 \times 10^7 A^2 - 1.12128 \times 10^7 A^3 \right. \\ & \left. + 4.04448 \times 10^6 A^4 - 774144.A^5 + 61440.A^6\right) t^2 \\ & + \left(-8.77232 \times 10^6 + 2.74991 \times 10^7 A - 3.5815 \times 10^7 A^2 + 2.4819 \times 10^7 A^3 \right. \\ & \left. - 9.6544 \times 10^6 A^4 + 1.99885 \times 10^6 A^5 - 172032.A^6\right) t^3 \\ & + \left(-2.58244 \times 10^6 + 5.81422 \times 10^6 A - 4.81691 \times 10^6 A^2 + 1.5741 \times 10^6 A^3 \right. \\ & \left. + 19840.A^4 - 124928.A^5 + 20480.A^6\right) t^4 \\ & + \left(-9.08616 \times 10^6 + 2.22009 \times 10^7 A - 2.16173 \times 10^7 A^2 + 1.04902 \times 10^7 A^3 \right. \\ & \left. - 2.53939 \times 10^6 A^4 + 245760.A^5\right) t^5 \\ & + \left(9.41664 \times 10^6 - 2.56844 \times 10^7 A + 2.77974 \times 10^7 A^2 - 1.49338 \times 10^7 A^3 \right. \\ & \left. + 3.98355 \times 10^6 A^4 - 421888.A^5\right) t^6 \\ & + \left(-756210. + 2.80065 \times 10^6 A - 3.7805 \times 10^6 A^2 + 2.41325 \times 10^6 A^3 - 744320.A^4 + 90112.A^5\right) t^7 \\ & + \left(7.53916 \times 10^6 - 1.50387 \times 10^7 A + 1.12769 \times 10^7 A^2 - 3.79831 \times 10^6 A^3 + 501120.A^4 - 6144.A^5\right) t^8 \\ & + \left(-6.76986 \times 10^6 + 1.50038 \times 10^7 A - 1.23278 \times 10^7 A^2 + 4.45402 \times 10^6 A^3 - 597120.A^4\right) t^9 \\ & + \left(2.17202 \times 10^6 - 4.54449 \times 10^6 A + 3.61115 \times 10^6 A^2 - 1.29184 \times 10^6 A^3 + 175808.A^4\right) t^{10} \\ & + \left(-3.73234 \times 10^6 + 5.82617 \times 10^6 A - 3.11178 \times 10^6 A^2 + 610496.A^3 - 21888.A^4\right) t^{11} \end{aligned}$$

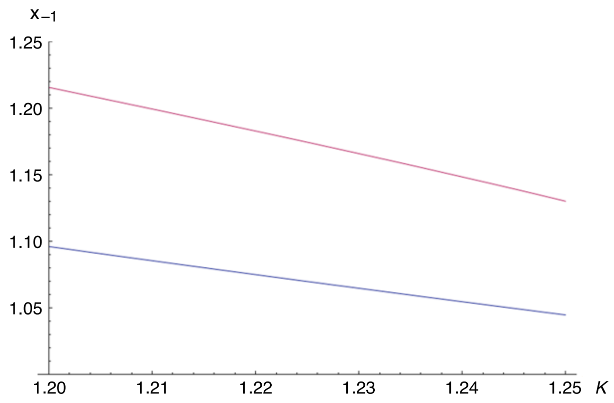

Fig. 1. Values of x_{-1} in which our criteria are satisfied but previous one not.

$$
\begin{aligned}
&+\left(3.19462\times 10^6-5.29662\times 10^6 A+2.89849\times 10^6 A^2-525552.A^3+960.A^4\right)t^{12}\\
&+\left(-1.37609\times 10^6+2.10122\times 10^6 A-1.08288\times 10^6 A^2+188608.A^3\right)t^{13}\\
&+\left(1.1962\times 10^6-1.35413\times 10^6 A+443442.A^2-32864.A^3\right)t^{14}\\
&+\left(-936214.+1.03558\times 10^6 A-288748.A^2+2688.A^3\right)t^{15}\\
&+\left(443797.-451436.A+116011.A^2-56.A^3\right)t^{16}\\
&+\left(-244910.+176412.A-25470.A^2\right)t^{17}\\
&+\left(158193.-91668.A+2919.A^2\right)t^{18}\\
&+\left(-75352.+38702.A-130.A^2\right)t^{19}\\
&+\left(29454.-10048.A+A^2\right)t^{20}\\
&+(-12920.+1454.A)t^{21}\\
&+(5475.-100.A)t^{22}\\
&+(-1602.+2.A)t^{23}\\
&+279.t^{24}\\
&-26.t^{25}\\
&+t^{26}.
\end{aligned}
$$

In other words for every value of x_{-1} between the two lines that appear in Fig. 1, the conditions given in [11] are not satisfied but our new conditions are satisfied.

For example, we choose $A = 1.22$ and $x_{-1} = 1.15$ we obtain

$$L_0 = 3.4725\ldots,$$
$$\lambda = 0.287977\ldots,$$
$$\rho = 0.0633549\ldots$$

and

$$\alpha = 0.15.$$

Notice that the old hypothesis $a_0^2 < (1-a_1)^2$ is not satisfied since

$$a_0^2 = 0.241602 > 0.189621 = (1-a_1)^2$$

but with the new definitions of a_0 and a_1 given in Section 2 conditions of Theorem 2.7 are satisfied since

$$a_0^2 = 0.101598 < 0.401839 = (1-a_1)^2,$$
$$a_{-1} = 0.296946\ldots < 0.464109\ldots = (1-a_0)^2$$

and

$$R = 0.119061 < 1.$$

So, Secant method starting from $x_0 \in B(x_0, 0.8)$ converges to the solution of $F(x) = 0$ from Theorem 2.7.

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Conflict of Interest: The authors declare that they have no conflict of interest.

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