ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations

J.E. Macías-Díaz [a,*], J.A. Guerrero [b]

[a] Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Avenida Universidad 940, Ciudad Universitaria, Aguascalientes 20131, Mexico

[b] Departamento de Estadística, Universidad Autónoma de Aguascalientes, Avenida Universidad 940, Ciudad Universitaria, Aguascalientes 20131, Mexico

ARTICLE INFO

Article history:
Received 21 July 2015
Received in revised form 16 November 2015

MSC:
65M06
35K57
35K61

Keywords:
Two-dimensional model
Nonlinear time-dependent partial differential equation
Finite-difference scheme
Symmetry preservation
Boundedness and positivity preservation

ABSTRACT

We propose a numerical method to approximate the solutions of generalized forms of two bi-dimensional models of mathematical physics, namely, the Burgers–Fisher and the Burgers–Huxley equations. In one-dimensional form, the literature in the area gives account of the existence of analytical solutions for both models, in the form of traveling-wave fronts bounded within an interval I of the real numbers. Motivated by this fact, we propose a finite-difference methodology that guarantees that, under certain analytical conditions on the model and computer parameters, estimates within I will evolve discretely into new estimates which are likewise bounded within I. Additionally, we establish the preservation in the discrete domain of the skew-symmetry of the solutions of the models under study. Our computational implementation of the method confirms numerically that the properties of positivity and boundedness are preserved under the analytical constraints derived theoretically. Our simulations show a good agreement between the analytical solutions derived in the present work and the corresponding numerical approximations.

1. Introduction

In the computational investigation of mathematical models that describe the dynamics of physical systems, it is important that the numerical techniques reflect reliably the phenomena that they describe. In these terms, the design of consistent and computationally stable methods to approximate the solutions of some particular equation in mathematical physics, could be of little interest to a scientist who wishes to guarantee that certain physical properties of interest are observed in the numerical simulations. For example, a scientist who investigates computationally the dynamics of traveling waves in shallow waters [1] may be interested in guaranteeing that the height of the surface of the fluid be nonnegative. In the context of thermodynamics, one could study numerically the propagation of a forest fire using the temperature measured in an absolute scale as the variable of interest [2], in which case, negative values of the variable are meaningless. A chemist could be interested in predicting numerically the evolution of multi-compound, chemical systems which satisfy conservation constraints [3], in such a way that the sum of the contributions of the compounds that make up the total mass be constant.

All the applications mentioned above share a feature in common: they are problems that require the use of methods that preserve numerically some physical characteristics of the solutions. Of course, many other physical problems share the same

* Corresponding author. Tel.: +52 449 910 8400; fax: +52 449 910 8401.
E-mail addresses: jemacias@correo.uaa.mx (J.E. Macías-Díaz), jaguerrero@correo.uaa.mx (J.A. Guerrero).

http://dx.doi.org/10.1016/j.cam.2016.01.032
0377-0427/

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032

characteristic, be it in the context of population dynamics [4,5], in the development of biological films [6], or in problems arising in fluid mechanics [7] among other scenarios. However, in most of them, the mathematical and physical properties of positivity, boundedness and the conservation of mass and energy, are the crucial features of concern. In fact, the literature gives evidence of a vast amount of works that have dealt with the conditions of positivity and boundedness both analytically and numerically [8–11]. Similar remarks hold for the conservation of mass, momentum and energy of systems [12–14].

From a numerical perspective, many computational methods have been designed to guarantee the conservation of the properties of positivity and boundedness of solutions; however, most of the techniques proposed in the literature follow non-conventional criteria. Some of the positivity-preserving methods available in the literature follow the non-standard methodology employed by R.E. Mickens in many of his works [8,9,15]. In fact, this and other non-traditional approaches have proved fruitful in many other problems where the preservation of the positivity and even the boundedness of solutions are two important criteria in the design of reliable computational methods [16,17].

In the present work, we consider a two-dimensional time-dependent partial differential equation that generalizes many models from mathematical physics. Among other interesting features, our equation contemplates the inclusion of a nonlinear term of advection/convection, and two possible nonlinear reaction laws which generalize the reaction term used by R.A. Fisher [18], and A. Kolmogorov, I. Petrovsky and N. Piscounov [19] in their seminal works. In the one-dimensional scenario, the existence of positive and bounded solutions of the model in the form of traveling-wave fronts, is a well-known fact [20]. For the two-dimensional case, we employ the traveling-wave solutions of the one-dimensional equation, and extend them to obtain wave fronts for the higher dimensional model. Then, we propose a linear finite-difference method to approximate the solutions of the equations under investigation (see [7,10,11,21–24] and other works in which the first author has followed linear approaches), and establish sufficient conditions on the model and computational constants under which the properties of positivity and boundedness of the solutions are preserved. These conditions are relatively flexible, and the method performs well when approximating the analytical, traveling-wave solutions obtained in this manuscript. Additionally, our numerical technique is capable of preserving the skew-symmetry property of solutions.

Our work is sectioned as follows. In Section 2, we introduce the mathematical models that motivate our study. As it will be noticed, these models are generalizations of the well-known two-dimensional Burgers–Fisher and Burgers–Huxley equations, and we derive here some analytical solutions in order to validate the numerical performance of the method designed in this work. Additionally, we establish some skew-symmetry properties of the solutions of our models. Section 3 introduces the finite-difference technique to approximate solutions of our equations. We give therein a convenient vector representation of the methodology. In Section 4, we establish the properties of interest of our technique. Next, we present some numerical simulations which show that our scheme provides good approximations to the exact solutions of our models, and that the positive and the bounded characters of the solutions are preserved in the discrete domain.

2. Preliminaries

Throughout this work, we use the symbol $\mathbb{R}^+$ to represent the set of non-negative numbers. Let α be a real number, let γ belong in $(0, 1)$, and let p be a positive integer. Let $u = u(x, y, t)$ be a real function defined for every $(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+$, which is twice differentiable in the interior of its domain. In this work, we consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} + \alpha u^p \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) - \nabla^2 u - uf(u) = 0, \tag{1}$$

where ∇^2 represents the Laplacian operator in the spatial variables x and y.

The reaction factor f is a real function defined in all of $\mathbb{R}$ which, for purposes of the present work, may assume one of two possible nonlinear forms: On the one hand, it will be given by the expression

$$f(u) = 1 - u^p, \tag{2}$$

in which case, our model will be called the *Burgers–Fisher equation*; on the other, it will be defined as

$$f(u) = (1 - u^p)(u^p - \gamma) \tag{3}$$

and, in this scenario, the corresponding model under investigation will be denominated the *Burgers–Huxley equation* (here, we follow the classification presented in [20,25]). Clearly, the constant α assumes the role of a coefficient of advection/convection; meanwhile, the exponent p is introduced in order to generalize the reaction law of the models proposed by Fisher [18], and Kolmogorov, Petrovsky and Piscounov [19].

For the sake of meaningfulness, it is important to realize that the models considered in this work are generalizations of several partial differential equations from mathematical biology and physics. To start with, what we have called the Burgers–Fisher equation is an advective/convective generalization of the classical Fisher's equation from population dynamics [18,19] with a more generic logistic law; also, it is a version of the Newell–Whitehead–Segel equation from fluid mechanics with convection effects included [26,27]; finally, the resulting model is the classical heat equation if f is identically equal to zero and α equals zero. In the case of the Burgers–Huxley model, it is an advective version of the famous FitzHugh–Nagumo equation studied in the transmission of electric signals in the nervous system [28]; the model is also an extension of the Huxley's equation investigated in the context of electrodynamics [29], with a generalized nonlinear factor.

It is worth noticing that the models considered in the present work possess symmetry properties which will be preserved in the discrete scenario of Section 3. For every function $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^+$ which is twice differentiable in the interior of its domain, let $\mathcal{L}(u)$ be the left-hand side of Eq. (1). With these conventions, the following result summarizes the symmetry properties of interest on the solutions of our model; its statement is valid for a class of functions f which encompasses the nonlinear reaction laws introduced above when p is an even number.

Proposition 1. *Let f be an even function, let p be an even number, and let u be a twice differentiable function. Then $\mathcal{L}(-u) = -\mathcal{L}(u)$.*

As a consequence, for an even function f and an even integer p, a function u satisfies (1) if and only if $-u$ satisfies it, too. This property of the solutions of our model will be referred to as the property of *skew-symmetry*, and together with the fact that our partial differential equation possesses positive and bounded solutions, will be one of the main properties of the solutions of (1).

It is important to poses exact solutions of the Burgers–Fisher and the Burgers–Huxley equations presented above. In particular, we are interested in finding bounded solutions of the model (1). In the one-dimensional scenario, it is well known that the Burgers–Fisher and the Burgers–Huxley equations have analytical solutions in the form of traveling-wave fronts. More precisely, let $u = u(x, t)$ be a function defined in $\mathbb{R} \times \mathbb{R}^+$. In the one-dimensional case, the partial differential equation (1) becomes

$$\frac{\partial u}{\partial t} + \alpha u^p \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - uf(u) = 0. \tag{4}$$

The following paragraphs quote some analytic solutions of (4) corresponding to the Burgers–Fisher or the Burgers–Huxley forms of the nonlinear reaction factor.

Remark 2. Eq. (4) with reaction function f given by (2) has a traveling-wave solution of the form

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2} \tanh \left[\frac{-\alpha p}{2(p+1)} \left(x - \left(\frac{\alpha}{p+1} + \frac{p+1}{\alpha} \right) t \right) \right] \right)^{1/p} \tag{5}$$

(see [20]). This expression represents a wave-front bounded within $(0, 1)$, which connects the constant solutions $u = 0$ and $u = 1$.

Remark 3. As in the previous example, we consider a nonlinear reaction factor of the form (2). In this case, we fix $p = 2$ and $\alpha = 0$. The resulting model is the well-known Newell–Whitehead–Segel equation from fluid mechanics [26,27], which has an exact traveling-wave solution with horizontal asymptotes equal to -1 and 1, given by the formula

$$u(x,t) = \frac{C_1 \exp\left(\frac{1}{\sqrt{2}}x\right) - C_2 \exp\left(-\frac{1}{\sqrt{2}}x\right)}{C_1 \exp\left(\frac{1}{\sqrt{2}}x\right) + C_2 \exp\left(-\frac{1}{\sqrt{2}}x\right) + C_3 \exp\left(-\frac{3}{2}t\right)}, \tag{6}$$

for suitable values of the real constants C_1, C_2 and C_3 [28].

Remark 4. If f is given by the formula (3) and $p = 1$, then the associated Burgers–Huxley model possesses a traveling-wave solution bounded within $(0, 1)$, with horizontal asymptotes equal to 0 and 1. This solution is given by the expression

$$u(x,t) = \frac{1}{2} - \frac{1}{2} \tanh \left[\frac{1}{r - \alpha} (x - vt) \right], \tag{7}$$

where

$$r = \sqrt{\alpha^2 + 8}, \qquad v = \frac{(\alpha - r)(2\gamma - 1) + 2\alpha}{4}. \tag{8}$$

This family of solutions is obtained as the result of employing symbolic computations and some relevant nonlinear transformations [20,30,31].

Remark 5. For the one-dimensional Burgers–Huxley model, a traveling-wave solution bounded within the horizontal asymptotes $u = 0$ and $u = \gamma^{1/p}$, is provided by the expression (see [20])

$$u(x,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} \tanh[a_1(x - a_2 t)] \right)^{1/p}, \tag{9}$$

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032

with constants

$$a_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4(1+p)}}{4(1+p)} p\gamma, \tag{10}$$

$$a_2 = \frac{2\alpha\gamma - (1+p-\gamma)(-\alpha + \sqrt{\alpha^2 + 4(1+p)})}{2(1+p)}. \tag{11}$$

For the two-dimensional case, we let φ be any real number, and introduce

$$\xi_\pm = x \sin\varphi \pm y \cos\varphi. \tag{12}$$

In terms of this new spatial variable, the two-dimensional model (1) reduces to the one-dimensional equation (4), with $x = \xi_\pm$ and the new advection/convection coefficient given in terms of α and φ by $\alpha(\sin\varphi \pm \cos\varphi)$. In view of this observation, any solution $u(\xi_\pm, t)$ of the one-dimensional model yields a family of solutions of the two-dimensional equation (1) when we let $\xi_\pm$ be given by (12).

3. Computational technique

In order to develop a finite-difference method to approximate the solutions of (1), we will restrict our attention to a bounded rectangular spatial domain $D = [a, b] \times [c, d]$ of the Cartesian plane, and we will approximate the exact solutions of the model at a given time $T > 0$. For the sake of simplicity, we fix uniform partitions $a = x_0 < x_1 < \cdots < x_M = b$ and $c = y_0 < y_1 < \cdots < y_N = d$ of the spatial intervals $[a, b]$ and $[c, d]$, respectively, with respective norms $\Delta x = (b - a)/M$ and $\Delta y = (d - c)/N$; additionally, we fix a uniform partition $0 = t_0 < t_1 < \cdots < t_K = T$ of the temporal interval $[0, T]$, with partition norm $\Delta t = T/K$. Finally, we let $u_{m,n}^k$ be an approximation of the exact value of $u(x_m, y_n, t_k)$, for every $m \in \{0, 1, \ldots, M\}$, every $n \in \{0, 1, \ldots, N\}$ and every $k \in \{0, 1, \ldots, K\}$.

In order to present conveniently a numerical method to approximate the solutions of (1), we will employ the following standard linear operators:

$$\delta_t u_{m,n}^k = \frac{u_{m,n}^{k+1} - u_{m.n}^k}{\Delta t}, \tag{13}$$

$$\delta_x^{(1)} u_{m,n}^k = \frac{u_{m+1,n}^k - u_{m-1,n}^k}{2\Delta x}, \tag{14}$$

$$\delta_y^{(1)} u_{m,n}^k = \frac{u_{m,n+1}^k - u_{m,n-1}^k}{2\Delta y}, \tag{15}$$

$$\delta_x^{(2)} u_{m,n}^k = \frac{u_{m+1,n}^k - 2u_{m,n}^k + u_{m-1,n}^k}{(\Delta x)^2}, \tag{16}$$

$$\delta_y^{(2)} u_{m,n}^k = \frac{u_{m,n+1}^k - 2u_{m,n}^k + u_{m,n-1}^k}{(\Delta y)^2}. \tag{17}$$

Clearly, Eq. (13) provides a first-order approximation in time of the exact value of u_t at both (x_m, y_n, t_k) and (x_m, y_n, t_{k+1}); here, the subindex t in u_t denotes derivation with respect to t. On the other hand, the formulas (14)–(17) yield second-order approximations in space of the exact values of u_x, u_y, u_{xx} and u_{yy}, respectively, at the point (x_m, y_n, t_k).

With these conventions, the finite-difference scheme employed in this manuscript to approximate the solutions of (1) is provided by the set of equations

$$\delta_t u_{m,n}^k + \alpha (u_{m,n}^k)^p \left(\delta_x^{(1)} u_{m,n}^{k+1} + \delta_y^{(1)} u_{m,n}^{k+1}\right) - \delta_x^{(2)} u_{m,n}^{k+1} - \delta_y^{(2)} u_{m,n}^{k+1} - u_{m.n}^{k+1} f(u_{m,n}^k) = 0, \tag{18}$$

where $m \in \{1, \ldots, M-1\}$, $n \in \{1, \ldots, N-1\}$ and $k \in \{0, 1, \ldots, K-1\}$. The method is a two-step discrete technique, for which the initial approximation at the time $t = 0$ will be given by the initial profile of a suitable exact solution u, that is, $u_{m,n}^0 = u(x_m, y_n, 0)$, for every $m \in \{0, 1, \ldots, M\}$ and every $n \in \{0, 1, \ldots, N\}$. Additionally, suitable conditions must be imposed on the boundary of D. In this work, we will consider boundary data of the form

$$u_{0,n}^k = \psi_a(y_n, t_k), \qquad u_{M,n}^k = \psi_b(y_n, t_k), \tag{19}$$

$$u_{m,0}^k = \chi_c(x_m, t_k), \qquad u_{m,N}^k = \chi_d(x_m, t_k), \tag{20}$$

where $\psi_a, \psi_b : [c, d] \times [0, T] \to \mathbb{R}$ and $\chi_c, \chi_d : [a, b] \times [0, T] \to \mathbb{R}$ are the driving functions at the boundary of D, which must satisfy the four compatibility identities $\psi_a(c, t_k) = \chi_c(a, t_k)$, $\psi_b(c, t_k) = \chi_c(b, t_k)$, $\psi_a(d, t_k) = \chi_d(a, t_k)$, $\psi_b(d, t_k) = \chi_d(b, t_k)$, for every $k \in \{0, 1, \ldots, K\}$. In our simulations, these functions will be the exact solution of the problem under consideration evaluated at the boundary points of the spatial domain, so that the compatibility conditions will be automatically satisfied.

In order to establish the main properties of the finite-difference method presented above, we need to represent our technique in a convenient vector form. To do it, it is important to notice that Eq. (18) may be rewritten as

$$b_1 u^{k+1}_{m-1,n} + b_2 u^{k+1}_{m,n-1} + b_3 u^{k+1}_{m,n} + b_4 u^{k+1}_{m,n+1} + b_5 u^{k+1}_{m+1,n} = u^k_{m,n}, \tag{21}$$

where $m \in \{1, \ldots, M-1\}$, $n \in \{1, \ldots, N-1\}$ and $k \in \{0, 1, \ldots, K-1\}$. The coefficients b_1, b_2, b_3, b_4 and b_5 are, in general, functions of the indexes m, n and k through the actual values of x_m, y_n and t_k, but these dependencies have been omitted for the sake of briefness. The exact expressions of the coefficient in (21) are provided by the following equations:

$$\begin{cases} b^k_{1,m,n} = -\dfrac{\alpha r_x (u^k_{m,n})^p}{2} - R_x, \qquad b^k_{2,m,n} = -\dfrac{\alpha r_y (u^k_{m,n})^p}{2} - R_y, \\ b^k_{3,m,n} = 1 + 2R_x + 2R_y - \Delta t f(u^k_{m,n}), \\ b^k_{4,m,n} = \dfrac{\alpha r_y (u^k_{m,n})^p}{2} - R_y, \qquad b^k_{5,m,n} = \dfrac{\alpha r_x (u^k_{m,n})^p}{2} - R_x, \end{cases} \tag{22}$$

where

$$r_x = \frac{\Delta t}{\Delta x}, \qquad r_y = \frac{\Delta t}{\Delta y}, \qquad R_x = \frac{\Delta t}{(\Delta x)^2}, \qquad R_y = \frac{\Delta t}{(\Delta y)^2}. \tag{23}$$

Let P be the integer $(M+1)(N+1)$. For every $k \in \{0, 1, \ldots, K\}$, we will represent by $\mathbf{u}^k$ the vector form of the numerical approximation to the exact solution of (1) at the time t_k. More precisely, the approximate solution at the kth time-step will be the P-dimensional vector

$$\mathbf{u}^k = (u^k_{0,0}, u^k_{0,1}, \ldots, u^k_{0,N}, u^k_{1,0}, u^k_{1,1}, \ldots, u^k_{1,N}, \ldots, u^k_{M,0}, u^k_{M,1}, \ldots, u^k_{M,N}). \tag{24}$$

For every $k \in \{0, 1, \ldots, K-1\}$, let $\mathbf{b}^{k+1}$ be the P-dimensional vector consisting of the data of the solutions on the boundary at the $(k+1)$st time-step, as provided by Eqs. (19)–(20). More precisely, let

$$\begin{aligned} \mathbf{b}^{k+1} = (&\psi_a(y_0, t_{k+1}), \psi_a(y_1, t_{k+1}), \ldots, \psi_a(y_{N-1}, t_{k+1}), \psi_a(y_N, t_{k+1}), \chi_c(x_1, t_{k+1}), \underbrace{0, \ldots, 0}_{(N-1)\text{-times}}, \chi_d(x_1, t_{k+1}), \\ &\vdots \\ &\chi_c(x_{M-1}, t_{k+1}), \underbrace{0, \ldots, 0}_{(N-1)\text{-times}}, \chi_d(x_{M-1}, t_{k+1}), \\ &\psi_b(y_0, t_{k+1}), \psi_b(y_1, t_{k+1}), \ldots, \psi_b(y_{N-1}, t_{k+1}), \psi_b(y_N, t_{k+1})). \end{aligned} \tag{25}$$

We will represent the identity matrix of size $(N+1) \times (N+1)$ by I. For every $m \in \{1, \ldots, M-1\}$ and every $k \in \{0, 1, \ldots, K-1\}$, we introduce the following matrices of the same size of I:

$$D^k_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b^k_{1,m,1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b^k_{1,m,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b^k_{1,m,N-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & b^k_{1,m,N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

$$E^k_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b^k_{5,m,1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b^k_{5,m,2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b^k_{5,m,N-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & b^k_{5,m,N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

and

$$A_m^k = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_{2,m,1}^k & b_{3,m,1}^k & b_{4,m,1}^k & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_{2,m,2}^k & b_{3,m,2}^k & b_{4,m,2}^k & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{2,m,N-2}^k & b_{3,m,N-2}^k & b_{4,m,N-2}^k & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{2,m,N-1}^k & b_{3,m,N-1}^k & b_{4,m,N-1}^k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

In the following equation, let 0 represent the null matrix of size equal to $(N+1) \times (N+1)$. We let A^{k+1} be the square real matrix of size $P \times P$ defined by blocks through

$$A^{k+1} = \begin{pmatrix} I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ D_1^k & A_1^k & E_1^k & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & D_2^k & A_2^k & E_2^k & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D_{M-2}^k & A_{M-2}^k & E_{M-2}^k & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & D_{M-1}^k & A_{M-1}^k & E_{M-1}^k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I \end{pmatrix}. \tag{29}$$

With this notation, our finite-difference method can be represented in vector form as the following system of equations, where $k \in \{0, 1, \ldots, K-1\}$:

$$A^{k+1}\mathbf{u}^{k+1} = \mathbf{u}^k + \mathbf{b}^{k+1}. \tag{30}$$

In the linear regime, the method introduced in this work is consistent of first order in time, and second order in space. Computationally, the solution of the sparse system described in (30) is carried out using an implementation of the stabilized bi-conjugate gradient method. Here, preconditioning is not required for convergence in view that we will guarantee conditions under which A^{k+1} is strictly diagonally dominant, for each $k \in \{0, 1, \ldots, K-1\}$.

Finally, it is worthwhile to notice that the coefficients of the matrix A^{k+1} are constant for each k. More importantly, the expressions of the coefficients do not depend on the solution at time t_{k+1}. Indeed, a careful look at the definitions of all the coefficients involved in the expression of A^{k+1} shows that, in the worst-case scenario, they depend on the solution $\mathbf{u}^k$ at the time t_k (which is assumed to be known in this iterative process), but also that they are independent of the approximation $\mathbf{u}^{k+1}$.

4. Numerical properties

The aim of the present section is to establish the most important properties of the method introduced above. In particular, we are interested in proving that the method satisfies the same symmetry property of (1) recorded in Proposition 1, as well as the preservation of the positive and the bounded characters of some solutions of our mathematical model. Throughout this stage, we will use the nomenclature and conventions introduced previously.

Let $\mathbf{0}$ represent the P-dimensional zero vector. In order to state the symmetry properties of our finite-difference scheme, let us express the system of Eqs. (30) as $\mathbf{L}_k(\mathbf{u}^k, \mathbf{u}^{k+1}) = \mathbf{0}$ for $k \in \{0, 1, \ldots, K-1\}$, where

$$\mathbf{L}_k(\mathbf{u}^k, \mathbf{u}^{k+1}) = A^{k+1}\mathbf{u}^{k+1} - B\mathbf{u}^k - \mathbf{b}^{k+1}. \tag{31}$$

With this notation, the following results is valid for both forms of the function f (namely, (2) and (3)), when p is a positive even integer.

Proposition 6. *Let $k \in \{0, 1, \ldots, K-1\}$, let f be an even function, and let p be an even number. Then* $\mathbf{L}(-\mathbf{u}^k, -\mathbf{u}^{k+1}) = -\mathbf{L}(\mathbf{u}^k, \mathbf{u}^{k+1})$.

Proof. The proof is straightforward. □

As in the continuous case and as a consequence of this last result, it follows that the pair $(\mathbf{u}^k, \mathbf{u}^{k+1})$ satisfies (30) if and only if $(-\mathbf{u}^k, -\mathbf{u}^{k+1})$ satisfies it, too. More precisely, the property of skew-symmetry of the solutions of the partial differential equation (1) is preserved by out finite-difference method if f is an even function and p is an even integer. Thus, this property of our method is consistent with the dynamics of the continuous model, and it will be illustrated in the section of simulations.

Recall that a square matrix is a *Z-matrix* if its off-diagonal elements are all real numbers which are less than or equal to zero. A special subclass of the class of Z-matrices is the collection of M-matrices. Here, by an *M-matrix* we understand a Z-matrix A which satisfies the following properties:

(i) The diagonal elements of A are positive numbers, and
(ii) there exists a diagonal matrix D of the same size as A, all of whose diagonal entries are positive numbers, such that AD is strictly diagonally dominant.

The importance of the M-matrices in our investigation lies in the facts that they are all non-singular, and that all the entries of their inverse matrices are positive numbers [32].

Let A be a matrix (or vector) of any size. We say that A is *positive* if all its entries are positive numbers, in which case, we employ the notation $A > 0$. If s is a real number, we represent by $A < s$ the fact that all the entries of A are less than s, and we say that A is *bounded from above* by this number. Clearly, this condition holds if and only if $s\mathbf{e} - A$ is positive, where $\mathbf{e}$ is the matrix (or vector) of the same size of A, all of whose entries are equal to 1. We say that A is *bounded from below* by s if $-A$ is bounded from above by $-s$, in which case, we write $s < A$. Finally, $|A|$ will be the matrix (or vector) of the same size of A, whose entries are the absolute values of the respective entries of A.

The following result is valid for any choice of the function f which satisfies the requirement (b) below.

Lemma 7. *Let $k \in \{0, 1, \dots, K-1\}$, and suppose that there exists a positive, real number $s \leq 1$ such that $|\mathbf{u}^k| < s^{1/p}$. Then the matrix A^{k+1} in (29) is an M-matrix if the following inequalities are satisfied:*

(a) *$s|\alpha| \max\{\Delta x, \Delta y\} \leq 2$, and*
(b) *$\Delta t f(u^k_{m,n}) < 1$ for every $m \in \{1, \dots, M-1\}$ and every $n \in \{1, \dots, N-1\}$.*

Proof. First, we check that the off-diagonal entries of the matrix A^{k+1} are non-positive under the hypothesis (a). This statement is true for the off-diagonal elements of the first $N+1$ and the last $N+1$ rows, as it is in the case of the rows $m(N+1)+1$ and $(m+1)(N+1)$, for every $m \in \{1, \dots, M-1\}$. Indeed, in these rows the off-diagonal entries are all equal to zero. For all of the remaining rows, notice that the only non-zero off-diagonal elements are the variables $b^k_{1,m.n}$, $b^k_{2,m.n}$, $b^k_{4,m.n}$ and $b^k_{5,m.n}$, and that

$$\begin{aligned} \max\left\{b^k_{l,m,n}\right\}_{l=1,2,4,5} &\leq \max\left\{\frac{|\alpha| r_x s}{2} - R_x, \frac{|\alpha| r_y s}{2} - R_y\right\} \\ &= \frac{1}{2}\max\left\{(s|\alpha|\Delta x - 2)R_x, (s|\alpha|\Delta y - 2)R_y\right\} \leq 0. \end{aligned} \tag{32}$$

Next, we show that the diagonal entries of A^{k+1} are positive, and that this matrix is strictly diagonally dominant. Again, we only need to check the rows labeled $m(N+1)+n$, where $m \in \{1, \dots, M-1\}$ and $n \in \{2, \dots, N\}$. Using (b) and the fact that A^{k+1} is a Z-matrix, we obtain that

$$\sum_{l=1,2,4,5} |b^k_{l,m,n}| = 2R_x + 2R_y < b^k_{3,m,n}, \tag{33}$$

which is what we needed. We conclude that A^{k+1} is an M-matrix. □

Proposition 8. *Let $k \in \{0, 1, \dots, K-1\}$, and suppose that there exists a positive number $s \leq 1$ such that $0 < \mathbf{u}^k < s^{1/p}$. Assume that the boundary conditions at the time t_{k+1} are positive. Then the vector $\mathbf{u}^{k+1}$ is positive if the inequalities* (a) *and* (b) *of Lemma 7 are satisfied.*

Proof. Under the hypotheses, the matrix A^{k+1} of (30) is an M-matrix, so it is non-singular and has a positive inverse. Meanwhile, the right-hand side of the same equation is a positive vector, which yields that $\mathbf{u}^{k+1}$ is positive, too. □

Let s be a positive number. In the following, we will consider separately the problem of approximating solutions of (1) bounded within $(0, s^{1/p})$ and bounded within $(-s^{1/p}, s^{1/p})$. The proofs will be independent of Proposition 8; however, Lemma 7 will be the crucial result.

Proposition 9. *Let $k \in \{0, 1, \dots, K-1\}$, suppose that there exists a positive number $s \leq 1$ such that $|\mathbf{u}^k| < s^{1/p}$, and assume that $\mathbf{b}^{k+1} < s^{1/p}$. Then $\mathbf{u}^{k+1} < s^{1/p}$ holds if* (a) *and* (b) *of Lemma 7 hold and if, additionally, the following inequality is satisfied:*

(c) *$s^{1/p}[1 - \Delta t f(u^k_{m,n})] - u^k_{m,n} > 0$, for every $m \in \{1, \dots, M-1\}$ and every $n \in \{1, \dots, N-1\}$.*

Proof. Let $\mathbf{w}^{k+1} = s^{1/p}\mathbf{e} - \mathbf{u}^{k+1}$, where $\mathbf{e}$ is the P-dimensional vector, all of whose components are equal to 1. In terms of $\mathbf{w}^{k+1}$, Eq. (30) becomes

$$A^{k+1}\mathbf{w}^{k+1} = As^{1/p}\mathbf{e} - \mathbf{u}^k - \mathbf{b}^{k+1}. \tag{34}$$

For the sake of simplicity, let $\mathbf{d}^{k+1}$ be the vector in the right-hand side of (34). The first $N+1$ components of $\mathbf{d}^{k+1}$, as well as the last $N+1$, are identical to the corresponding components of the vector $s^{1/p}\mathbf{e} - \mathbf{b}^{k+1}$, which are positive. Similarly, the $[m(N+1)+1]$st and the $[(m+1)(N+1)]$th components of $\mathbf{d}^{k+1}$ are positive, for every $m \in \{1, \dots, M-1\}$. The remaining entries of $\mathbf{d}^{k+1}$ assume the expression on the left side of the inequality (c), for some $m \in \{1, \dots, M-1\}$ and $n \in \{1, \dots, N-1\}$; consequently, $\mathbf{d}^{k+1}$ is positive. This and the fact that A^{k+1} is an M-matrix yield that $\mathbf{w}^{k+1}$ is a positive vector or, equivalently, that $\mathbf{u}^{k+1}$ is bounded from above by $s^{1/p}$. □

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032

Following our nomenclature, let s be a positive number such that $s \leq 1$ and let $F : (-s^{1/p}, s^{1/p}) \to \mathbb{R}$ be the function given by

$$F(u) = s^{1/p}[1 - \Delta t f(u)] - u. \tag{35}$$

Before we establish our next result, it is necessary to verify the feasibility of condition (c) in Proposition 9, at least for the functions f given by Eqs. (2) and (3). In the two scenarios considered below, u is a real number satisfying the inequality $|u| < s^{1/p}$.

- Consider first the case when f takes on the expression (2). Our need to approximate the solutions of (1) which are bounded from above by 1 leads us to fix s in such value. If we impose the constraint $p\Delta t < 1$, then $\Delta t(1+u+u^2+\cdots+u^{p-1}) < 1$. As a consequence,

$$F(u) = 1 - u + \Delta t(u^p - 1) = (1-u)\left[1 - \Delta t(1 + u + u^2 + \cdots + u^{p-1})\right] > 0. \tag{36}$$

- For a reaction factor of the form (3), we consider two different scenarios, namely, when the solutions are bounded from above by either $\gamma^{1/p}$ or 1. In either case, we suppose that the solutions are positive. Notice first that $F(u) = (\gamma^{1/p} - u) + \gamma^{1/p}\Delta t(1-u^p)(\gamma - u^p)$, so F is positive within its domain when s is equal to γ. Now, when s is equal to 1, the inequality in the expression

$$F(u) = (1-u)\left[1 - \Delta t(1 + u + u^2 + \cdots + u^{p-1})(u^p - \gamma)\right] > 0 \tag{37}$$

is satisfied when $(1-\gamma)p\Delta t < 1$.

Next, we want to show that the conditions given by Proposition 9 also guarantee the boundedness from above of new approximations.

Proposition 10. *Let $k \in \{0, 1, \ldots, K-1\}$, suppose that there exists a positive number $s \leq 1$ such that $|\mathbf{u}^k| < s^{1/p}$ and $|\mathbf{b}^{k+1}| < s^{1/p}$. Moreover, let f be an even function, and let p be an even integer. If* (a) *and* (b) *of Lemma 7 are satisfied as well as the inequality*

(c)′ $s^{1/p}[1 - \Delta t f(u^k_{m,n})] - |u^k_{m,n}| > 0$, *for every $m \in \{1, \ldots, M-1\}$ and every $n \in \{1, \ldots, N-1\}$,*

then $|\mathbf{u}^{k+1}| < s^{1/p}$.

Proof. Beforehand, notice that (c) of Proposition 9 and the inequality $s^{1/p}[1 - \Delta t f(-u^k_{m,n})] - (-u^k_{m,n}) > 0$ are satisfied for every $m \in \{1, \ldots, M-1\}$ and every $n \in \{1, \ldots, N-1\}$. In particular, Proposition 9 guarantees that $\mathbf{u}^{k+1} < s^{1/p}$. The discussion after Proposition 6 establishes that the pair of vectors $\mathbf{v}^k = -\mathbf{u}^k$ and $\mathbf{v}^{k+1} = -\mathbf{u}^{k+1}$ satisfies (30). On the other hand, the facts that $|\mathbf{v}^k| < s^{1/p}$ and that f is an even function guarantee that the hypotheses of Proposition 9 are satisfied for $\mathbf{v}^k$ and $\mathbf{v}^{k+1}$. It follows that the inequality $-s^{1/p} < \mathbf{u}^{k+1}$ also holds, whence the conclusion follows. □

It is important to point out that the constants s satisfying the hypotheses of all these propositions do exist for the cases of the Burgers–Fisher and the Burgers–Huxley equations. In the former case $s = 1$, while in the second $s \in \{1, \gamma\}$. In view of these considerations, the numerical method (18) preserves the positivity and the boundedness of approximations within $(0, s^{1/p})$ or $(-s^{1/p}, s^{1/p})$ for s being equal to either 1 or $\gamma^{1/p}$. It is important to notice also that the respective continuous models possess analytical results on the existence and uniqueness of solutions bounded within the same interval [33]. The existence of such constants is a cornerstone in this report, and it is the reason why this manuscript focuses on the Burgers–Fisher and the Burgers–Huxley models.

We summarize in the following theorem the most important properties on the existence and uniqueness of positive and bounded solutions of our methodology for the cases of the Burgers–Fisher and Burgers–Huxley equations.

Proposition 11 (*Existence and Uniqueness*)**.** *Let $s \in \{1, \gamma\}$, and suppose that* (a) *and* (b) *of Lemma 7 are satisfied.*

(i) *If $0 < \mathbf{u}^0 < s^{1/p}$, and if ψ_a, ψ_b, χ_c and χ_b are all bounded in $(0, s^{1/p})$, then there exists a unique sequence $(\mathbf{u}^k)_{k=0}^K$ satisfying (30), as well as the inequalities $0 < \mathbf{u}^k < s^{1/p}$ for each $k \in \{0, 1, \ldots, K\}$.*
(ii) *Let p be an even number, and suppose that $|\mathbf{u}^0| < s^{1/p}$ and that ψ_a, ψ_b, χ_c and χ_b are all bounded in $(-s^{1/p}, s^{1/p})$. If* (c)′ *of Proposition 10 holds then there exists a unique sequence $(\mathbf{u}^k)_{k=0}^K$ satisfying (30), such that $|\mathbf{u}^k| < s^{1/p}$ for each $k \in \{0, 1, \ldots, K\}$.*

Proof. The proof follows from Propositions 8 and 10. □

Finally, we explore the feasibility of the inequality (c)′ in Proposition 10, at least for an even reaction law associated to the Newell–Whitehead–Segel equation. Let

$$\begin{aligned} G(u) &= 1 - \Delta t(1 - u^{2p}) - |u| \\ &= (1 - |u|)\left[1 - \Delta t(1 + |u| + |u|^2 + \cdots + |u|^{2p-1})\right], \end{aligned} \tag{38}$$

with $|u| < 1$. Clearly, a sufficient condition for $G(u)$ to be positive is that $2p\Delta t < 1$. Clearly, the method proposed in this work is a linear technique [11,21–24].

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032

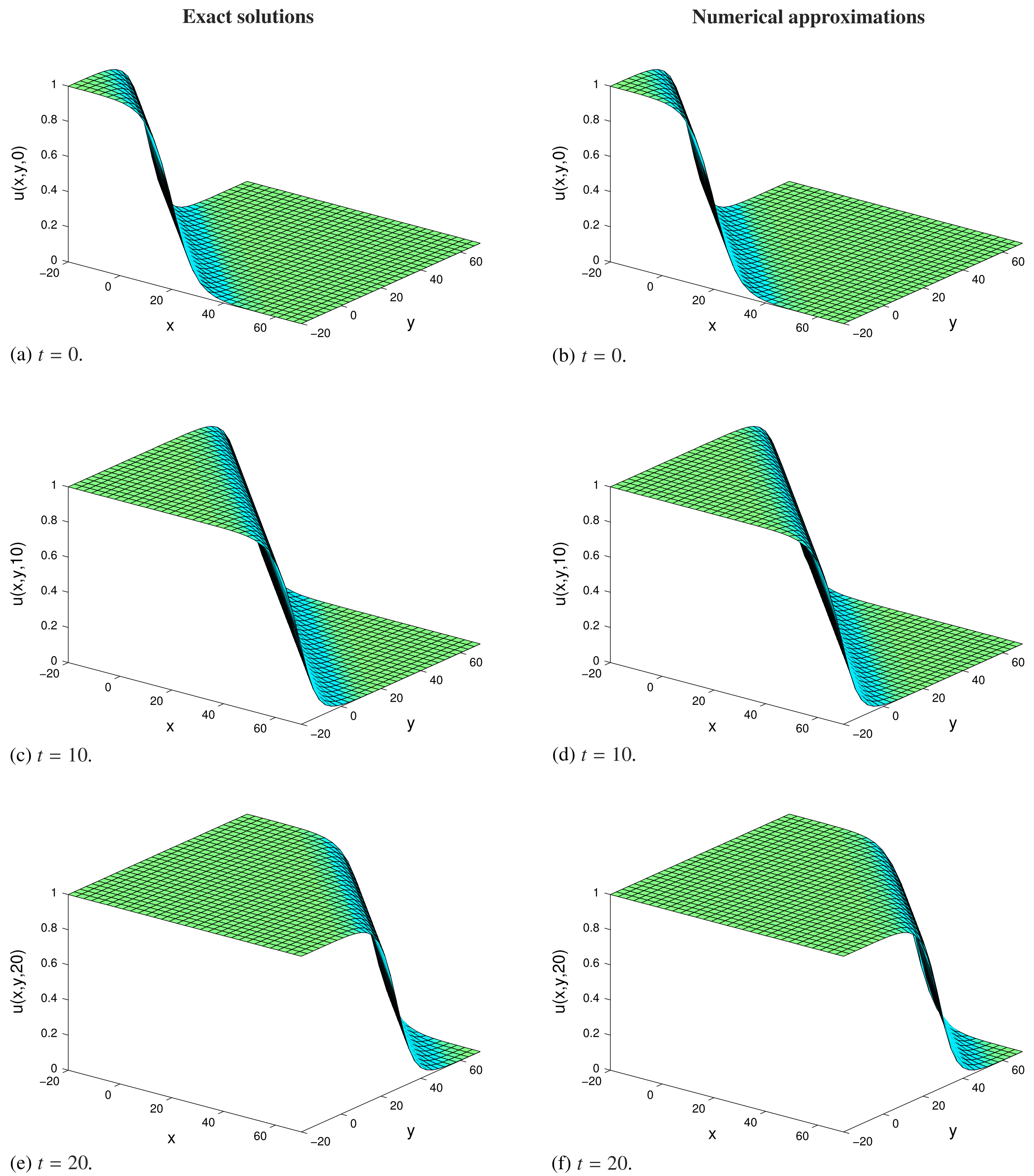
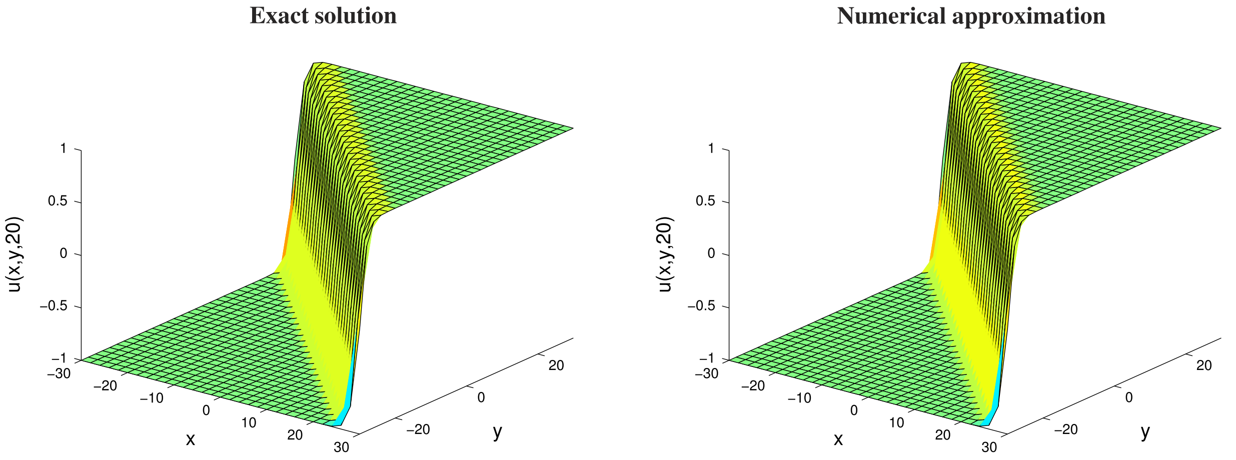


Fig. 1. Graphs of the exact solutions (left column) and the corresponding numerical approximation (right column) versus the spatial coordinates x and y at the times $t = 0$ (top row), $t = 10$ (middle row) and $t = 20$ (bottom row) of the Burgers–Fisher model (1) subject to initial and boundary conditions given by the exact solution (5), using the parameters $\alpha = 0.45$ and $p = 1$. The simulations were performed using the finite-difference method (18) over the spatial domain $D = [-20, 70] \times [-20, 70]$, with step sizes $\Delta x = \Delta y = 3$ and $\Delta t = 0.01$.

5. Illustrative simulations

Beforehand, it is important to mention that each iteration of the computer implementation of our technique requires to solve the sparse system (30) using the stabilized bi-conjugate gradient method. Here, preconditioning is not required for convergence in view that A^{k+1} is strictly diagonally dominant.

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032



(a) $t = 20$.

(b) $t = 20$.

Fig. 2. Graphs of the exact solution (left column) and the corresponding, numerical approximation (right column) versus the spatial coordinates x and y at the time $t = 20$ for the Burgers–Fisher model (1) subject to initial and boundary conditions given by the exact solution (6), using the parameters $\alpha = 0$, $p = 2$ and $C_1 = C_2 = C_3 = 1$. The simulations were performed using the finite-difference method (18) over the spatial domain $D = [-30, 30] \times [-30, 30]$, with step sizes $\Delta x = \Delta y = 2$ and $\Delta t = 0.1$.

In the present section, we validate our method using the analytic solutions of the one-dimensional model quoted in Section 2. In the two-dimensional scenario, the spatial variable will be ξ_+ as given by (12) with $\varphi = \pi/4$. It is important to keep in mind that the model is the Burgers–Fisher equation if the nonlinear reaction factor f is given by (2), and it is the Burgers–Huxley equation if f is provided by (3). In the examples below, the initial and boundary conditions will be provided by each of the four particular solutions. In all of them, the relevant boundedness conditions summarized by the results of Section 4 will be satisfied by the parameter values.

Example 12. Fix the model parameters $\alpha = 0.45$ and $p = 1$, and consider the particular solution (5). Computationally, fix the spatial domain $D = [-20, 70] \times [-20, 70]$, and the parameters $\Delta x = \Delta y = 3$ and $\Delta t = 0.01$. Fig. 1 presents the exact solution (left column) and the corresponding numerical approximation (right column) of the problem at three different times, namely, $t = 0$ (top row), $t = 10$ (middle row) and $t = 20$ (bottom row). The simulations show a good qualitative agreement between the analytical and the computational solutions of the problem, in addition to the fact that the positive and the bounded characters of the exact solutions are preserved by the numerical technique. □

Example 13. Let $\alpha = 0$ and $p = 2$, and consider the exact solution (6) with $C_1 = C_2 = C_3 = 1$. Fix the spatial domain $D = [-30, 30] \times [-30, 30]$, and step sizes $\Delta x = \Delta y = 2$ and $\Delta t = 0.1$. Fig. 2(a) presents the exact solution at time $t = 20$, while graph Fig. 2(b) shows the corresponding, numerical approximation computed through (18). We notice that the approximate solution is both bounded within $(-1, 1)$ and qualitatively similar to the exact solution at that time. This similarity will be further analyzed quantitatively in other scenarios. □

For comparison purposes, we define the *point-wise, absolute error* between the exact solution u at time t_k and a corresponding numerical approximation $\mathbf{u}^k$ by

$$E(x_m, y_n, t_k) = |u(x_m, y_n, t_k) - u^k_{m,n}|, \tag{39}$$

for each $m \in \{0, 1, \ldots, M\}$ and $n \in \{0, 1, \ldots, N\}$. Likewise, we define the *absolute error* at the kth time step by

$$\epsilon(t_k) = \max\{E(x_m, y_n, t_k) : m = 0, 1, \ldots, M, n = 0, 1, \ldots, N\}. \tag{40}$$

Example 14. Consider the exact solution (7) of the Burgers–Huxley model with $\alpha = -0.1$, $\gamma = 0.4$ and $p = 1$. Computationally, let $D = [-30, 30] \times [-30, 30]$, $\Delta x = \Delta y = 2$ and $\Delta t = 0.1$. The left column of Fig. 3 presents the temporal evolution of the approximate solution at times $t = 50$ (top row), $t = 100$ (middle row) and $t = 250$ (bottom row); meanwhile, the right column presents the corresponding point-wise, absolute errors. Clearly, the numerical solutions remain bounded within the interval $(0, 1)$, as expected. We must mention here that we have performed simulations with smaller values of the step sizes. Our results (not presented here) showed that the point-wise, absolute error decreases as these values are decreased. □

It is important to notice that the error increases as time increases, a fact which is entirely expected. On the other hand, it is also interesting to notice that the error is localized around the wave front. This is also expected, in view that the numerical approximation and the exact solution are both approximately equal to 0 or 1 away from the front, and the absolute error

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032

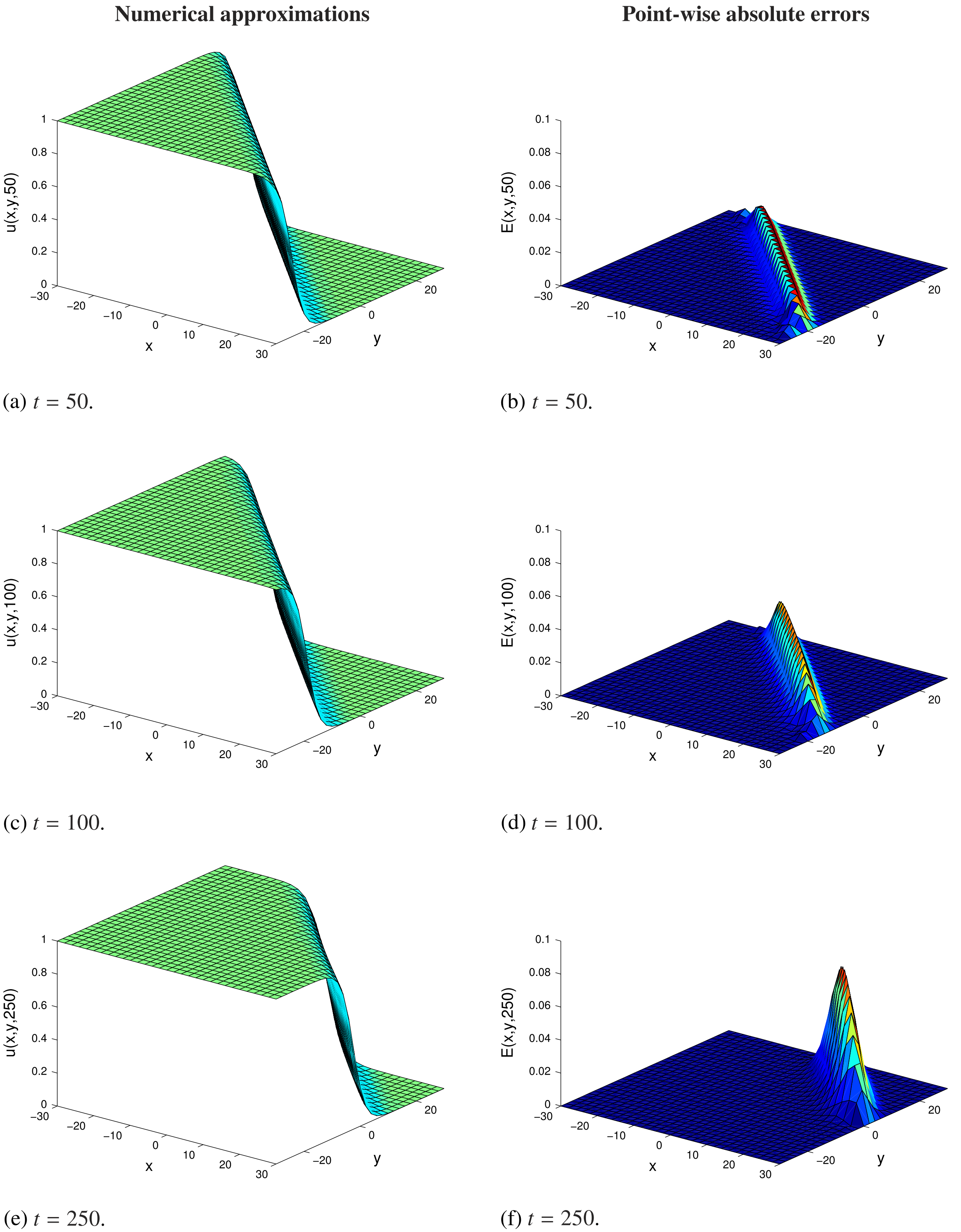
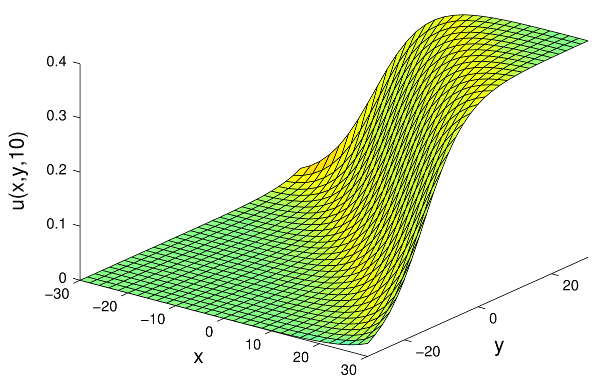
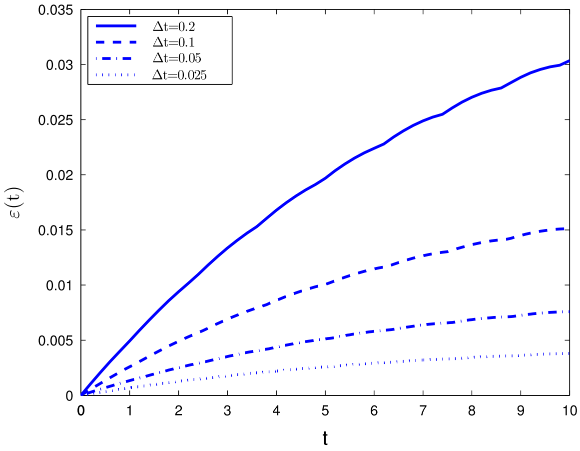


Fig. 3. Graphs of the numerical approximations (left column) and the corresponding point-wise, absolute errors (right column) versus the spatial coordinates x and y at the times $t = 50$ (top row), $t = 100$ (middle row) and $t = 250$ (bottom row) for the Burgers–Huxley model (1) subject to initial and boundary conditions given by the exact solution (7), using the parameters $\alpha = -0.1$, $\gamma = 0.4$ and $p = 1$. The simulations were performed using the finite-difference method (18) over the spatial domain $D = [-30, 30] \times [-30, 30]$, with step sizes $\Delta x = \Delta y = 2$ and $\Delta t = 0.1$.



(a) Numerical approximation ($t = 10$).



(b) Absolute error.

Fig. 4. Graphs of the numerical approximation at time $t = 10$ with $\Delta t = 0.1$ versus the spatial coordinates x and y (left column) and the temporal evolution of the absolute error (right column) with $\Delta x = \Delta y = 2$ for the Burgers–Huxley model (1) subject to initial and boundary conditions given by the exact solution (9), using the parameters $\alpha = -0.2$, $\gamma = 0.4$ and $p = 1$. The simulations were performed using the finite-difference method (18) over the spatial domain $D = [-30, 30] \times [-30, 30]$.

in those points is insignificant. As we will notice in the following example, this difference decreases as the values of the computational parameters tend to zero.

Example 15. We show now that the method performs well with solutions of the Burgers–Huxley equation which are bounded within $(0, \gamma^{1/p})$. Consider the particular solution (9) with $\alpha = -0.2$, $\gamma = 0.4$ and $p = 1$, and fix the computational parameters $\Delta x = \Delta y = 2$. Fig. 4(a) presents the approximate solution at time $t = 10$ for a temporal step-size equal to 0.1; evidently, it shows that the approximation at that time is bounded within $(0, 0.4)$, as desired. On the other hand, Fig. 4(b) shows the temporal behavior of the absolute error for several values of Δt, namely, 0.2, 0.1, 0.05 and 0.01. The results show a decrease in the propagation of the absolute error as the temporal step-size is decreased. □

Our last simulations illustrate the property of preservation of the skew-symmetry of our method. To that end, we will take p to be an even positive integer in the reaction factor of the Burgers–Fisher equation, so f will be an even function.

Example 16. Let us fix the parameters $\alpha = 0.45$ and $p = 2$, and let us consider the exact solution u given by Remark 2 on the spatial domain $D = [-20, 70] \times [-20, 70]$. Computationally, let $\Delta x = \Delta y = 3$ and $\Delta t = 0.005$. Each of the graphs of Fig. 5 presents snapshots of the approximate solution of (1) at the times 0, 2.5, 5, 7.5, 10 and 12.5, for initial–boundary conditions prescribed by u (top graph) and $-u$ (lower graph). These figures corroborate qualitatively the fact that the method preserves the skew-symmetry of approximate solutions. □

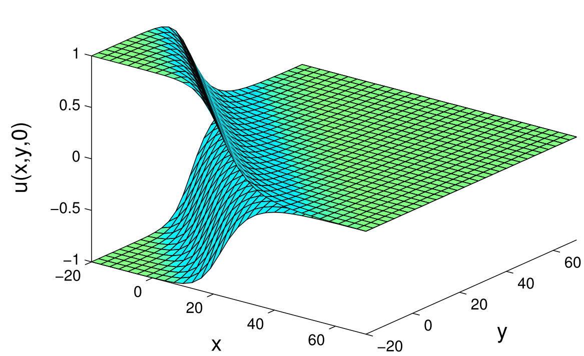
6. Conclusions

In this work, we introduced a finite-difference method to approximate the solutions of a two-dimensional generalization of the classical Burgers–Fisher and Burgers–Huxley equations. The numerical technique proposed is non-standard in the way that it approximates the nonlinear terms of advection/convection and reaction, and the linearized form of the method is consistent of first order in time, and of second order in space. The finite-difference scheme can be rewritten in vector form in terms of a square matrix that, under suitable conditions, is an M-matrix. The facts that such matrix is non-singular and that the components of its inverse are positive numbers, follow at once from the elementary properties of M-matrices. Moreover, some relatively flexible constraints were also imposed in order to guarantee that new bounded and positive estimates are derived from old bounded and positive approximations.

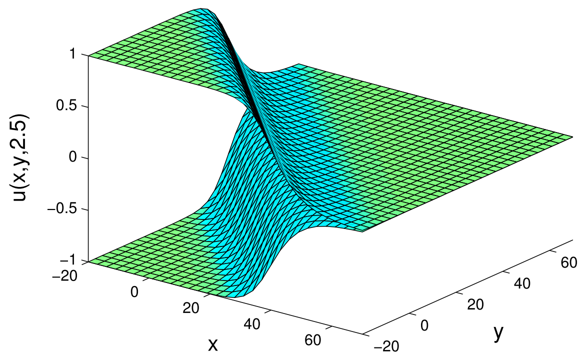
We derived some supplemental properties of the method presented in this work, like the property of the preservation of the anti-symmetry of solutions. Computer simulations were performed in order to assess the validity of our results, particularly those related to the preservation of the positivity and the boundedness of the numerical solutions. Our results indicate that our finite-difference scheme approximates well the solutions of the models under investigation, and that the sufficient conditions for the positivity and the boundedness of the approximations are valid in the cases considered. The qualitative comparisons were performed against some traveling-wave solutions for the one-dimensional case.

After these results, several avenues of research open ahead in the near future. For instance, the authors are interested in designing positivity- and boundedness-preserving differential quadrature methods to approximate the solutions of two-dimensional generalized Burgers–Fisher and Burgers–Huxley equation. The design of differential quadrature techniques has been popularized mainly by S. Tomasiello in various of her works [34–37]. The extensions of the results reported in this

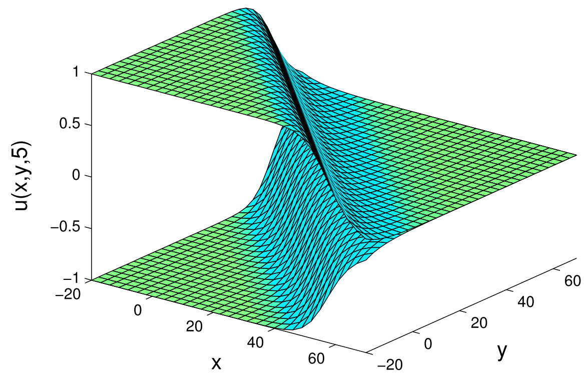
Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032



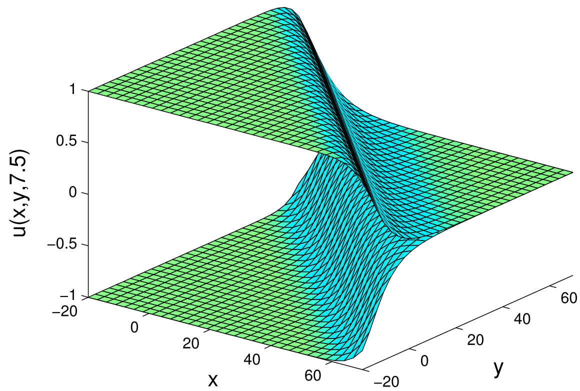
(a) $t = 0$.



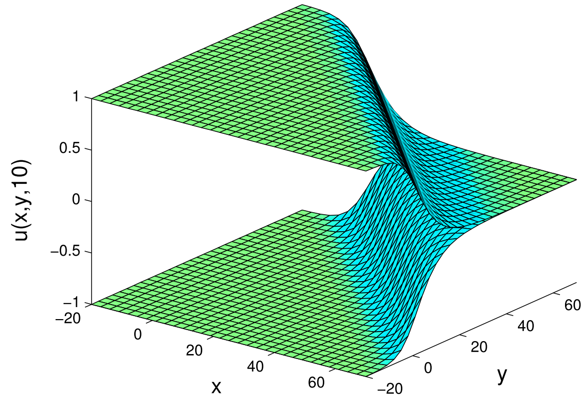
(b) $t = 2.5$.



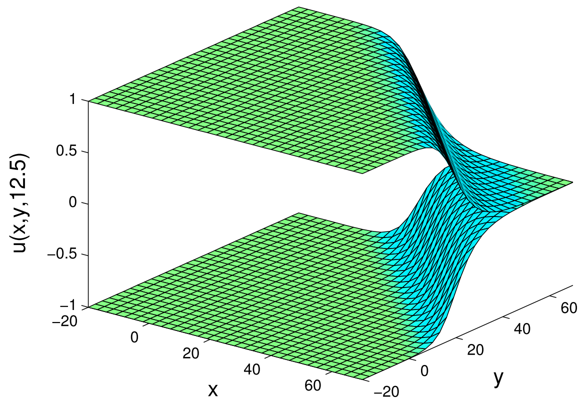
(c) $t = 5$.



(d) $t = 7.5$.



(e) $t = 10$.



(f) $t = 12.5$.

Fig. 5. Graphs of the numerical approximations versus the spatial coordinates x and y at six different times for the Burgers–Fisher model (1). Each graph provides the approximations for two sets of initial–boundary conditions: the top graph in each graph corresponds to initial–boundary conditions given by the exact solution u of (7), while the bottom graph corresponds to $-u$. We have used the parameters $\alpha = 0.45$ and $p = 2$. The simulations were performed using the finite-difference method (18) over the spatial domain $D = [-20, 70] \times [-20, 70]$, with step sizes $\Delta x = \Delta y = 3$ and $\Delta t = 0.005$.

work to the design of positivity- and boundedness-preserving finite-element and differential quadrature-element methods are also topics of interest to the authors of the present manuscript. Of course, the design of positivity- and boundedness-preserving techniques for advection–diffusion equations may be expanded in order to consider different forms of the reaction law. In this work, we explored two particular forms of reaction which are relevant in the physical sciences, namely, the well-known Fisher's law from population dynamics and Huxley's law from fluid dynamics. However, other different expressions of the reaction term are important in various branches of the sciences and engineering. In any case, we aim at designing dynamically consistent techniques for those systems.

Acknowledgments

The authors wish to thank the organizers of the symposium "Advances in the Numerical Solution of Nonlinear Time-Dependent Partial Differential Equations" of the 15th International Conference on Computational and Mathematical Methods in Science and Engineering, for the kind invitation to submit a paper. Also, the authors want to thank the anonymous reviewers and the editor in charge of handling this manuscript for all their comments and suggestions.

References

[1] M.E. Vázquez-Cendón, L. Cea, Analysis of a new kolgan-type scheme motivated by the shallow water equations, Appl. Numer. Math. 62 (2012) 489–506.
[2] V. Méndez, J. Camacho, Hyperbolic reaction–diffusion equations for a forest fire model, Phys. Rev. E 56 (1997) 6557–6563.
[3] J. Kim, K. Kang, A numerical method for the ternary cahn-hilliard system with a degenerate mobility, Appl. Numer. Math. 59 (2009) 1029–1042.
[4] J. Fort, V. Méndez, Time-delayed theory of the Neolithic transition in Europe, Phys. Rev. Lett. 82 (1999) 867–870.
[5] J. Fort, V. Méndez, Reaction–diffusion waves of advance in the transition to agricultural economies, Phys. Rev. E 60 (1999) 5894–5901.
[6] H.J. Eberl, D.F. Parker, M.C.M. van Loosdrecht, A new deterministic spatio-temporal continuum model for biofilm development, Comput. Math. Methods Med. 3 (2001) 161–175.
[7] J.E. Macías-Díaz, J. Ruiz-Ramírez, A non-standard symmetry-preserving method to compute bounded solutions of a generalized Newell–Whitehead–Segel equation, Appl. Numer. Math. 61 (2011) 630–640.
[8] R.E. Mickens, P.M. Jordan, A positivity-preserving nonstandard finite difference scheme for the damped wave equation, Numer. Methods Partial Differential Equations 20 (2004) 639–649.
[9] R.E. Mickens, P.M. Jordan, A new positivity-preserving nonstandard finite difference scheme for the DWE, Numer. Methods Partial Differential Equations 21 (2005) 976–985.
[10] J.E. Macías-Díaz, A. Puri, A boundedness-preserving finite-difference scheme for a damped nonlinear wave equation, Appl. Numer. Math. 60 (2010) 934–948.
[11] J.E. Macías-Díaz, A. Puri, On some explicit non-standard methods to approximate nonnegative solutions of a weakly hyperbolic equation with logistic nonlinearity, Int. J. Comput. Math. 88 (2011) 3308–3323.
[12] D. Furihata, Finite-difference schemes for nonlinear wave equation that inherit energy conservation property, J. Comput. Appl. Math. 134 (2001) 37–57.
[13] D. Furihata, Finite difference schemes that inherit energy conservation or dissipation property, J. Comput. Phys. 156 (1999) 181–205.
[14] J.E. Macías-Díaz, A. Puri, An energy-based computational method in the analysis of the transmission of energy in a chain of coupled oscillators, J. Comput. Appl. Math. 214 (2008) 393–405.
[15] R.E. Mickens, Nonstandard finite-difference schemes for reaction–diffusion equations, Numer. Methods Partial Differential Equations 15 (1999) 201–214.
[16] S.M. Moghadas, A.B. Gumel, Dynamical and numerical analyses of a generalized food-chain model, Appl. Math. Comput. 142 (2003) 35–49.
[17] S.M. Moghadas, M.E. Alexander, B.D. Corbett, A nonstandard numerical scheme for a generalized Gause-type predator–prey model, Physica D 188 (2004) 134–151.
[18] R.A. Fisher, The wave of advance of advantageous genes, Ann. Eugenics 7 (1937) 355–369.
[19] A. Kolmogorov, I. Petrovsky, N. Piscounov, Étude de l'équations de la diffusion avec croissance de la quantité de matière et son application a un problème biologique, Bull. Univ. Moskou, Ser. Internat. 1A (1937) 1–25.
[20] X.Y. Wang, Z.S. Zhu, Y.K. Lu, Solitary wave solutions of the generalised Burgers–Huxley equation, J. Phys. A: Math. Gen. 23 (1990) 271.
[21] J. Ruiz-Ramírez, J.E. Macías-Díaz, A finite-difference scheme to approximate non-negative and bounded solutions of a FitzHugh–Nagumo equation, Int. J. Comput. Math. 88 (15) (2011) 3186–3201.
[22] J.E. Macías-Díaz, On a boundedness-preserving semi-linear discretization of a two-dimensional nonlinear diffusion–reaction model, Int. J. Comput. Math. 89 (12) (2012) 1678–1688.
[23] I.E. Medina-Ramírez, J.E. Macías-Díaz, On a fully discrete finite-difference approximation of a nonlinear diffusion–reaction model in microbial ecology, Int. J. Comput. Math. 90 (9) (2013) 1915–1937.
[24] J.E. Macías-Díaz, R.E. Landry, A. Puri, A finite-difference scheme in the computational modelling of a coupled substrate-biomass system, Int. J. Comput. Math. 91 (10) (2014) 2199–2214.
[25] H.N.A. Ismail, K. Raslan, A.A. Abd Rabboh, Adomian decomposition method for Burger's–Huxley and Burger's-Fisher equations, Appl. Math. Comput. 159 (2004) 291–301.
[26] A.C. Newell, J.A. Whitehead, Stability of stationary periodic structures for weakly supercritical convection and related problems, J. Fluid Mech. 38 (1969) 279–303.
[27] L.A. Segel, Distant side-walls cause slow amplitude modulation of cellular convection, J. Fluid Mech. 38 (1969) 203–224.
[28] A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, first ed., Chapman & Hall CRC Press, Boca Raton, Fla, 2004.
[29] X. Zhou, Exp-function method for solving Huxley equation, Math. Probl. Eng. 2008 (2008) 1–7.
[30] Y.N. Kyrychko, M.V. Bartuccelli, K.B. Blyuss, Persistence of travelling wave solutions of a fourth order diffusion system, J. Comput. Appl. Math. 176 (2005) 433–443.
[31] E.G. Fan, Traveling wave solutions for nonlinear equations using symbolic computation, Comput. Math. Appl. 43 (2002) 671–680.
[32] T. Fujimoto, R.R. Ranade, Two characterizations of inverse-positive matrices: The Hawkins–Simon condition and the Le Chatelier–Braun principle, Electron. J. Linear Algebra 11 (2004) 59–65.
[33] V.J. Ervin, J.E. Macías-Díaz, J. Ruiz-Ramírez, A positive and bounded finite element approximation of the generalized Burgers–Huxley equation, J. Math. Anal. Appl. 424 (2) (2015) 1143–1160.
[34] S. Tomasiello, Numerical solutions of the Burgers–Huxley equation by the IDQ method, Int. J. Comput. Math. 87 (1) (2010) 129–140.
[35] S. Tomasiello, Stability and accuracy of the iterative differential quadrature method, Internat. J. Numer. Methods Engrg. 58 (9) (2003) 1277–1296.
[36] S. Tomasiello, Numerical stability of dq solutions of wave problems, Numer. Algorithms 57 (3) (2011) 289–312.
[37] S. Tomasiello, A new least-squares dq-based method for the buckling of structures with elastic supports, Appl. Math. Model. 39 (9) (2015) 2809–2814.

Please cite this article in press as: J.E. Macías-Díaz, J.A. Guerrero, A bounded linear integrator for some diffusive nonlinear time-dependent partial differential equations, Journal of Computational and Applied Mathematics (2016), http://dx.doi.org/10.1016/j.cam.2016.01.032