



On an optimal quadrature formula for approximation of Fourier integrals in the space $L_2^{(1)}$



Abdullo R. Hayotov^{a,b,*}, Soomin Jeon^a, Chang-Ock Lee^a

^a Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 34141, Republic of Korea

^b V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 81, M. Ulugbek str., Tashkent 100170, Uzbekistan

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ABSTRACT

This paper deals with the construction of an optimal quadrature formula for approximation of Fourier integrals in the Sobolev space $L_2^{(1)}[a, b]$ of non-periodic, complex valued functions which are square integrable with first order derivative. Here the quadrature sum consists of linear combination of the given function values in a uniform grid. The difference between the integral and the quadrature sum is estimated by the norm of the error functional. The optimal quadrature formula is obtained by minimizing the norm of the error functional with respect to coefficients. Analytic formulas for optimal coefficients can also be obtained using discrete analogue of the differential operator d^2/dx^2 . In addition, the convergence order of the optimal quadrature formula is studied. It is proved that the obtained formula is exact for all linear polynomials. Thus, it is shown that the convergence order of the optimal quadrature formula for functions of the space $C^2[a, b]$ is $O(h^2)$. Moreover, several numerical results are presented and the obtained optimal quadrature formula is applied to reconstruct the X-ray Computed Tomography image by approximating Fourier transforms.

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1. Introduction

In practice, since we have discrete values of an integrand, the Fourier transforms are reduced to an approximation of the integral of type

$$I(\varphi) = \int_0^1 e^{2\pi i \omega x} \varphi(x) dx \quad (1.1)$$

with $\omega \in \mathbb{R}$. For example, the problem of X-ray Computed Tomography (CT) is to reconstruct the function from its Radon transform. One of the widely used analytic methods in CT image reconstruction is the filtered back-projection method in which the Fourier transforms are used (see [1, Chapter 3] or formulas (3.9)–(3.11) of Section 3.2).

It should be recalled that integrals of type (1.1) with strongly oscillating integrands are used in applications of mathematics and other sciences. They are mainly calculated using special effective methods of numerical integration (for review see, for example, [2–12], and references therein).

Based on Sobolev's method, the problem of the construction of optimal quadrature formulas for numerical calculation of Fourier coefficients (1.1) with $\omega \in \mathbb{Z}$ in Hilbert spaces $L_2^{(m)}$ and $W_2^{(m, m-1)}$ was studied in [13] and [14], respectively. In

* Corresponding author at: V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 81, M. Ulugbek str., Tashkent 100170, Uzbekistan.

E-mail addresses: hayotov@mail.ru (A.R. Hayotov), soominjeon@kaist.ac.kr (S. Jeon), colee@kaist.edu (C.-O. Lee).

these works, explicit formulas of optimal coefficients were obtained for $m \geq 1$. In particular, for $m = 1$, the convergence order of optimal quadrature formulas was studied.

Recently, in [15] the optimal quadrature formulas were studied for integrals with arbitrary weights in Sobolev space $H^1([0, 1])$. General formulas were obtained for the worst-case error depending on nodes. Especially, when calculating Fourier coefficients of the form (1.1) with real ω , it was proved that equidistant nodes are optimal if $n \geq 2.7|\omega| + 1$, where n is the number of nodes in the quadrature formula.

It should be noted that for numerical calculation of the integral (1.1) with real ω , a quadrature formula with explicit coefficients is needed. Therefore, in this paper, we study constructing of optimal quadrature formulas in the sense of Sard for approximation of Fourier integrals (1.1) with $\omega \in \mathbb{R}$ in the Sobolev space of non-periodic square integrable functions with the first order derivative. We obtain explicit formulas for optimal coefficients and calculate the norm of the error functional of the optimal quadrature formula. We note that the obtained optimal quadrature formula can be used to approximate Fourier integrals and reconstruct a function from its discrete Radon transform.

The rest of the paper is organized as follows. In Section 2, an optimal quadrature formula in the sense of Sard is constructed to approximate Fourier integrals in the space $L_2^{(1)}[a, b]$. In Section 3 the obtained quadrature formula is applied to the approximation of Fourier transforms of a function using the given values of the function and to the reconstruction of the X-ray CT image.

2. Construction of optimal quadrature formulas

Consider the quadrature formula

$$\int_0^1 e^{2\pi i \omega x} \varphi(x) dx \cong \sum_{\beta=0}^N C_\beta \varphi(h\beta) \quad (2.1)$$

with the error

$$(\ell, \varphi) = \int_0^1 e^{2\pi i \omega x} \varphi(x) dx - \sum_{\beta=0}^N C_\beta \varphi(h\beta), \quad (2.2)$$

where

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx,$$

and the corresponding error functional

$$\ell(x) = e^{2\pi i \omega x} \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N C_\beta \delta(x - h\beta). \quad (2.3)$$

Here, C_β are coefficients of the formula (2.1), $h = 1/N$, $N \in \mathbb{N}$, $i^2 = -1$, $\omega \in \mathbb{R}$ with $\omega \neq 0$, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0, 1]$, and δ is the Dirac's delta-function. The function φ belongs to the Sobolev space $L_2^{(1)}[a, b]$ of complex valued functions which are defined in the interval $[a, b]$ and square integrable with the first order derivative. In this space, the inner product is defined as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi'(x) \bar{\psi}'(x) dx, \quad (2.4)$$

where $\bar{\psi}$ is the complex conjugate function for the function ψ and the semi-norm of the function φ is denoted by

$$\|\varphi\|_{L_2^{(1)}[a,b]} = \langle \varphi, \varphi \rangle^{1/2}.$$

We note that the coefficients C_β in the formula (2.1) vary by ω and h , that is $C_\beta = C_\beta(\omega, h)$.

The error (2.2) in the quadrature formula (2.1) is a linear functional in $L_2^{(1)*}[0, 1]$, where $L_2^{(1)*}[0, 1]$ is the conjugate space for the space $L_2^{(1)}[0, 1]$.

The absolute value of the error (2.2) is estimated by Cauchy–Schwarz inequality as

$$|(\ell, \varphi)| \leq \|\varphi\|_{L_2^{(1)}[0,1]} \cdot \|\ell\|_{L_2^{(1)*}[0,1]},$$

where

$$\|\ell\|_{L_2^{(1)*}[0,1]} = \sup_{\|\varphi\|_{L_2^{(1)}[0,1]}=1} |(\ell, \varphi)| \quad (2.5)$$

is the norm of the error functional (2.3).

In the sense of Sard [16], the problem of construction of the optimal quadrature formula (2.1) is to find the minimum of the norm (2.5) of the error functional ℓ by coefficients C_β when nodes are fixed. Here, we note that distances between adjacent nodes in the formula (2.1) are the same. For the quadrature formulas of the form (2.1) with $\omega = 0$, this problem was first studied by Sard in $L_2^{(m)}$ space for some m , where $L_2^{(m)}$ is the space of real-valued functions which are square integrable with m th generalized derivative. Also this problem for the case $\omega = 0$ has been investigated by many authors using splines, ϕ -function and Sobolev methods. For example, see [17–24] and references therein.

Therefore, in order to construct optimal quadrature formulas of the form (2.1) in the sense of Sard in the space $L_2^{(1)}[0, 1]$, the following problem needs to be solved.

Problem 1. Find the coefficients \check{C}_β that satisfy the equality

$$\|\check{\ell}\|_{L_2^{(1)*}[0,1]} = \inf_{C_\beta} \|\ell\|_{L_2^{(1)*}[0,1]}. \quad (2.6)$$

In this section we solve Problem 1 for the case $\omega \in \mathbb{R}$ with $\omega \neq 0$ by finding the norm (2.5) and minimizing it by coefficients C_β .

2.1. The norm of the error functional (2.3)

To find the norm (2.5), we use the extremal function ψ_ℓ for the error functional ℓ (see [23,24]) that satisfies the following equality:

$$(\ell, \psi_\ell) = \|\ell\|_{L_2^{(1)*}[0,1]} \cdot \|\psi_\ell\|_{L_2^{(1)}[0,1]}. \quad (2.7)$$

Since $L_2^{(1)}[0, 1]$ is a Hilbert space, we obtain

$$(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle \quad (2.8)$$

using the Riesz theorem for ψ_ℓ , where $\langle \psi_\ell, \varphi \rangle$ is the inner product of the functions ψ_ℓ and φ defined by (2.4) and $\varphi \in L_2^{(1)}[0, 1]$, respectively. In addition, the equality $\|\ell\|_{L_2^{(1)*}[0,1]} = \|\psi_\ell\|_{L_2^{(1)}[0,1]}$ is achieved. Then we obtain

$$(\ell, \psi_\ell) = \|\ell\|_{L_2^{(1)*}[0,1]}^2 \quad (2.9)$$

from (2.7). In order for the error functional (2.3) to be defined in the space $L_2^{(1)}[0, 1]$, the condition

$$(\ell, 1) = 0 \quad (2.10)$$

must be imposed which means that the quadrature formula (2.1) is exact for any constant term.

For ψ_ℓ in (2.8) we have

$$\psi_\ell''(x) = -\bar{\ell}(x), \quad (2.11)$$

$$\psi_\ell'(0) = 0, \quad \psi_\ell'(1) = 0, \quad (2.12)$$

where $\bar{\ell}$ is the complex conjugate to ℓ . Then the following theorem holds.

Theorem 1. The solution of the boundary value problem (2.11)–(2.12) is the extremal function ψ_ℓ of the error functional ℓ , expressed as

$$\psi_\ell(x) = -\bar{\ell}(x) * G_1(x) + p_0, \quad (2.13)$$

where

$$G_1(x) = \frac{|x|}{2}, \quad (2.14)$$

$p_0 = p_0^R + ip_0^I$, a complex number, and $*$ is the convolution operation.

From Sobolev's result (see [23,24]) on the extremal function of quadrature formulas in the space $L_2^{(m)}$, we can get the statement of Theorem 1, especially when $m = 1$.

Next, we assume that

$$C_\beta = C_\beta^R + iC_\beta^I, \quad (2.15)$$

where C_β^R and C_β^I are real numbers. Then, using (2.10) and (2.13) for the norm of the error functional ℓ with (2.9), we get

$$\begin{aligned}\|\ell\|_{L_2^{(1)*}[0,1]}^2 &= (\ell, \psi_\ell) \\ &= \int_{-\infty}^{\infty} \ell(x) \psi_\ell(x) dx = - \int_{-\infty}^{\infty} \ell(x) \cdot (\bar{\ell}(x) * G_1(x)) dx.\end{aligned}$$

Therefore, by direct calculation with (2.15), we get

$$\begin{aligned}\|\ell\|_{L_2^{(1)*}[0,1]}^2 &= - \left[\sum_{\beta=0}^N \sum_{\gamma=0}^N (C_\beta^R C_\gamma^R + C_\beta^I C_\gamma^I) G_1(h\beta - h\gamma) \right. \\ &\quad - 2 \sum_{\beta=0}^N C_\beta^R \int_0^1 \cos 2\pi \omega x \cdot G_1(x - h\beta) dx \\ &\quad - 2 \sum_{\beta=0}^N C_\beta^I \int_0^1 \sin 2\pi \omega x \cdot G_1(x - h\beta) dx \\ &\quad \left. + \int_0^1 \int_0^1 \cos[2\pi \omega(x - y)] \cdot G_1(x - y) dx dy \right].\end{aligned}\quad (2.16)$$

Then from (2.10) with (2.15), we obtain the following equalities:

$$\sum_{\beta=0}^N C_\beta^R = \int_0^1 \cos 2\pi \omega x \, dx, \quad (2.17)$$

$$\sum_{\beta=0}^N C_\beta^I = \int_0^1 \sin 2\pi \omega x \, dx. \quad (2.18)$$

Thus, we get the expression (2.16) for the norm of the error functional (2.3).

Further, in the next section we will solve Problem 1.

2.2. Minimization of the expression (2.16) by coefficients C_β

Problem 1 is equivalent to the problem minimizing (2.16) in C_β^R and C_β^I using Lagrange method under the conditions (2.17) and (2.18).

Now we consider the function

$$\begin{aligned}\Psi(C_0^R, C_1^R, \dots, C_N^R, C_0^I, C_1^I, \dots, C_N^I, p_0^R, p_0^I) \\ = \|\ell\|_{L_2^{(1)*}[0,1]}^2 + 2p_0^R \left(\int_0^1 \cos 2\pi \omega x dx - \sum_{\beta=0}^N C_\beta^R \right) + 2p_0^I \left(\int_0^1 \sin 2\pi \omega x dx - \sum_{\beta=0}^N C_\beta^I \right).\end{aligned}$$

By making the partial derivatives of Ψ with respect to C_β^R , C_β^I , $(\beta = \overline{0, N})$, p_0^R and p_0^I equal to zero, we get the following system of linear equations:

$$\sum_{\gamma=0}^N C_\gamma^R G_1(h\beta - h\gamma) + p_0^R = \int_0^1 \cos 2\pi \omega x G_1(x - h\beta) dx, \beta = 0, \dots, N, \quad (2.19)$$

$$\sum_{\gamma=0}^N C_\gamma^R = \int_0^1 \cos 2\pi \omega x \, dx, \quad (2.20)$$

$$\sum_{\gamma=0}^N C_\gamma^I G_1(h\beta - h\gamma) + p_0^I = \int_0^1 \sin 2\pi \omega x G_1(x - h\beta) dx, \beta = 0, \dots, N, \quad (2.21)$$

$$\sum_{\gamma=0}^N C_\gamma^I = \int_0^1 \sin 2\pi \omega x \, dx. \quad (2.22)$$

We multiply both sides of (2.21) and (2.22) by i and add these to (2.19) and (2.20), respectively, to obtain a system of $(N + 2)$ linear equations with $(N + 2)$ unknowns C_γ , $\gamma = 0, 1, \dots, N$, and p_0 :

$$\sum_{\gamma=0}^N C_\gamma G_1(h\beta - h\gamma) + p_0 = \int_0^1 e^{2\pi i \omega x} G_1(x - h\beta) dx, \quad \beta = 0, \dots, N, \quad (2.23)$$

$$\sum_{\gamma=0}^N C_\gamma = \int_0^1 e^{2\pi i \omega x} dx, \quad (2.24)$$

where $G_1(x)$ is defined in (2.14). The system (2.23)–(2.24) has a unique solution. The uniqueness of the solution of this system can be proved by the uniqueness of the solution of the system (3.1)–(3.2) in [25]. The solution of the system (2.23)–(2.24) provides the minimum of $\|\ell\|_{L_2^{(1)*}[0,1]}^2$ at $C_\beta = \hat{C}_\beta$. The quadrature formula of the form (2.1) with coefficients \hat{C}_β is called the *optimal quadrature formula* in the sense of Sard, and \hat{C}_β are said to be the *optimal coefficients*. For convenience, the optimal coefficients \hat{C}_β will be denoted as C_β .

The purpose of this section is to obtain an analytic solution for the system (2.23)–(2.24). To do this, we use the concept of discrete argument functions and operations. The theory of discrete argument functions is given in [23,24]. We give the definition for the function of discrete argument. Suppose that node x_β has uniform spacing (i.e., $x_\beta = h\beta$, h is a small positive parameter), and functions $\varphi(x)$ and $\psi(x)$ are complex-valued and defined on the real line \mathbb{R} or on an interval of \mathbb{R} .

The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β . The inner product of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \bar{\psi}(h\beta),$$

if the series on the right hand side of the last equality converges absolutely. The convolution of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \bar{\psi}(h\beta - h\gamma).$$

We also use the discrete analogue $D_1(h\beta)$ for the operator d^2/dx^2 , that satisfies

$$hD_1(h\beta) * G_1(h\beta) = \delta_d(h\beta), \quad (2.25)$$

where $G_1(h\beta) = \frac{|h\beta|}{2}$, $\delta_d(h\beta)$ is equal to 0 when $\beta \neq 0$, and 1 when $\beta = 0$.

It should be noted that the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m} was first introduced and investigated by Sobolev [23,24] and it was constructed in [26]. In particular, from the results of [26] for $m = 1$, the following are obtained.

Theorem 2. The discrete analogue $D_1(h\beta)$ to the operator d^2/dx^2 satisfying (2.25) has the form

$$D_1(h\beta) = \frac{1}{h^2} \begin{cases} 0, & |\beta| \geq 2, \\ 1, & |\beta| = 1, \\ -2, & \beta = 0 \end{cases} \quad (2.26)$$

and satisfies

$$D_1(h\beta) * 1 = 0, \quad D_1(h\beta) * (h\beta) = 0. \quad (2.27)$$

Now we return to our problem.

We regard the coefficients C_β as a discrete argument function and assume $C_\beta = 0$ for $\beta = -1, -2, \dots$ and $\beta = N + 1, N + 2, \dots$. Then, considering the above definitions, we rewrite the system (2.23)–(2.24) in the convolution form as

$$C_\beta * G_1(h\beta) + p_0 = f_1(h\beta), \quad \beta = 0, 1, \dots, N, \quad (2.28)$$

$$\sum_{\beta=0}^N C_\beta = g_0, \quad (2.29)$$

where

$$f_1(h\beta) = -\frac{h\beta}{2(2\pi i\omega)}(e^{2\pi i\omega} + 1) + \frac{1}{2(2\pi i\omega)^2}(2e^{2\pi i\omega h\beta} + (2\pi i\omega - 1)e^{2\pi i\omega} - 1), \quad (2.30)$$

$$g_0 = \frac{1}{2\pi i\omega}(e^{2\pi i\omega} - 1) \quad (2.31)$$

and $G_1(x)$ is defined by (2.14).

Now we have the following problem.

Problem 2. Find C_β , $\beta = 0, 1, \dots, N$, and p_0 satisfying the system (2.28)–(2.29) for given $f_1(h\beta)$ and g_0 .

Note that Problem 2 is equivalent to Problem 1. The main result of this section is as follows.

Theorem 3. For $\omega \in \mathbb{R}$ with $\omega \neq 0$, coefficients of the optimal quadrature formulas of the form (2.1) in the sense of Sard in the space $L_2^{(1)*}[0, 1]$ have the form

$$\begin{aligned} C_0 &= h \frac{(1 + 2\pi i\omega h - e^{2\pi i\omega h})}{(2\pi \omega h)^2}, \\ C_\beta &= h \frac{2(1 - \cos 2\pi \omega h)}{(2\pi \omega h)^2} e^{2\pi i\omega h\beta}, \quad \beta = 1, 2, \dots, N-1, \\ C_N &= h \frac{(1 - 2\pi i\omega h - e^{-2\pi i\omega h})}{(2\pi \omega h)^2} e^{2\pi i\omega}. \end{aligned} \quad (2.32)$$

In addition, for the square of the norm of the error functional (2.3) of the optimal quadrature formula (2.1) in the space $L_2^{(1)*}[0, 1]$, the following holds:

$$\|\ell\|_{L_2^{(1)*}[0, 1]}^2 = \frac{1}{(2\pi \omega)^2} \left(1 - \frac{2(1 - \cos 2\pi \omega h)}{(2\pi \omega h)^2} \right). \quad (2.33)$$

Proof. We consider a discrete argument function

$$u_1(h\beta) = C_\beta * G_1(h\beta) + p_0. \quad (2.34)$$

Then, considering (2.25) and (2.27), we have

$$C_\beta = hD_1(h\beta) * u_1(h\beta). \quad (2.35)$$

Calculating the convolution (2.35) requires the representation of the function $u_1(h\beta)$ for all integer values of β . From (2.28) we have

$$u_1(h\beta) = f_1(h\beta) \text{ for } \beta = 0, 1, \dots, N. \quad (2.36)$$

Now we need to find the representation of $u_1(h\beta)$ for $\beta < 0$ and $\beta > N$. Using (2.14) and (2.29) for $\beta \leq 0$ and $\beta \geq N$, respectively, we get

$$u_1(h\beta) = \begin{cases} -\frac{h\beta}{2} g_0 + \frac{1}{2} \sum_{\gamma=0}^N C_\gamma h\gamma + p_0, & \beta \leq 0, \\ \frac{h\beta}{2} g_0 - \frac{1}{2} \sum_{\gamma=0}^N C_\gamma h\gamma + p_0, & \beta \geq N, \end{cases} \quad (2.37)$$

where g_0 is defined as (2.31), and $\sum_{\gamma=0}^N C_\gamma h\gamma$ and p_0 are unknowns. Then from the last two equalities when $\beta = 0$ and $\beta = N$, we get the following system of two linear equations for these unknowns:

$$\begin{aligned} p_0 + \frac{1}{2} \sum_{\gamma=0}^N C_\gamma h\gamma &= f_1(0), \\ p_0 - \frac{1}{2} \sum_{\gamma=0}^N C_\gamma h\gamma + \frac{1}{2} g_0 &= f_1(1). \end{aligned}$$

Therefore, solving this system using (2.30) and (2.31), we get

$$p_0 = 0, \quad (2.38)$$

$$\sum_{\gamma=0}^N C_{\gamma} h_{\gamma} = \frac{e^{2\pi i \omega}}{2\pi i \omega} - \frac{e^{2\pi i \omega} - 1}{(2\pi i \omega)^2}. \quad (2.39)$$

With (2.38) and (2.39) in mind, the combination of (2.36) and (2.37) results in

$$u_1(h\beta) = \begin{cases} -\frac{h\beta}{2} g_0 + \frac{(2\pi i \omega - 1)e^{2\pi i \omega} + 1}{2(2\pi i \omega)^2}, & \beta \leq 0, \\ f_1(h\beta), & 0 \leq \beta \leq N, \\ \frac{h\beta}{2} g_0 - \frac{(2\pi i \omega - 1)e^{2\pi i \omega} + 1}{2(2\pi i \omega)^2}, & \beta \geq N. \end{cases}$$

The analytic formula (2.32) is now obtained from (2.35) by taking into account (2.26) and (2.27), using the last representation of $u_1(h\beta)$, and by direct calculation of the optimal coefficients C_{β} , $\beta = 0, 1, \dots, N$.

Now we are going to get (2.33). We rewrite (2.16) in the following form:

$$\begin{aligned} \|\tilde{\ell}\|_{L_2^{(1)*}[0,1]}^2 = & - \left[\sum_{\beta=0}^N C_{\beta}^R \left(\sum_{\gamma=0}^N C_{\gamma}^R G_1(h\beta - h\gamma) - \int_0^1 \cos 2\pi \omega x G_1(x - h\beta) dx \right) \right. \\ & + \sum_{\beta=0}^N C_{\beta}^I \left(\sum_{\gamma=0}^N C_{\gamma}^I G_1(h\beta - h\gamma) - \int_0^1 \sin 2\pi \omega x G_1(x - h\beta) dx \right) \\ & - \sum_{\beta=0}^N C_{\beta}^R \int_0^1 \cos 2\pi \omega x G_1(x - h\beta) dx - \sum_{\beta=0}^N C_{\beta}^I \int_0^1 \sin 2\pi \omega x G_1(x - h\beta) dx \\ & \left. + \int_0^1 \int_0^1 \cos[2\pi \omega(x - y)] G_1(x - y) dx dy \right]. \end{aligned} \quad (2.40)$$

Since $p_0 = p_0^R + ip_0^I$, considering (2.38), we have

$$p_0^R = 0 \text{ and } p_0^I = 0.$$

Therefore, these two last equalities are used in (2.19) and (2.21) to obtain

$$\sum_{\gamma=0}^N C_{\gamma}^R G_1(h\beta - h\gamma) - \int_0^1 \cos 2\pi \omega x G_1(x - h\beta) dx = 0, \quad \beta = 0, \dots, N$$

and

$$\sum_{\gamma=0}^N C_{\gamma}^I G_1(h\beta - h\gamma) - \int_0^1 \sin 2\pi \omega x G_1(x - h\beta) dx = 0, \quad \beta = 0, \dots, N.$$

Then the expression (2.40) for $\|\tilde{\ell}\|_{L_2^{(1)*}[0,1]}^2$ takes the form

$$\begin{aligned} \|\tilde{\ell}\|_{L_2^{(1)*}[0,1]}^2 = & \sum_{\beta=0}^N C_{\beta}^R \int_0^1 \cos 2\pi \omega x G_1(x - h\beta) dx + \sum_{\beta=0}^N C_{\beta}^I \int_0^1 \sin 2\pi \omega x G_1(x - h\beta) dx \\ & - \int_0^1 \int_0^1 \cos[2\pi \omega(x - y)] G_1(x - y) dx dy. \end{aligned}$$

Therefore calculating the definite integrals, keeping (2.15) in mind and using (2.32), we get (2.33) after some simplifications. Theorem 3 has been proved. \square

We note that in Theorem 3, the formulas for the optimal coefficients C_{β} are decomposed into two parts: real and imaginary parts. Therefore from the formulas (2.32) of Theorem 3, we get the following results.

Corollary 1. For $\omega \in \mathbb{R}$ with $\omega \neq 0$, coefficients of the optimal quadrature formula of the form

$$\int_0^1 \cos 2\pi \omega x \cdot \varphi(x) dx \cong \sum_{\beta=0}^N C_{\beta}^R \varphi(h\beta)$$

in the sense of Sard in $L_2^{(1)}[0, 1]$ have the form

$$C_0^R = h \frac{1 - \cos 2\pi \omega h}{(2\pi \omega h)^2},$$

$$C_\beta^R = h \frac{2(1 - \cos 2\pi\omega h)}{(2\pi\omega h)^2} \cos 2\pi\omega h\beta, \quad \beta = 1, 2, \dots, N-1,$$

$$C_N^R = h \frac{(1 - \cos 2\pi\omega h) \cos 2\pi\omega + (2\pi\omega h - \sin 2\pi\omega h) \sin 2\pi\omega}{(2\pi\omega h)^2}.$$

Corollary 2. For $\omega \in \mathbb{R}$ with $\omega \neq 0$, coefficients of the optimal quadrature formula of the form

$$\int_0^1 \sin 2\pi\omega x \cdot \varphi(x) dx \cong \sum_{\beta=0}^N C_\beta^I \varphi(h\beta)$$

in the sense of Sard in $L_2^{(1)}[0, 1]$ have the form

$$C_0^I = h \frac{2\pi\omega h - \sin 2\pi\omega h}{(2\pi\omega h)^2},$$

$$C_\beta^I = h \frac{2(1 - \cos 2\pi\omega h)}{(2\pi\omega h)^2} \sin 2\pi\omega h\beta, \quad \beta = 1, 2, \dots, N-1,$$

$$C_N^I = h \frac{(1 - \cos 2\pi\omega h) \sin 2\pi\omega - (2\pi\omega h - \sin 2\pi\omega h) \cos 2\pi\omega}{(2\pi\omega h)^2}.$$

It is easy to see that for $\omega \rightarrow 0$ Sard's following result [16] on the optimality of the trapezoidal quadrature formula in $L_2^{(1)}[0, 1]$ is obtained from Theorem 3.

Corollary 3. Coefficients of the optimal quadrature formula of the form

$$\int_0^1 \varphi(x) dx \cong \sum_{\beta=0}^N C_\beta \varphi(h\beta) \tag{2.41}$$

in the space $L_2^{(1)}[0, 1]$ have the form

$$C_0 = \frac{h}{2},$$

$$C_\beta = h, \quad \beta = 1, 2, \dots, N-1,$$

$$C_N = \frac{h}{2}$$

and for the norm of the error functional of the optimal quadrature formula (2.41) in the space $L_2^{(1)*}[0, 1]$, the following holds

$$\|\dot{\ell}\|_{L_2^{(1)*}[0,1]}^2 = \frac{h^2}{12}.$$

In addition, for $\omega h \in \mathbb{Z}$ with $\omega \neq 0$ we obtain the following corollary from (2.32) and (2.33).

Corollary 4. For $\omega h \in \mathbb{Z}$ with $\omega \neq 0$, coefficients of the optimal quadrature formula of the form (2.1) in the sense of Sard in the space $L_2^{(1)}[0, 1]$ have the form

$$C_0 = \frac{1}{2\pi i\omega},$$

$$C_\beta = 0, \quad \beta = 1, 2, \dots, N-1,$$

$$C_N = -\frac{1}{2\pi i\omega}$$

and for the norm of the error functional (2.3) of the optimal quadrature formula (2.1) in the space $L_2^{(1)*}[0, 1]$, the following holds:

$$\|\dot{\ell}\|_{L_2^{(1)*}[0,1]}^2 = \frac{1}{(2\pi\omega)^2},$$

i.e., the convergence order of the optimal quadrature formula of the form (2.1) is $O(|\omega|^{-1})$ for $\omega h \in \mathbb{Z}$ with $\omega \neq 0$.

Remark 1. It should be noted that for a fixed ω , we obtain

$$\|\dot{\ell}\|_{L_2^{(1)*}[0,1]}^2 = \frac{1}{12}h^2 - \frac{1}{90}\pi^2\omega^2h^4 + \frac{1}{1260}\pi^4\omega^4h^6 + O(h^8),$$

from (2.33), i.e., the convergence order of the optimal quadrature formula of the form (2.1) is $O(h)$.

Remark 2. In particular, in the case $\omega \in \mathbb{Z}$ with $\omega \neq 0$, the results of [27] and of Section 6 of [13] are obtained from Theorem 3.

Remark 3. The equality (2.39) means that the optimal quadrature formula of the form (2.1) with coefficients (2.32) is exact to $\varphi(x) = x$ because

$$\int_0^1 e^{2\pi i \omega x} x \, dx = \frac{e^{2\pi i \omega} - 1}{2\pi i \omega} - \frac{e^{2\pi i \omega} - 1}{(2\pi i \omega)^2}.$$

The equality (2.39) together with (2.29) provides the exactness of our optimal quadrature formula for all linear functions. Furthermore, it is easy to check that this optimal quadrature formula is exact for all piecewise linear functions. Since functions with a continuous second derivative can be expanded in the first Taylor polynomial about a point, the convergence order of the optimal quadrature formula (2.1) with coefficients (2.32) is concluded as $O(h^2)$.

It should be noted that optimal quadrature formulas for the interval $[a, b]$ are obtained by a linear transform from the optimal quadrature formulas for the interval $[0, 1]$.

Now we consider a quadrature formula of the form

$$\int_a^b e^{2\pi i \omega x} \varphi(x) \, dx \cong \sum_{\beta=0}^N C_{\beta, \omega}[a, b] \varphi(h\beta + a) \quad (2.42)$$

in the Sobolev space $L_2^{(1)}[a, b]$. Here $C_{\beta, \omega}[a, b]$ are coefficients, $\omega \in \mathbb{R}$, $i^2 = -1$, and $h = \frac{b-a}{N}$ for $N \in \mathbb{N}$.

Then, by applying Theorem 3 and Corollary 3, we have the following main result of the present work.

Theorem 4. For $\omega \in \mathbb{R}$ with $\omega \neq 0$, coefficients of the optimal quadrature formula of the form (2.42) in the sense of Sard in the space $L_2^{(1)}[a, b]$ have the form

$$\begin{aligned} C_{0, \omega}[a, b] &= h \frac{(1 + 2\pi i \omega h - e^{2\pi i \omega h})}{(2\pi \omega h)^2} e^{2\pi i \omega a}, \\ C_{\beta, \omega}[a, b] &= h \frac{2(1 - \cos 2\pi \omega h)}{(2\pi \omega h)^2} e^{2\pi i \omega(h\beta + a)}, \quad \beta = 1, 2, \dots, N-1, \\ C_{N, \omega}[a, b] &= h \frac{(1 - 2\pi i \omega h - e^{-2\pi i \omega h})}{(2\pi \omega h)^2} e^{2\pi i \omega b}, \end{aligned} \quad (2.43)$$

and for $\omega = 0$, the coefficients take the form

$$\begin{aligned} C_{0, 0}[a, b] &= \frac{h}{2}, \\ C_{\beta, 0}[a, b] &= h, \quad \beta = 1, 2, \dots, N-1, \\ C_{N, 0}[a, b] &= \frac{h}{2}, \end{aligned} \quad (2.44)$$

where $h = \frac{b-a}{N}$.

3. Approximation of Fourier transforms by optimal quadrature formula

Here we consider some numerical results confirming the theoretical results of the previous sections. The present section consists of two parts. In the first part, using the optimal quadrature formula (2.42), we approximate the integrals

$$g_{\alpha, \omega}[-1, 1] = \int_{-1}^1 e^{2\pi i \omega x} x^\alpha \, dx, \quad \alpha = 0, 1, 2,$$

where

$$g_{0, \omega}[-1, 1] = \begin{cases} \frac{1}{\pi \omega} \sin 2\pi \omega, & \omega \neq 0, \\ 2, & \omega = 0, \end{cases} \quad (3.1)$$

$$g_{1, \omega}[-1, 1] = \begin{cases} \frac{2i}{(2\pi \omega)^2} (\sin 2\pi \omega - 2\pi \omega \cos 2\pi \omega), & \omega \neq 0, \\ 0, & \omega = 0, \end{cases} \quad (3.2)$$

$$g_{2, \omega}[-1, 1] = \begin{cases} \frac{4}{(2\pi \omega)^3} ((2\pi^2 \omega^2 - 1) \sin 2\pi \omega + 2\pi \omega \cos 2\pi \omega), & \omega \neq 0, \\ \frac{2}{3}, & \omega = 0. \end{cases} \quad (3.3)$$

In the second part, using the given function f , the optimal quadrature formula (2.42) is applied to the approximation of the Fourier Transforms

$$F(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} f(x) dx, \quad (3.4)$$

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i \omega x} F(\omega) d\omega \quad (3.5)$$

and thereby resulting in the approximate reconstruction of the function f .

3.1. Approximation of the Fourier integral

The error of the optimal quadrature formula (2.42) is denoted by

$$R_{\varphi, \omega}[a, b] = \int_a^b e^{2\pi i \omega x} \varphi(x) dx - \sum_{\beta=0}^N C_{\beta, \omega}[a, b] \varphi(h\beta + a).$$

Consider the functions f_α , $\alpha = 0, 1, 2$, obtained by extending monomials x^α , $\alpha = 0, 1, 2$, respectively, with zeros outside the interval $[-1, 1]$, i.e., we have

$$f_\alpha(x) = \begin{cases} x^\alpha & \text{for } x \in [-1, 1], \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \alpha = 0, 1, 2. \quad (3.6)$$

First the optimal quadrature formula (2.42) is applied to approximate the integrals

$$\int_{-1}^1 e^{2\pi i \omega x} f_\alpha(x) dx$$

for functions f_α , $\alpha = 0, 1, 2$, in (3.6).

Since the optimal quadrature formula (2.42) is exact for linear functions, for functions f_0 and f_1 , we obtain

$$g_{0, \omega}[-1, 1] = \sum_{\beta=0}^N C_{\beta, \omega}[-1, 1], \quad g_{1, \omega}[-1, 1] = \sum_{\beta=0}^N C_{\beta, \omega}[-1, 1] \cdot (h\beta - 1),$$

where $h = 2/N$. Therefore, $R_{f_0, \omega}[-1, 1] = 0$ and $R_{f_1, \omega}[-1, 1] = 0$. The graphs for the absolute values of the real parts of the errors $R_{f_0, \omega}[-1, 1]$ and $R_{f_1, \omega}[-1, 1]$ in the first and the second columns of Fig. 1 confirm the real parts of the last equalities numerically. For the function f_2 , there is the error

$$R_{f_2, \omega}[-1, 1] = g_{2, \omega}[-1, 1] - \sum_{\beta=0}^N C_{\beta, \omega}[-1, 1] (h\beta - 1)^2, \quad (3.7)$$

where $C_{\beta, \omega}[-1, 1]$ are defined by (2.43) and (2.44) with $h = 2/N$. For $h = 0.1$ and $h = 0.01$ with $\omega \in [-1, 1]$ the graphs for absolute values of the real part of the error (3.7) are shown in the third column of Fig. 1. From the graphs in the third and fourth rows of the third column of Fig. 1, we can see that the error (3.7) of the optimal quadrature formula (2.42) for the function $f_2(x) = x^2$ is $O(h^2)$. This statement confirms Remark 3 numerically.

Then for an interval $[a, b]$ containing the interval $[-1, 1]$, the error $R_{f_\alpha, \omega}[a, b]$, $\alpha = 0, 1, 2$, of the optimal quadrature formula (2.42) corresponding to the functions (3.6) takes the form

$$\begin{aligned} R_{f_\alpha, \omega}[a, b] &= \int_a^b e^{2\pi i \omega x} f_\alpha(x) dx - \sum_{\beta=0}^N C_{\beta, \omega}[a, b] f_\alpha(h\beta + a) \\ &= \int_{-1}^1 e^{2\pi i \omega x} x^\alpha dx - \sum_{\beta=0}^N C_{\beta, \omega}[a, b] f_\alpha(h\beta + a) \\ &= g_{\alpha, \omega}[-1, 1] - \sum_{\beta=0}^N C_{\beta, \omega}[a, b] f_\alpha(h\beta + a), \end{aligned} \quad (3.8)$$

where $g_{\alpha, \omega}[-1, 1]$, $\alpha = 0, 1, 2$, are defined by (3.1)–(3.3). We provide numerical results for intervals $[-10, 10]$ and $[-100, 100]$. In these intervals, from (3.8) for the errors of the optimal quadrature formula (2.42), we get

$$R_{f_\alpha, \omega}[-10, 10] = g_{\alpha, \omega}[-1, 1] - \sum_{\beta=0}^N C_{\beta, \omega}[-10, 10] f_\alpha(h\beta - 10)$$

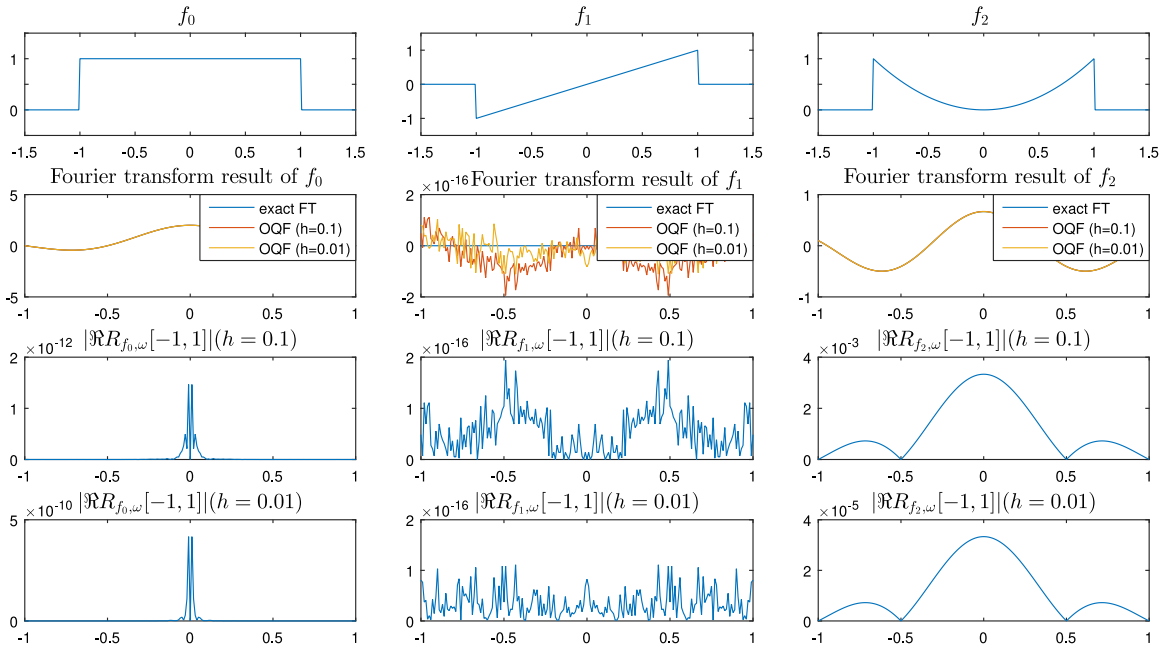


Fig. 1. Graphs of functions f_α , $\alpha = 0, 1, 2$, defined by (3.6) (the first row), graphs of the exact Fourier transforms for functions f_α , $\alpha = 0, 1, 2$ and their approximations by OQF applied for the interval $[-1, 1]$ with steps $h = 0.1$ and $h = 0.01$ (the second row) as well as graphs of $|\Re R_{f_\alpha, \omega}[-1, 1]|$, $\alpha = 0, 1, 2$ when $h = 0.1$ (the third row) and $h = 0.01$ (the fourth row) with $\omega \in [-1, 1]$.

and

$$R_{f_\alpha, \omega}[-100, 100] = g_{\alpha, \omega}[-1, 1] - \sum_{\beta=0}^N C_{\beta, \omega}[-100, 100] f_\alpha(h\beta - 100),$$

respectively. For functions f_α , $\alpha = 0, 1, 2$, defined by (3.6), Figs. 2 and 3 show the graphs of f_α (the first rows), graphs of the exact Fourier transforms for functions f_α and their approximations by OQF applied for the intervals $[-10, 10]$ and $[-100, 100]$ with steps $h = 0.1$ and $h = 0.01$ (the second rows) as well as graphs of $|\Re R_{f_\alpha, \omega}[-10, 10]|$ and $|\Re R_{f_\alpha, \omega}[-100, 100]|$ when $h = 0.1$ (the third rows) and $h = 0.01$ (the fourth rows).

We note that the functions f_α defined by (3.6) are piecewise continuous and do not belong to the space $L_2^{(1)}[a, b]$ when the interval $[a, b]$ contains the interval $[-1, 1]$ and wider than it. Nevertheless, from the numerical results in the first and the third column of Figs. 2 and 3, we conclude that the convergence order of the optimal quadrature formula (2.42) for these functions is $O((h^{-1} + |\omega|)^{-1})$.

Note that the real part of the function $g_{1, \omega}[-1, 1]$ is zero. Due to the symmetry of the considered intervals $[-1, 1]$, $[-10, 10]$ and $[-100, 100]$ and the oddness of the function f_1 as well as the evenness of the optimal coefficients, the real part of the corresponding quadrature sum is also zero. This means that the absolute values of the real part of the error $R_{f_1, \omega}[a, b]$ are zero. This assertion confirms the numerical results (machine zero) given in the second columns of Figs. 1–3.

It is easy to see that the error of the optimal quadrature formula (2.42) is less than the error of the Discrete Fourier Transform for the integral $\int_a^b e^{2\pi i \omega x} \varphi(x) dx$.

3.2. Reconstruction of a function using approximate direct and inverse Fourier transforms

It is known that when complete continuous X-ray data are available then CT image can be reconstructed exactly using the filtered back-projection formula (see, for instance, [1,28,29]). This formula gives interactions between the Radon transform, the Fourier transform and the back-projection transform. A description of the filtered back-projection formula along [1, Chapter 3] is provided below.

In the Cartesian system with x, y -axes consider a unit vector $(\cos \theta, \sin \theta)$. Then the line perpendicular to this vector with the distance t to the origin can be expressed as $\ell_{t, \theta}: x \cos \theta + y \sin \theta = t$. Assume the object is represented by a two variable function $\mu(x, y)$, which denotes the attenuation coefficient in X-ray CT applications. Then, the θ -view projection along the line $\ell_{t, \theta}$ can be expressed as

$$P(t, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) \delta(x \cos \theta + y \sin \theta - t) dx dy,$$

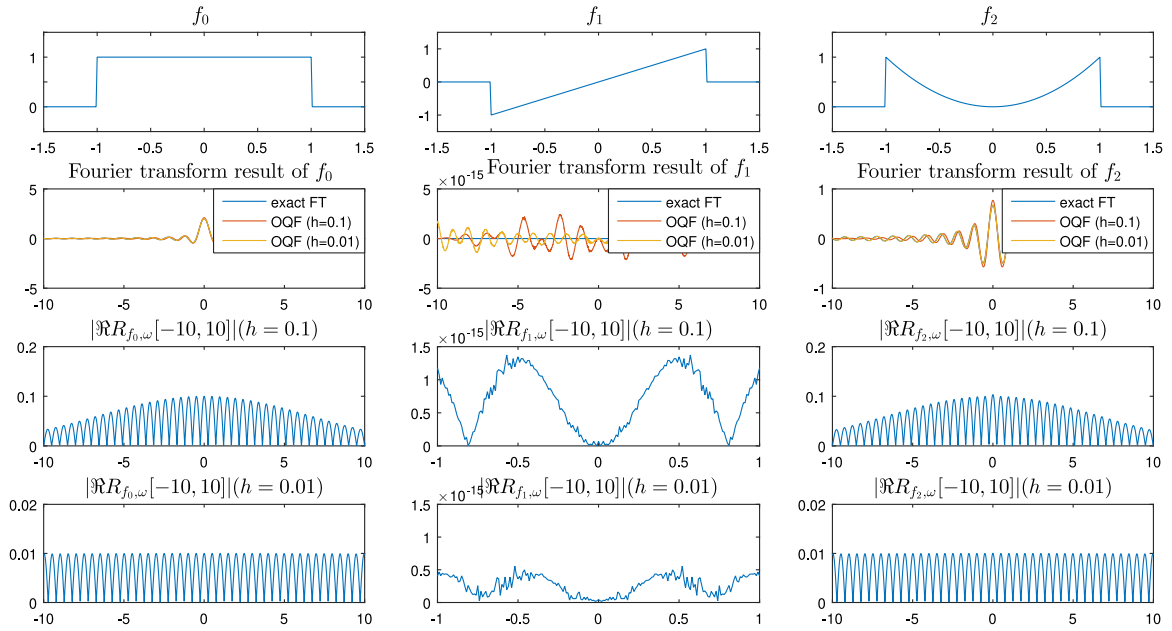


Fig. 2. Graphs of functions f_α , $\alpha = 0, 1, 2$, defined by (3.6) (the first row), graphs of the exact Fourier transforms for functions f_α , $\alpha = 0, 1, 2$ and their approximations by OQF applied for the interval $[-10, 10]$ with steps $h = 0.1$ and $h = 0.01$ (the second row) as well as graphs of $|\Re R_{f_\alpha, \omega}[-10, 10]|$, $\alpha = 0, 1, 2$ when $h = 0.1$ (the third row) and $h = 0.01$ (the fourth row) with $\omega \in [-10, 10]$.

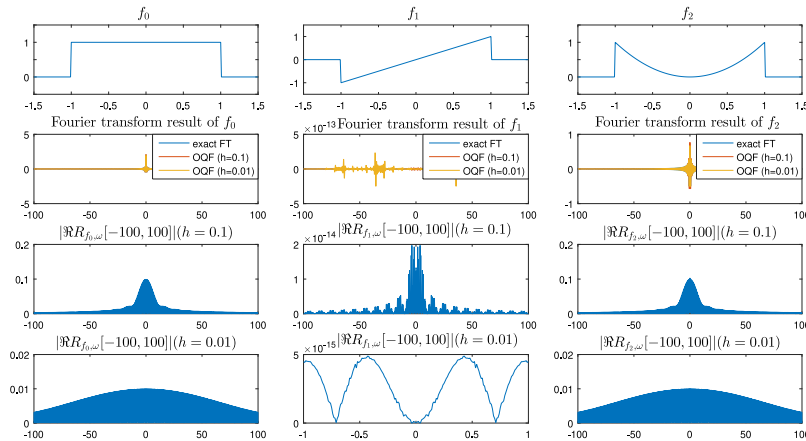


Fig. 3. Graphs of functions f_α , $\alpha = 0, 1, 2$, defined by (3.6) (the first row), graphs of the exact Fourier transforms for functions f_α , $\alpha = 0, 1, 2$ and their approximations by OQF applied for the interval $[-100, 100]$ with steps $h = 0.1$ and $h = 0.01$ (the second row) as well as graphs of $|\Re R_{f_\alpha, \omega}[-100, 100]|$, $\alpha = 0, 1, 2$ when $h = 0.1$ (the third row) and $h = 0.01$ (the fourth row) with $\omega \in [-100, 100]$.

where δ denotes the Dirac delta-function. The function $P(t, \theta)$ is known as the Radon transform of $\mu(x, y)$. A projection is formed by combining a set of line integrals. The simplest projection is a collection of parallel ray integrals as is given by $P(t, \theta)$ for a constant θ . This is known as a parallel beam projection. It should be noted that there are fan-beam in 2D and cone-beam in 3D projections [1,28,29].

The problem of CT is to reconstruct the function $\mu(x, y)$ from its projections $P(t, \theta)$. There are analytic and iterative methods for CT reconstruction. One of the widely used analytic methods of CT reconstruction is the filtered back-projection method. It can be modeled by

$$\mu(x, y) = \int_0^\pi \int_{-\infty}^\infty S(\omega, \theta) |\omega| e^{2\pi i \omega (x \cos \theta + y \sin \theta)} d\omega d\theta, \quad (3.9)$$

where

$$S(\omega, \theta) = \int_{-\infty}^{\infty} P(t, \theta) e^{-2\pi i \omega t} dt \quad (3.10)$$

is the 1D Fourier transform of $P(t, \theta)$. The inner integral of (3.9) can be regarded as a 1D inverse Fourier transform of the product $S(\omega, \theta) |\omega|$, i.e.,

$$Q(t, \theta) = \int_{-\infty}^{\infty} S(\omega, \theta) |\omega| e^{2\pi i \omega t} d\omega \quad (3.11)$$

which represents a projection filtered by a 1D filter whose frequency representation is $|\omega|$. The outer integral performs back-projection. Therefore, the filtered back-projection consists of two steps: filtration and then back-projection.

Thus, in (3.9)–(3.11) the Fourier transforms play the main role. But in practice, due to the fact that we have discrete values of the Radon transform, we have to approximately calculate the Fourier transforms in the filtered back-projection.

Here, in the examples of two functions, we first show that the optimal quadrature formula (2.42) can be used for approximation of the Fourier transforms of these functions and reconstruction of them. Then, the optimal quadrature formula (2.42) is applied for an approximate reconstruction of the 512×512 size Shepp–Logan phantom from its Radon transform.

Suppose that we are given the values $f(h\beta + a)$, where $\beta = 0, 1, \dots, N$, $h = \frac{b-a}{N}$ for $N = 2, 3, \dots$. The values of the function f are assumed to be zero outside the interval $[a, b]$. Fourier transform (3.4) of f is then approximated by the optimal quadrature formula (2.42) using $f(h\beta + a)$, $\beta = 0, 1, \dots, N$, as follows:

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} e^{-2\pi i \omega x} f(x) dx \\ &= \int_a^b e^{-2\pi i \omega x} f(x) dx \cong \sum_{\beta=0}^N C_{\beta, -\omega} [a, b] f(h\beta + a). \end{aligned}$$

Since the coefficients $C_{\beta, -\omega} [a, b]$, $\beta = 0, 1, \dots, N$, defined by (2.43)–(2.44) are continuous functions of the variable ω , the following approximation for the Fourier transform is obtained from the last relations

$$F(\omega) \cong F_{\text{app}}(\omega), \quad (3.12)$$

where

$$F_{\text{app}}(\omega) = \sum_{\beta=0}^N C_{\beta, -\omega} [a, b] f(h\beta + a) \text{ for } \omega \in \mathbb{R}.$$

Now for the approximate reconstruction of f in the interval $[a, b]$, we approximate the inverse Fourier transform (3.5) using the values $F_{\text{app}}(\tau\gamma + a)$, $\gamma = 0, 1, \dots, M$, $\tau = \frac{b-a}{M}$, $M = 2, 3, \dots$, of the function $F_{\text{app}}(\omega)$ in the interval $[a, b]$ for ω and truncate the integral outside $[a, b]$ as follows:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} e^{2\pi i \omega x} F(\omega) d\omega \cong \int_{-\infty}^{\infty} e^{2\pi i \omega x} F_{\text{app}}(\omega) d\omega \\ &\cong \int_a^b e^{2\pi i \omega x} F_{\text{app}}(\omega) d\omega \\ &\cong \sum_{\gamma=0}^M C_{\gamma, x} [a, b] F_{\text{app}}(\tau\gamma + a), \end{aligned}$$

where $C_{\gamma, x} [a, b]$ are optimal coefficients defined by (2.43)–(2.44). Hence, due to the continuity of the coefficients $C_{\gamma, x} [a, b]$ for any $x \in \mathbb{R}$, we obtain an approximation

$$f(x) \cong f_{\text{app}}(x), \quad (3.13)$$

where

$$f_{\text{app}}(x) = \sum_{\gamma=0}^M C_{\gamma, x} [a, b] F_{\text{app}}(\tau\gamma + a) \text{ for } x \in \mathbb{R}.$$

Thus, the function $f(x)$ can be approximately reconstructed, especially in the interval $[a, b]$. Therefore, formulas (3.12) and (3.13) can be used for approximate reconstruction of a function from a set of its values.

We now demonstrate this in the example of two piecewise continuous functions

$$f_0(x) = \begin{cases} 1 & \text{for } x \in [-1, 1], \\ 0 & \text{otherwise} \end{cases}$$

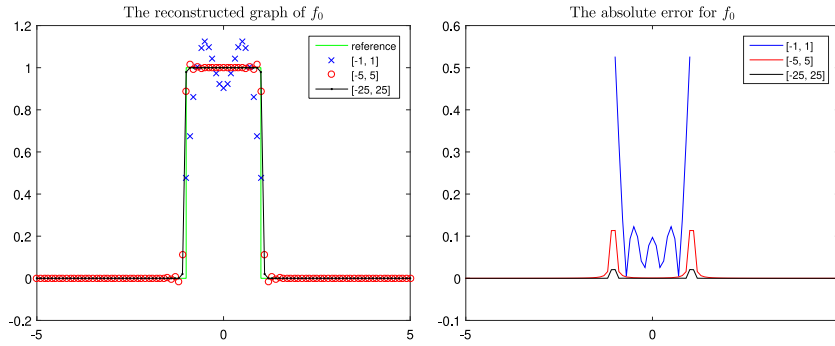


Fig. 4. The reconstructed graphs and corresponding absolute errors of the function f_0 for the intervals $[-1, 1]$, $[-5, 5]$ and $[-25, 25]$ using approximation formulas (3.12) and (3.13) with steps $h = 0.1$ and $\tau = 0.01$.

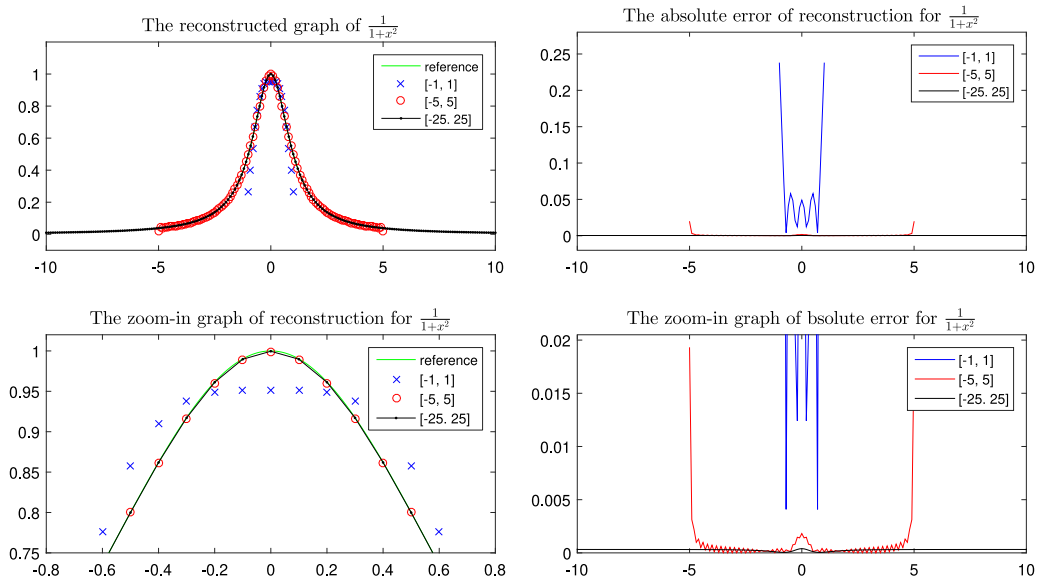


Fig. 5. The reconstructed graphs and corresponding absolute errors of the function ϕ for the intervals $[-1, 1]$, $[-5, 5]$ and $[-25, 25]$ using approximation formulas (3.12) and (3.13) with $h = 0.1$ and $\tau = 0.01$, respectively.

and

$$\phi(x) = \begin{cases} \frac{1}{1+x^2} & \text{for } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

For reconstruction of these functions, we use the approximation formulas (3.12) and (3.13). In numerical calculations we take the intervals $[-1, 1]$, $[-5, 5]$ and $[-25, 25]$ as an interval of integration $[a, b]$ in (3.12) and (3.13) with $h = 0.1$ (for x) and $\tau = 0.01$ (for ω), respectively. Then we obtain the reconstructed graphs and graphs of corresponding absolute errors of the functions f_0 and ϕ for the intervals $[-1, 1]$, $[-5, 5]$ and $[-25, 25]$, shown in Figs. 4 and 5, respectively. These graphs show that we get more accurate reconstructions by taking wider intervals. For the function f_0 , there are the maximum errors around the jump points -1 and 1 in Fig. 4, while for the function ϕ in Fig. 5 the maximum errors are found around the end points of the integration intervals $[-1, 1]$, $[-5, 5]$ and $[-25, 25]$. Thus, the optimal quadrature formula (2.42) constructed using coefficients (2.43) and (2.44) can be effectively applied to the approximation of Fourier integrals.

Finally, we provide the results of applying the optimal quadrature formula (2.42) for approximate reconstruction of the 512×512 size Shepp–Logan phantom from its Radon transform.

We generate the sinogram using half rotation sampling with sampling angle 0.5° . We compare the result of CT image reconstruction using the optimal quadrature formula for Fourier integrals with the result of *iradon*, a built-in function of MATLAB R2019a, which uses *fft* and *ifft* for Fourier integrals. For the image quality analysis, we compare maximum

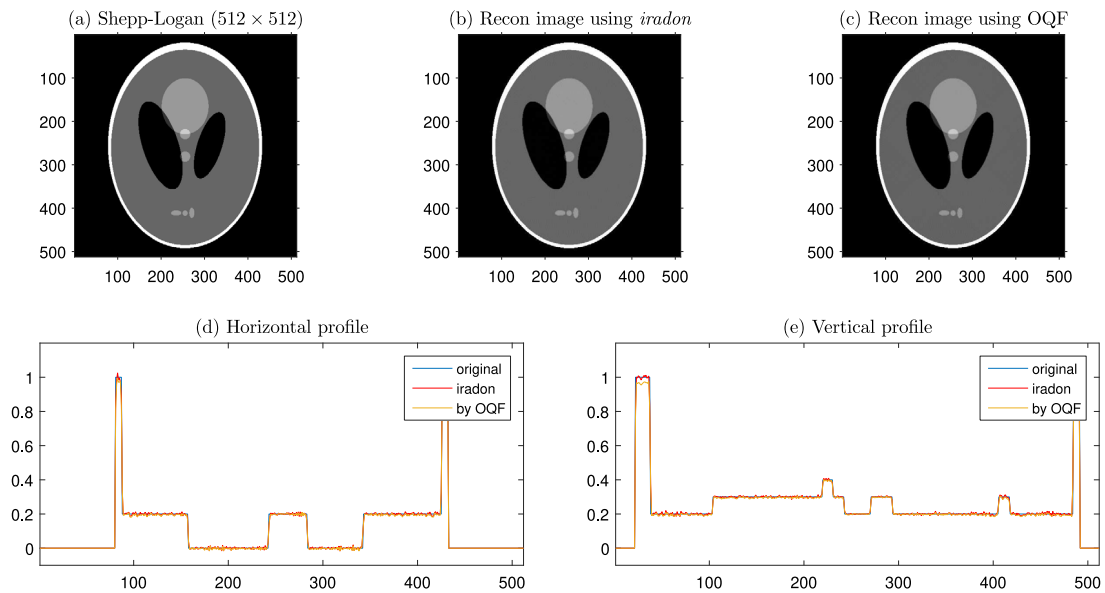


Fig. 6. Comparison result of CT image reconstruction: (a) Shepp-Logan phantom, (b) reconstructed CT image using MATLAB built-in function *iradon*, (c) reconstructed CT image using optimal quadrature formula (OQF), (d) profile of horizontal centerline, (e) profile of vertical centerline.

Table 1
Quantitative image analysis.

	E_{\max} (inner part)	MSE (inner part)	PSNR (inner part)
Reconstruction result using MATLAB built-in function <i>iradon</i>	0.3472 (0.2812)	9.2103e-04 (1.9613e-04)	30.357 (37.075)
Reconstruction result using the proposed optimal quadrature formula	0.3895 (0.2689)	10.8548e-04 (1.9265e-04)	29.644 (37.152)

error (E_{\max}), mean squared error (MSE), and the peak signal-to-noise ratio (PSNR):

$$E_{\max}(I) = \max_{i,j} |I(i,j) - I_{\text{ref}}(i,j)|,$$

$$\text{MSE}(I) = \frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n |I(i,j) - I_{\text{ref}}(i,j)|^2,$$

$$\text{PSNR}(I) = 10 \log_{10} \left(\frac{I_{\max}^2}{\text{MSE}(I)} \right),$$

where I_{\max} is the maximum pixel value of the image I . For I_{ref} , we adopt a Shepp-Logan phantom (Fig. 6(a)).

As shown in Fig. 6(c), the CT image reconstruction algorithm using optimal quadrature formula produces a clear reconstruction image which has the same structures with the original phantom. It also has almost the same appearance with the result of *iradon* as shown in Fig. 6(b). From Fig. 6(d) and (e), we see that the results using the optimal quadrature formula and *iradon* are almost the same except the outer ring. Table 1 shows E_{\max} , MSE, and PSNR for two reconstruction results. The numbers written without parentheses are measured errors in the whole image domain and those in parentheses are measured ones only inside the outer ring. The reconstruction result by using *iradon* seems better than the result by using our optimal quadrature formula if we consider the whole image domain for the error measurement. However, in most CT applications, we are interested in interior structures of an object rather than its outer part, hence we exclude the outer ring, then the opposite holds true. Note that unlike *iradon* which uses various optimized image processing techniques as a MATLAB built-in function, no image processing technique has been applied to our image reconstruction process. We expect that the performance can be improved further if we use the higher order optimal quadrature formula, which is our next research topic.

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