



On generalization of classical Hurwitz stability criteria for matrix polynomials

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ABSTRACT

In this paper we associate a class of Hurwitz matrix polynomials with Stieltjes positive definite matrix sequences. This connection leads to an extension of two classical criteria of Hurwitz stability for real polynomials to matrix polynomials: tests for Hurwitz stability via positive definiteness of block-Hankel matrices built from matricial Markov parameters and via matricial Stieltjes continued fractions. We obtain further conditions for Hurwitz stability in terms of block-Hankel minors and quasiminors, which may be viewed as a weak version of the total positivity criterion.

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1. Introduction

Consider a high-order differential system

$$A_0 y^{(n)}(t) + A_1 y^{(n-1)}(t) + \dots + A_n y(t) = u(t), \quad (1.1)$$

where A_0, \dots, A_n are complex matrices, $y(t)$ is the output vector and $u(t)$ denotes the control input vector. The asymptotic stability of such a system is determined by the Hurwitz stability of its characteristic matrix polynomial

$$F(z) = A_0 z^n + A_1 z^{n-1} + \dots + A_n. \quad (1.2)$$

That is to say, the system (1.1) is asymptotically stable if all roots of $\det F(z)$ lie in the open left half-plane $\Re z < 0$. The effective numerical calculation of the roots may be done via such tools as linearization (we briefly touch upon this below). At the same time there are many algebraic techniques for testing the Hurwitz stability of matrix polynomials avoiding computing its determinant or zeros: the LMI approach [1–4], the Anderson–Jury Bezoutian [5,6], matrix Cauchy indices [7], lossless positive real property [8], block Hurwitz matrix [9], extended Routh–Hurwitz array [10], argument principle [11] and so on.

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However, the authors are so far unaware of a suitable extension to the following classical landmarks which one will inevitably encounter in the abundant study of the scalar case. Gantmacher's monograph [12] gives a comprehensive overview of issues related to Hurwitz polynomials, among which there is a characterization of the Hurwitz stability through the corresponding Stieltjes continued fraction [12, Theorem 16, Chapter XV, p232]:

Theorem (Stability Criterion Via Continued Fractions). *A real polynomial $f(z)$ of degree $n = 2m$ or $n = 2m + 1$ is a Hurwitz polynomial if and only if its even part $f_e(z)$ and its odd part $f_o(z)$ satisfying $f_e(z^2) + zf_o(z^2) = f(z)$ admit the Stieltjes continued fraction*

$$\frac{f_o(z)}{f_e(z)} = c_0 + \frac{1}{zc_1 + \frac{1}{\dots + \frac{1}{c_{2m-2} + \frac{1}{zc_{2m-1} + c_{2m}}}}}, \quad (1.3)$$

where $c_k > 0$ for $k = 1, \dots, n$ and, when $n = 2m + 1$, additionally $c_0 > 0$.

Theorem 17 of [12, Chapter XV] connects Hurwitz polynomials with positive definite Hankel matrices built from Markov parameters:

Theorem (Stability Criterion Via Markov Parameters). *Given a real polynomial $f(z)$ of degree $n = 2m$ or $n = 2m + 1$, define its Markov parameters $(s_k)_{k=-1}^\infty$ as the coefficients in the expansion of the ratio*

$$\frac{f_o(z)}{f_e(z)} = s_{-1} + \sum_{k=0}^{\infty} \frac{(-1)^k s_k}{z^{k+1}},$$

where $f_e(z)$ and $f_o(z)$ are as above. Then $f(z)$ is a Hurwitz polynomial if and only if both Hankel matrices $[s_{j+k}]_{j,k=0}^{m-1}$ and $[s_{j+k+1}]_{j,k=0}^{m-1}$ are positive definite and, for $n = 2m + 1$, additionally $s_{-1} > 0$.

The positive definiteness of the pair of matrices $[s_{j+k}]_{j,k=0}^{m-1}$ and $[s_{j+k+1}]_{j,k=0}^{m-1}$ turns out to be equivalent to the total positivity of rank m of the infinite Hankel matrix $[s_{j+k}]_{j,k=0}^\infty$, by which we mean that all minors of $[s_{j+k}]_{j,k=0}^\infty$ of order $\leq m$ are positive and all minors of order $> m$ are zero. Theorem 20 of [12, Chapter XV] gives the corresponding alternative criterion for Hurwitz stability: *A real polynomial $f(z)$ is a Hurwitz polynomial if and only if $[s_{j+k}]_{j,k=0}^\infty$ is totally positive of rank m and, for $n = 2m + 1$, additionally $s_{-1} > 0$.*

Certain obstacles arise when adopting these classical tools to test Hurwitz stability of matrix polynomials. For example, noncommutativity of the matrix product limits the determinant approach to algebraic constructs built from coefficients of matrix polynomials. The applicability of scalar methods to the matrix case is also influenced by lack of suitable correlations between the matrix coefficients of a matrix polynomial (even of lower degree) and its zeros.

We follow alternative lines to deal with the matrix extension. Based on the spectral theory of matrix polynomials, the papers [5,6] solve some zero-separation problems for matrix polynomials in terms of the Anderson–Jury Bezoutian. Converting the solution from [5,6] into the form of matricial Markov parameters is the key step for our matricial refinement of the stability criterion via Markov parameters.

The so-called “matrix Hurwitz type polynomials”, which are defined via a matricial analogue of Stieltjes continued fraction as in (1.3), are studied by Choque Rivero [13] in connection with the matricial Stieltjes moment problem. To extend the stability criterion via continued fractions to the matrix case or, in other words, to uncover the relation between matrix Hurwitz type polynomials and Hurwitz matrix polynomials, we link the Markov parameters with the properties of Stieltjes moment matrix sequences.

Characteristic polynomials (1.2) may be generalized to matrix entire functions corresponding to infinite-order differential problems (for example, fixed matrices times scalar exponentials arise in systems with fixed delays [14]). Like in the scalar case, our tools have partial extensions to matrix entire functions, which are mostly of academic interest (they involve an infinite series of conditions). For practical purposes, one should exploit particular properties of the problems or characteristic functions. For instance, the Lyapunov-type methods allow one to treat a wide range of dynamical systems including systems with delay [15–17]. Specially tuned methods are needed for systems with non-smooth coefficients, e.g. [18,19].

Applications raise further related questions concerning, for example, the robustness and stability of numerical methods [1,14,20]. Here we only consider the stability region $\Re z < 0$; the other important regions are the unit disc for discrete-time systems [21, p. 451] and sectors of the complex plane for fractional-order equations [22,23]. Instead of dealing with matrix polynomials, one can also apply linearization [24]: the spectrum of a matrix polynomial $F(z)$ as in (1.2) coincides with the spectrum of the linear pencil $z \operatorname{diag}(A_0, I) - C$, where C is its companion matrix (we use such a matrix in (4.2)). A detailed review of stability concepts related to linear systems may be found in [25].

Linearization turns out to be a very universal method also suitable for determining the eigenvectors of (1.2). However, it does have certain drawbacks that may lead to less accurate numerical results. In particular the linear pencil corresponding to a polynomial allows more types of perturbations. Moreover, it often breaks a special structure (Hermitian, antisymmetric etc.) of a matrix polynomial. To overcome some of the drawbacks, one can look for more sophisticated linearizations

(e.g. [26]), or use “quadrification” or the more general “ ℓ -ification” technique introduced in [27]: the spectrum of a matrix polynomial is studied through another matrix polynomial of lower degree but higher dimension. See also a brief review of specific examples in [28, Section 4].

Our approach more straightforwardly resembles the theory for real scalar polynomials. Its inherited limitation is that the Markov parameters need to be Hermitian matrices, which in turn provides simpler tests of stability: the block Hankel matrices generalizing the above $[s_{j+k}]_{j,k=0}^{m-1}$ and $[s_{j+k+1}]_{j,k=0}^{m-1}$ turn out to be structured and of size smaller than the corresponding linearization.

Let us conclude the introduction with the outline of the paper. Section 2 brings forth two natural questions concerning the extension of the stability criterion via Markov parameters and via Stieltjes continued fractions. It turns out that to give such extensions for all complex/real matrix polynomials is impossible. In Section 3, we derive an inertia representation for matrix polynomials in terms of the matricial Markov parameters. Our main results are provided in Section 4 where we deal with a relationship between Hurwitz matrix polynomials and an important type of matricial Stieltjes moment sequences. This leads us to matricial extensions of the stability criteria via Markov parameters and via continued fractions to a special class of matrix polynomials. Further conditions for Hurwitz stability are obtained in terms of Hankel minors and Hankel quasiminors built from matricial Markov parameters. Unfortunately, the block-Hankel total positivity is not generally guaranteed for Hurwitz matrix polynomials.

2. Matricial Markov parameters, Stieltjes continued fractions and Hurwitz stability

We begin with some basic notation. Denote by \mathbb{C} , \mathbb{R} , \mathbb{N}_0 and \mathbb{N} , respectively, the sets of all complex, real, nonnegative integer, and positive integer numbers. Unless explicitly noted, we assume in this paper that $p, q \in \mathbb{N}$. Let $\mathbb{C}^{p \times q}$ stand for the set of all complex $p \times q$ matrices. Let also 0_p and I_p be, respectively, the zero and the identity $p \times p$ matrices. Given a matrix A we denote its transpose by A^T , and its conjugate transpose by A^* . If $A \in \mathbb{C}^{p \times p}$, then we write $A \succ 0$ if it is positive definite, and $A \succeq 0$ if A is nonnegative definite.

We denote by $\mathbb{C}[z]^{p \times p}$ the set of $p \times p$ matrix polynomials, that is the ring of polynomials in z with coefficients from $\mathbb{C}^{p \times p}$; in particular $\mathbb{C}[z] = \mathbb{C}[z]^{1 \times 1}$. So, each $F(z) \in \mathbb{C}[z]^{p \times p}$ may be written as

$$F(z) = \sum_{k=0}^n A_k z^{n-k}, \quad \text{with } A_0, \dots, A_n \in \mathbb{C}^{p \times p}, \quad A_0 \neq 0_p \quad (2.1)$$

for certain $n \in \mathbb{N}_0$, which is called the degree of $F(z)$. In particular, $F(z)$ is *monic* if $A_0 = I_p$. Furthermore, $F(z)$ may be represented as a $p \times p$ matrix whose entries are scalar polynomials in z , and $\deg F$ then equals the maximal degree of the entries. In most cases, the degree will be assumed to be at least two, since the results for linear polynomials are trivial.

Given a matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$ written as in (2.1), define $F^\vee(z) \in \mathbb{C}[z]^{p \times p}$ by

$$F^\vee(z) := \sum_{k=0}^n A_k^* z^{n-k}.$$

A matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$ is said to be *regular* if $\det F(z)$ is not identically zero. It is clear that all monic matrix polynomials are regular. Given a regular matrix polynomial $F(z)$, we say that $\lambda \in \mathbb{C}$ is a *zero* of $F(z)$ if $\det F(\lambda) = 0$. Its *multiplicity* is the multiplicity of λ as a zero of $\det F(z)$. The spectrum $\sigma(F)$ of $F(z)$ is the set of all zeros of $F(z)$.

Definition 2.1. A matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$ may be split into the *even part* $F_e(z)$ and the *odd part* $F_o(z)$ so that $F(z) = F_e(z^2) + zF_o(z^2)$. For $F(z)$ of degree n written as in (2.1), they are defined by

$$F_e(z) := \sum_{k=0}^m A_{2k} z^{m-k} \quad \text{and} \quad F_o(z) := \sum_{k=1}^m A_{2k-1} z^{m-k}$$

when $n = 2m$, and by

$$F_e(z) := \sum_{k=0}^m A_{2k+1} z^{m-k} \quad \text{and} \quad F_o(z) := \sum_{k=0}^m A_{2k} z^{m-k}$$

when $n = 2m + 1$.

Definition 2.2. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be a monic matrix polynomial with the even part $F_e(z)$ and the odd part $F_o(z)$.

(i) In the even case $n = 2m$, suppose that

$$G_1(z) := F_o(z) \cdot (F_e(z))^{-1} \quad (\text{resp. } G_1(z) := (F_e(z))^{-1} \cdot F_o(z)) \quad (2.2)$$

admits the Laurent expansion

$$G_1(z) = \sum_{k=0}^{\infty} (-1)^k z^{-(k+1)} \mathbf{s}_k \quad (2.3)$$

for large enough $|z|$, $z \in \mathbb{C}$. If so, we call the matrix sequence $(\mathbf{s}_k)_{k=0}^\infty$ the *sequence of right (resp. left) Markov parameters* of $F(z)$.

- (ii) In the odd case $n = 2m + 1$, we may also consider $G_1(z)$ as in (2.2) if $F_e(z)$ is regular. Its Laurent expansion then slightly differs: when $\deg F_e = m$,

$$G_1(z) = \mathbf{s}_{-1} + \sum_{k=0}^{\infty} (-1)^k z^{-(k+1)} \mathbf{s}_k \quad (2.4)$$

for large enough $|z|$, $z \in \mathbb{C}$. In this case, the matrix sequence $(\mathbf{s}_k)_{k=-1}^\infty$ is called the *sequence of right (resp. left) Markov parameters of first type* of $F(z)$.

- (iii) Another option for the odd case $n = 2m + 1$ is to consider

$$G_2(z) := F_e(z) \cdot (F_o(z))^{-1} \quad (\text{resp. } G_2(z) := (F_o(z))^{-1} \cdot F_e(z))$$

with the following Laurent representation:

$$G_2(z) = \sum_{k=0}^{\infty} (-1)^k z^{-k} \mathbf{s}_k \quad (2.5)$$

for large enough $|z|$, $z \in \mathbb{C}$. The matrix sequence $(\mathbf{s}_k)_{k=0}^\infty$ is then called the *sequence of right (resp. left) Markov parameters of second type* of $F(z)$.

Our Definition 2.2 is relevant to the matricial Markov parameters introduced in [13, Definition 2.10].

Remark 2.3. Observe that $(\mathbf{s}_j)_{j=0}^\infty$ is the sequence of left Markov parameters of $F(z) \in \mathbb{C}[z]^{p \times p}$ if and only if $(\mathbf{s}_j^*)_{j=0}^\infty$ is the sequence of right Markov parameters of $F^\vee(z)$.

If $F(z)$ is a scalar polynomial, then (i) and (ii) of Definition 2.2 coincide with the classical notion of Markov parameters. If $F(z)$ is a matrix polynomial of odd degree, the definition of the Markov parameters of second type given in (iii) avoids the restriction that the even part of $F(z)$ is regular, while the first type given in (ii) allows to study some of polynomials with degenerate leading coefficients (which is outside the scope of this paper). These matricial Markov parameters will play different roles in identifying Hurwitz matrix polynomials.

Supposing that $\mathcal{S} := (\mathbf{s}_j)_j$ is a sequence of $p \times p$ complex matrices, denote the finite or infinite block Hankel matrix associated with \mathcal{S} by

$$H_{j,k}^{(\mathcal{S})} := \begin{bmatrix} \mathbf{s}_j & \mathbf{s}_{j+1} & \cdots & \mathbf{s}_{j+k} \\ \mathbf{s}_{j+1} & \mathbf{s}_{j+2} & \cdots & \mathbf{s}_{j+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{s}_{j+k} & \mathbf{s}_{j+k+1} & \cdots & \mathbf{s}_{j+2k} \end{bmatrix},$$

where $j \in \mathbb{N}_0$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. For simplicity we write $H_k^{(\mathcal{S})}$ for $H_{0,k}^{(\mathcal{S})}$.

Given $A, B \in \mathbb{C}^{p \times p}$ such that B is nonsingular, denote

$$\frac{A}{B} := A \cdot B^{-1}.$$

Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be a monic matrix polynomial of degree $n = 2m$ or $n = 2m + 1$ (whose even part $F_e(z)$ is regular if $n = 2m + 1$). Let \mathcal{S} be the associated sequence of left or right Markov parameters (of first type when $n = 2m + 1$). Concerning the matricial extension of the above stability criteria, one might pose the following questions:

(Q1) For Hurwitz stability of $F(z)$, is it necessary or sufficient that all eigenvalues of the matrices $H_{m-1}^{(\mathcal{S})}$, $H_{1,m-1}^{(\mathcal{S})}$ and, when nontrivial, \mathbf{s}_{-1} are positive (or e.g. have positive real parts)?

(Q2) For Hurwitz stability of $F(z)$, is it necessary or sufficient that its even part $F_e(z)$ and odd part $F_o(z)$ satisfy

$$\frac{F_o(z)}{F_e(z)} = \mathbf{c}_0 + \frac{I_p}{z\mathbf{c}_1 + \cdots + \frac{I_p}{\mathbf{c}_{2m-2} + \frac{I_p}{z\mathbf{c}_{2m-1} + \mathbf{c}_{2m}^{-1}}}}, \quad (2.6)$$

where all eigenvalues of the matrices \mathbf{c}_k for $k = 1, \dots, n$ and, when $n = 2m + 1$, additionally \mathbf{c}_0 are positive (or e.g. have positive real parts)?

Let us show that this cannot be in general true, even when all coefficients of $F(z)$ are assumed to be real.

Claim 2.4. To the necessity part of (Q1) and (Q2), the answer is generally negative. As a counterexample, consider the matrix polynomial

$$F(z) := \begin{bmatrix} z^2 + 2z + 2 & 2z + 1 \\ z + \frac{1}{2} & z^2 + z + \frac{1}{2} \end{bmatrix}.$$

Both its right Markov parameters

$$\mathbf{s}_0 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ \frac{5}{2} & \frac{3}{2} \end{bmatrix}$$

are singular and have one positive eigenvalue each. Furthermore, no matricial Stieltjes continued fraction as in (2.6) exists. However, the zeros of $F(z)$ may be found explicitly, their approximate values are

$$\{-1, -1.876, -0.062 \pm 0.513i\};$$

so, $F(z)$ is a Hurwitz matrix polynomial. Another counterexample to the necessity part of (Q1) (with nondegenerate Markov parameters) may be found in formula (2.7) below.

Claim 2.5. The sufficiency in questions (Q1) and (Q2) does not generally hold. Indeed, if we take

$$F(z) := \begin{bmatrix} z^2 + z + 1 & \frac{z}{3} + \frac{1}{2} \\ 5z + 1 & z^2 + 2z + 1 \end{bmatrix},$$

the two initial right Markov parameters will be

$$\mathbf{s}_0 = \begin{bmatrix} 1 & \frac{1}{3} \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_1 = \begin{bmatrix} \frac{4}{3} & \frac{5}{6} \\ 7 & \frac{9}{2} \end{bmatrix},$$

with only positive eigenvalues. (In fact, the eigenvalues of all Markov parameters in this example are positive). At the same time, there exists a matricial Stieltjes continued fraction of the form (2.6), where

$$\mathbf{c}_1 = \begin{bmatrix} 6 & -1 \\ -15 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_2 = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 6 & -1 \end{bmatrix}.$$

All eigenvalues of both matrices \mathbf{c}_1 and \mathbf{c}_2 have positive real parts. Nevertheless, the zeros of $F(z)$ are approximately

$$\{-1.581 \pm 0.396i, 0.081 \pm 0.426i\},$$

and hence $F(z)$ cannot be a Hurwitz matrix polynomial.

Quadratic matrix polynomials with Hermitian coefficients may be considered as a close analogue of real scalar polynomials. Indeed, they are necessarily stable when their coefficients are positive definite, see e.g. [24, Chapter 13]. However, the converse is generally not true if the coefficient near the linear term fails to be positive definite: for example, the real symmetric matrix polynomial

$$\begin{bmatrix} z^2 + 2z + 16 & \frac{5}{2}z + 1 \\ \frac{5}{2}z + 1 & z^2 + 2z + 3 \end{bmatrix} \quad (2.7)$$

turns out to be Hurwitz stable. Furthermore, polynomials of higher degrees do not allow such a direct analogy between the cases of Hermitian and scalar coefficients.

Consequently, to give a proper extension of the stability criterion via Markov parameters for all complex/real matrix polynomials seems to be impossible. So, the following questions arise naturally:

(Q3) Which additional conditions on the coefficients or Markov parameters of a matrix polynomial may be posed to make (Q1) and (Q2) true?

(Q4) How can we formulate the stability tests via Markov parameters of second type?

Before answering them, we will find out in the next section how to calculate the number of zeros that a matrix polynomial has in different parts of the complex plane. Then Section 4 will be devoted to answers to (Q3) and (Q4), as well as connections between Hurwitz matrix polynomials and total positivity.

3. Zero localization of matrix polynomials

3.1. Matricial Hermite–Fujiwara theorem revisited

Let us recall the definition of the inertia of a matrix: For $A \in \mathbb{C}^{p \times p}$, the inertia of A with respect to the imaginary axis $i\mathbb{R}$ is defined by the triple

$$\text{In}(A) := (\pi(A), \nu(A), \delta(A)),$$

where $\pi(A)$, $\nu(A)$, and $\delta(A)$ stand for the number of eigenvalues (counting algebraic multiplicities) of A with positive, negative, and zero real parts, respectively.

For the inertia of regular matrix polynomials, we adopt the notation from [5], which is essentially as in [6] (see also [5, Proposition 2.2]).

Definition 3.1. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be regular. Denote by $\gamma_+(F)$, $\gamma_-(F)$, $\gamma_0(F)$ the number of zeros of $F(z)$ (counting with multiplicities), in the open upper half plane, the open lower half plane and on the real axis, respectively. The triple

$$\gamma(F) := (\gamma_+(F), \gamma_-(F), \gamma_0(F))$$

is called the *inertia of $F(z)$ with respect to \mathbb{R}* . Analogously, the triple

$$\gamma'(F) := (\gamma'_+(F), \gamma'_-(F), \gamma'_0(F)),$$

is called the *inertia of $F(z)$ with respect to $i\mathbb{R}$* , replacing the upper half plane by the right half plane, the lower half plane by the left half plane, and the real axis by the imaginary axis, respectively.

A regular matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$ of degree n is Hurwitz stable if and only if $\gamma'_-(F) = np$. So, our first step is to determine the inertia $\gamma'(F)$ of a matrix polynomial $F(z)$, that is to solve the Routh–Hurwitz problem for $F(z)$. In the scalar case, the relation between the inertia of $F(z)$ and the inertia of a special Hermitian matrix – the Bezoutian – is well known, see e.g. [29] or [21, pp. 466–467]. The matrix case may be found in [5,6,30]. We recall a particular refinement of the classical Hermite–Fujiwara theorem by Lerer and Tismenetsky [6]. It is stated in terms of the generalized Bezoutian matrices and greatest common divisors of matrix polynomials.

Let matrix polynomials $L(z)$, $\tilde{L}(z)$, $M(z)$, $\tilde{M}(z) \in \mathbb{C}[z]^{p \times p}$ satisfy

$$\tilde{M}(z)\tilde{L}(z) = M(z)L(z). \quad (3.1)$$

Then the Anderson–Jury Bezoutian matrix $\mathbf{B}_{\tilde{M},M}(L, \tilde{L}) \in \mathbb{C}^{n_1 p \times n_2 p}$ associated with the quadruple $(\tilde{M}, \tilde{L}, M, L)$ is defined by

$$\begin{bmatrix} I_p & zI_p & \cdots & z^{n_1-1}I_p \end{bmatrix} \cdot \mathbf{B}_{\tilde{M},M}(L, \tilde{L}) \cdot \begin{bmatrix} I_p \\ uI_p \\ \vdots \\ u^{n_2-1}I_p \end{bmatrix} = \frac{1}{z-u} (\tilde{M}(z)\tilde{L}(u) - M(z)L(u)), \quad (3.2)$$

where $n_1 := \max\{\deg M, \deg \tilde{M}\}$ and $n_2 := \max\{\deg L, \deg \tilde{L}\}$.

The Anderson–Jury Bezoutian matrix $\mathbf{B}_{\tilde{M},M}(L, \tilde{L})$ is skew-symmetric in the sense that

$$\mathbf{B}_{\tilde{M},M}(L, \tilde{L}) = -\mathbf{B}_{M,\tilde{M}}(\tilde{L}, L).$$

For commuting $L(z)$ and $\tilde{L}(z)$, i.e., when $L(z)\tilde{L}(z) = \tilde{L}(z)L(z)$, it is natural to choose $\tilde{M}(z) = L(z)$ and $M(z) = \tilde{L}(z)$. For a nontrivial choice of $\tilde{M}(z)$ and $M(z)$ in the general non-commutative case, we refer the reader to the construction of the common multiples via spectral theory of matrix polynomials (see [24, Theorem 9.11] for the monic case and [31, Theorem 2.2] for the comonic case).

Definition 3.2. Suppose that $F(z), L(z) \in \mathbb{C}[z]^{p \times p}$. $L(z)$ is called a *right* (resp. *left*) *divisor* of $F(z)$ if there exists an $M(z) \in \mathbb{C}[z]^{p \times p}$ such that

$$F(z) = M(z)L(z) \quad (\text{resp. } F(z) = L(z)M(z)).$$

Let additionally $\tilde{F}(z) \in \mathbb{C}[z]^{p \times p}$. Then $L(z)$ is called a *right* (resp. *left*) *common divisor* of $F(z)$ and $\tilde{F}(z)$ if $L(z)$ is a right (resp. left) divisor of $F(z)$ and also a right (resp. left) divisor of $\tilde{F}(z)$.

Furthermore, $L(z)$ is called a *greatest right* (resp. *left*) *common divisor* (GRCD (resp. GLCD)) of $F(z)$ and $\tilde{F}(z)$ if any other right (resp. left) common divisor is a right (resp. left) divisor of $L(z)$.

Some basic properties of GRCDs/GLCDs are given in [Appendix](#).

Lemma 3.3 ([6, Theorem 2.1]). Given a regular matrix polynomial $L(z) \in \mathbb{C}[z]^{p \times p}$, let $L_1(z) \in \mathbb{C}[z]^{p \times p}$ be also regular such that

$$L^\vee(z)L(z) = L_1^\vee(z)L_1(z).$$

Then

$$\gamma_+(L) = \pi(-i\mathbf{B}_{L_1^\vee, L^\vee}(L, L_1)) + \gamma_+(L_0),$$

$$\gamma_-(L) = \nu(-i\mathbf{B}_{L_1^\vee, L^\vee}(L, L_1)) + \gamma_-(L_0),$$

$$\gamma_0(L) = \delta(-i\mathbf{B}_{L_1^\vee, L^\vee}(L, L_1)) - \gamma_+(L_0) - \gamma_-(L_0),$$

where $L_0(z)$ is a GRCD of $L(z)$ and $L_1(z)$.

Lemma 3.3 describes the inertia of $L(z)$ in terms of the inertia of the matrix $-i\mathbf{B}_{L_1^\vee, L^\vee}(L, L_1)$. In general, the inertia of $-i\mathbf{B}_{L_1^\vee, L^\vee}(L, L_1)$ depends on the choice of $L_1(z)$. In the scalar case, the obvious choice $L_1(z) = L^\vee(z)$ leads to the classical Bezoutian which occurs in the Hermite–Fujiwara theorem.

3.2. Inertia representation of matrix polynomials in terms of Hermitian Markov parameters

Given a monic matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$, define a related composite polynomial $L(z) := F(iz)$. Then the inertia $\gamma'(F)$ may be obtained as $\gamma(L)$ in Lemma 3.3. The remaining question is how to choose an appropriate $L_1(z)$ and, accordingly, $L_0(z)$ such that the inertia of $-i\mathbf{B}_{L_1^\vee, L_1}^\vee(L, L_1)$ may be expressed via the Markov parameters of $F(z)$. Due to this reason, we consider the case when the Markov parameters are Hermitian matrices and apply a lemma from [32, Lemma 2.2] bridging the Anderson–Jury Bezoutian matrices and block Hankel matrices:

Lemma 3.4. Let $N_R(z)$, $D_R(z)$, $N_L(z)$ and $D_L(z) \in \mathbb{C}[z]^{p \times p}$ be regular such that $\deg D_R > \deg N_R$, $\deg D_L > \deg N_L$ and, for all large enough $z \in \mathbb{C}$,

$$(D_L(z))^{-1}N_L(z) = N_R(z)(D_R(z))^{-1} = \sum_{k=0}^{\infty} z^{-(k+1)} \mathbf{s}_k.$$

If $D_L(z)$ and $D_R(z)$ are written in the form

$$D_L(z) = \sum_{k=0}^{m_L} D_{L, m_L-k} z^k \quad \text{and} \quad D_R(z) = \sum_{k=0}^{m_R} D_{R, m_R-k} z^k,$$

where $D_{L,k} \in \mathbb{C}^{p \times p}$ for $k = 0, \dots, m_L$ and $D_{R,k} \in \mathbb{C}^{p \times p}$ for $k = 0, \dots, m_R$, then

$$\mathbf{B}_{D_L, N_L}(D_R, N_R) = \begin{bmatrix} D_{L, m_L-1} & \cdots & D_{L, 0} \\ \vdots & \ddots & \\ D_{L, 0} & & \end{bmatrix} \cdot \begin{bmatrix} \mathbf{s}_0 & \cdots & \mathbf{s}_{m_R-1} \\ \vdots & & \vdots \\ \mathbf{s}_{m_L-1} & \cdots & \mathbf{s}_{m_R+m_L-2} \end{bmatrix} \cdot \begin{bmatrix} D_{R, m_R-1} & \cdots & D_{R, 0} \\ \vdots & \ddots & \\ D_{R, 0} & & \end{bmatrix}.$$

Theorem 3.5. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be monic of degree n . Suppose that \mathcal{S} is the associated sequence of right (resp. left) Markov parameters for the case $n = 2m$ or right (resp. left) Markov parameters of second type for the case $n = 2m + 1$. If \mathcal{S} is a sequence of Hermitian matrices, then

(i) in the even case when $n = 2m$,

$$\gamma'_-(F) = \pi(H_{m-1}^{(\mathcal{S})}) + \pi(H_{1, m-1}^{(\mathcal{S})}) + \gamma_+(\widehat{F}), \quad (3.3)$$

$$\gamma'_+(F) = \nu(H_{m-1}^{(\mathcal{S})}) + \nu(H_{1, m-1}^{(\mathcal{S})}) + \gamma_-(\widehat{F}), \quad (3.4)$$

$$\gamma'_0(F) = \delta(H_{m-1}^{(\mathcal{S})}) + \delta(H_{1, m-1}^{(\mathcal{S})}) - \gamma_+(\widehat{F}) - \gamma_-(\widehat{F}), \quad (3.5)$$

(ii) in the odd case when $n = 2m + 1$,

$$\gamma'_-(F) = \pi(H_m^{(\mathcal{S})}) + \pi(H_{1, m-1}^{(\mathcal{S})}) + \gamma_+(\widehat{F}),$$

$$\gamma'_+(F) = \nu(H_m^{(\mathcal{S})}) + \nu(H_{1, m-1}^{(\mathcal{S})}) + \gamma_-(\widehat{F}),$$

$$\gamma'_0(F) = \delta(H_m^{(\mathcal{S})}) + \delta(H_{1, m-1}^{(\mathcal{S})}) - \gamma_+(\widehat{F}) - \gamma_-(\widehat{F}),$$

where $\widehat{F}(z)$ is a GRCD (resp. GLCD) of $F_e(-z^2)$ and $zF_o(-z^2)$, and $F_e(z)$ and $F_o(z)$ are the even and odd parts of $F(z)$, respectively.

Proof. We only prove the case of right Markov parameters: One can easily obtain the corresponding results for left Markov parameters with the help of Remarks 2.3 and A.4.

The proof for (i): Let

$$\widehat{F}_e(z) := F_e(-z^2),$$

$$\widehat{F}_o(z) := zF_o(-z^2),$$

$$L(z) := F(iz) = \widehat{F}_e(z) + i\widehat{F}_o(z),$$

$$L_1(z) := \widehat{F}_e(z) - i\widehat{F}_o(z).$$

It is clear that both $L(z)$ and $L_1(z)$ are regular. A combination of the fact that

$$\begin{bmatrix} L(z) \\ L_1(z) \end{bmatrix} = \begin{bmatrix} I_p & iI_p \\ I_p & -iI_p \end{bmatrix} \begin{bmatrix} \widehat{F}_e(z) \\ \widehat{F}_o(z) \end{bmatrix}$$

and Proposition A.5 shows that $\widehat{F}(z)$ is also a GRCD of $L(z)$ and $L_1(z)$.

In the even subcase $n = 2m$, the sequence \mathcal{S} consists of Hermitian matrices, and hence both $\mathbf{B}_{L_1^\vee, L_1}^\vee(L, L_1)$ and $\mathbf{B}_{\widehat{F}_e^\vee, -\widehat{F}_o^\vee}^\vee(\widehat{F}_e, -\widehat{F}_o)$ are well-defined. Moreover,

$$\mathbf{B}_{L_1^\vee, L_1}^\vee(L, L_1) = 2i\mathbf{B}_{\widehat{F}_e^\vee, -\widehat{F}_o^\vee}^\vee(\widehat{F}_e, -\widehat{F}_o). \quad (3.6)$$

As a result,

$$\gamma'_-(F) = \gamma_+(L) = \pi(-i\mathbf{B}_{L_1^\vee, L_1}^\vee(L, L_1)) + \gamma_+(\widehat{F}) = \pi(2\mathbf{B}_{\widehat{F}_e^\vee, -\widehat{F}_o}^\vee(\widehat{F}_e, -\widehat{F}_o)) + \gamma_+(\widehat{F}) \quad (3.7)$$

where the 2nd equality is due to Lemma 3.3 and the last is due to (3.6).

Assume that $\mathcal{S} := (\mathbf{s}_k)_{k=0}^\infty$. Lemma 3.4 shows that $\mathbf{B}_{\widehat{F}_e^\vee, -\widehat{F}_o}^\vee(\widehat{F}_e, -\widehat{F}_o)$ is congruent to the block Hankel matrix

$$\begin{bmatrix} \mathbf{s}_0 & 0_p & \mathbf{s}_1 & \cdots & \mathbf{s}_{m-1} & 0_p \\ 0_p & \mathbf{s}_1 & 0_p & \cdots & 0_p & \mathbf{s}_m \\ \mathbf{s}_1 & 0_p & \mathbf{s}_2 & \cdots & \mathbf{s}_m & 0_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{s}_{m-1} & 0_p & \mathbf{s}_m & \cdots & \mathbf{s}_{2m-2} & 0_p \\ 0_p & \mathbf{s}_m & 0_p & \cdots & 0_p & \mathbf{s}_{2m-1} \end{bmatrix}.$$

By reordering rows and columns, it is subsequently congruent to

$$\begin{bmatrix} H_{m-1}^{(\mathcal{S})} & \\ & H_{1, m-1}^{(\mathcal{S})} \end{bmatrix}.$$

Hence, (3.7) implies the equality

$$\gamma'_-(F) = \pi(H_{m-1}^{(\mathcal{S})}) + \pi(H_{1, m-1}^{(\mathcal{S})}) + \gamma_+(\widehat{F}),$$

which is (3.3). Analogously, we deduce (3.4) and (3.5), so Theorem 3.5 is verified for the even case (i).

The odd case (ii) follows in a similar way. \square

4. Main results

The following three subsections are devoted to the identification of Hurwitz matrix polynomials based on the inertia representation from Theorem 3.5.

4.1. Hurwitz stability and Stieltjes positive definiteness

Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be monic of degree n with the even part $F_e(z)$ and odd part $F_o(z)$. Assume that $\mathcal{S} := (\mathbf{s}_k)_{k=0}^\infty$ is the sequence of right or left Markov parameters of $F(z)$ (of the second type for $n = 2m + 1$). Then there exists a one-to-one correspondence between the coefficients of $F(z)$, \mathcal{S} and the truncated sequence $(\mathbf{s}_k)_{k=0}^{n-1}$. For simplicity, we only exhibit this correspondence for the case when $n = 2m$ and \mathcal{S} is the sequence of right Markov parameters. A comparison between (2.2) and (2.3) shows that

$$H_{j+k, m-1}^{(\mathcal{S})} = H_{j+k-1, m-1}^{(\mathcal{S})} C = \cdots = H_{j, m-1}^{(\mathcal{S})} C^k, \quad j, k \in \mathbb{N}_0, \quad (4.1)$$

and

$$\begin{bmatrix} A_{2m-1} \\ A_{2m-3} \\ \vdots \\ A_1 \end{bmatrix} = \begin{bmatrix} \mathbf{s}_0 & -\mathbf{s}_1 & \cdots & (-1)^{m-1} \mathbf{s}_{m-1} \\ & \mathbf{s}_0 & \cdots & (-1)^{m-2} \mathbf{s}_{m-2} \\ & & \ddots & \vdots \\ & & & \mathbf{s}_0 \end{bmatrix} \begin{bmatrix} A_{2m-2} \\ A_{2m-4} \\ \vdots \\ A_0 \end{bmatrix} \quad (4.2)$$

where $F(z)$ is written as in (2.1) and

$$C := \begin{bmatrix} 0_p & \cdots & 0_p & -A_{2m} \\ I_p & \cdots & 0_p & A_{2m-2} \\ \vdots & \ddots & \vdots & \vdots \\ 0_p & \cdots & I_p & (-1)^m A_2 \end{bmatrix} \quad (4.3)$$

is the so-called companion matrix of $F_e(-z)$. The correspondence is accordingly obtained.

We put special attention to the case that the truncated sequence of Markov parameters

$$(\mathbf{s}_k)_{k=0}^{n-1} \text{ is a sequence of Hermitian matrices.} \quad (4.4)$$

If so, \mathbf{s}_k is also Hermitian for $k = n, n + 1, \dots$. Indeed, without loss of generality we again consider the case of $n = 2m$ and \mathcal{S} being the sequence of right Markov parameters. As an immediate consequence of (4.1), we have

$$H_{2k+l, m-1}^{(\mathcal{S})} = (C^k)^* H_{l, m-1}^{(\mathcal{S})} C^k, \quad \text{for all } k, l \in \mathbb{N}_0. \quad (4.5)$$

The fact on \mathbf{s}_k follows since the left-hand side of (4.5) is Hermitian. So, these Hermitian matrices \mathbf{s}_k are a natural extension of classical real Markov parameters. It is important for us that the condition (4.4) allows to associate this sequence $(\mathbf{s}_k)_{k=0}^{n-1}$ with a truncated matricial Stieltjes moment problem.

Given a sequence of Hermitian matrices $(\mathbf{s}_k)_{k=0}^{n-1}$, the truncated matricial Stieltjes moment problem of first type (resp. second type) is to find all the nonnegative Hermitian $p \times p$ Borel measures τ on $[0, \infty)$ such that

$$\mathbf{s}_k = \int_0^\infty u^k d\tau(u), \quad k = 0, \dots, n-1 \quad \left(\text{resp. } \mathbf{s}_k = \int_0^\infty u^k d\tau(u), \quad k = 0, \dots, n-2, \quad \mathbf{s}_{n-1} - \int_0^\infty u^{n-1} d\tau(u) \geq 0 \right).$$

For the detailed study of these matrix moment problems, we refer the reader to [33–39]. Recall that the well-known solvability criteria for these problems (see e.g., Lemma 1.7 of [36], Lemma 1.2 of [37], Theorem 1.4 of [38], Theorem 1.1 of [39]) are related to some special matrix sequences as follows:

Definition 4.1. Let $l \in \mathbb{N}_0$ and let $(\mathbf{s}_k)_{k=0}^l$ be a finite sequence of $p \times p$ Hermitian matrices.

- (i) $(\mathbf{s}_k)_{k=0}^l$ is called *Stieltjes positive definite* (resp. *Stieltjes nonnegative definite*) if both block Hankel matrices $[\mathbf{s}_{j+k}]_{j,k=0}^{\lfloor \frac{l-1}{2} \rfloor}$ and $[\mathbf{s}_{j+k+1}]_{j,k=0}^{\lfloor \frac{l-1}{2} \rfloor}$ are positive definite (resp. nonnegative definite).
- (ii) $(\mathbf{s}_k)_{k=0}^l$ is called *Stieltjes nonnegative definite extendable* if there exists a complex $p \times p$ matrix \mathbf{s}_{l+1} such that $(\mathbf{s}_k)_{k=0}^{l+1}$ is Stieltjes nonnegative definite.

Theorem 4.2. Let $l \in \mathbb{N}_0$ and let $(\mathbf{s}_k)_{k=0}^l$ be a finite sequence of $p \times p$ Hermitian matrices. Then $(\mathbf{s}_k)_{k=0}^l$ corresponds to a solvable truncated matricial Stieltjes moment problem of first type if and only if $(\mathbf{s}_k)_{k=0}^l$ is Stieltjes nonnegative definite extendable.

Theorem 4.3. Let $l \in \mathbb{N}_0$ and let $(\mathbf{s}_k)_{k=0}^l$ be a finite sequence of $p \times p$ Hermitian matrices. Then $(\mathbf{s}_k)_{k=0}^l$ corresponds to a solvable truncated matricial Stieltjes moment problem of second type if and only if $(\mathbf{s}_k)_{k=0}^l$ is Stieltjes nonnegative definite.

Given a Stieltjes positive definite sequence $(\mathbf{s}_k)_{k=0}^l$, the related truncated matricial Stieltjes moment problem is solvable with an infinite number of solutions. The so-called Dyukarev matrix polynomials form the resolvent matrix for this problem via the Dyukarev–Stieltjes parameters of $(\mathbf{s}_k)_{k=0}^l$ (see [40, Theorem 7]). In what follows, we link Stieltjes positive definite sequences with Hurwitz stability of matrix polynomials.

Theorem 4.4. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be monic of degree n . Assume that \mathcal{S} is the related truncated Hermitian sequence of right Markov parameters (of second type when $n = 2m + 1$). Then $F(z)$ is a Hurwitz matrix polynomial if and only if \mathcal{S} is Stieltjes positive definite.

Proof. The proof for the “if” implication: Assume that \mathcal{S} is a Stieltjes positive definite matrix sequence. [Theorem 3.5](#) then yields that

$$np = \deg \det F(z) \geq \gamma'_-(F) \geq \pi(H_m^{(\mathcal{S})}) + \pi(H_{1,m-1}^{(\mathcal{S})}) = np$$

in the even case and

$$np = \deg \det F(z) \geq \gamma'_-(F) \geq \pi(H_m^{(\mathcal{S})}) + \pi(H_{1,m-1}^{(\mathcal{S})}) = np$$

in the odd case. Thus $\gamma'_-(F) = np$, which means that $F(z)$ is a Hurwitz matrix polynomial.

The proof for the “only if” implication: Let $F(z)$ be a Hurwitz matrix polynomial with the sequence of right Markov parameters \mathcal{S} . Then $\gamma'_-(F) = np$. Denote by $\widehat{F}(z)$ a GRCD of $F_e(-z^2)$ and $zF_o(-z^2)$, where $F_e(z)$ and $F_o(z)$ are the even part and odd part of $F(z)$, respectively. Suppose that $\sigma(\widehat{F}) \neq \emptyset$. One can see from [Proposition A.5](#) that $\widehat{F}(z)$ is a GRCD of

$$F(iz) = F_e(-z^2) + izF_o(-z^2) \quad \text{and} \quad F(-iz) = F_e(-z^2) + izF_o(-z^2).$$

Then for each zero $z_0 \in \sigma(\widehat{F})$, we have $iz_0 \in \sigma(F)$ and $-iz_0 \in \sigma(F)$, which contradicts the assumption that $F(z)$ is a Hurwitz matrix polynomial. Consequently, [Theorem 3.5](#) yields the expressions

$$np = \pi(H_m^{(\mathcal{S})}) + \pi(H_{1,m-1}^{(\mathcal{S})})$$

for the even case and

$$np = \pi(H_m^{(\mathcal{S})}) + \pi(H_{1,m-1}^{(\mathcal{S})})$$

for the odd case, from which the theorem follows. \square

Remark 4.5. As follows from [Remark 2.3](#), if the right Markov parameters in [Theorem 4.4](#) are substituted by the left Markov parameters, the corresponding results hold true as well.

We provide a few examples of testing the Hurwitz stability for matrix polynomials via [Theorem 4.4](#).

Example 4.6. Let $F(z) \in \mathbb{C}[z]^{2 \times 2}$ of degree 3 be given as

$$F(z) := \begin{bmatrix} z^3 + 3z^2 + (23 - 15i)z + 115 - 85i & 4z^2 + (33 + 35i)z + 170 + 165i \\ 4z^2 + (12 - 10i)z + 191 - 140i & z^3 + 8z^2 + (17 + 15i)z + 261 + 260i \end{bmatrix}.$$

The related right Markov parameters are

$$\mathbf{s}_0 = \begin{bmatrix} 3 & 4 \\ 4 & 8 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} 2 & -3 \\ -3 & 7 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 10 & 15 + 25i \\ 15 - 25i & 20 \end{bmatrix}.$$

They satisfy $\mathbf{s}_0 \succ 0$ and $\mathbf{s}_1 \succ 0$, however

$$\mathbf{s}_2 - \mathbf{s}_1 \mathbf{s}_0^{-1} \mathbf{s}_1 = \begin{bmatrix} -\frac{27}{8} & \frac{323}{8} + 25i \\ \frac{323}{8} - 25i & -\frac{227}{8} \end{bmatrix}$$

is not positive definite. So $F(z)$ cannot be a Hurwitz matrix polynomial.

Example 4.7. Let $F(z) \in \mathbb{C}[z]^{3 \times 3}$ of degree 3 be given as

$$F(z) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z^3 + \begin{bmatrix} 2 & -1-i & i \\ -1+i & 2 & -1 \\ -i & -1 & 2 \end{bmatrix} z^2 + \begin{bmatrix} 65+3i & -6 & 70i \\ -1+30i & 5-3i & -35-3i \\ 2-20i & 4i & 40 \end{bmatrix} z + \begin{bmatrix} 180-21i & -24-3i & 32+219i \\ -72+143i & 14-16i & -180-76i \\ 8-136i & -5+17i & 182+3i \end{bmatrix}.$$

The related right Markov parameters are

$$\mathbf{s}_0 = \begin{bmatrix} 2 & -1-i & i \\ -1+i & 2 & -1 \\ -i & -1 & 2 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} 1 & i & -i \\ -i & 2 & 0 \\ i & 0 & 3 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 15 & 1+5i & 3-5i \\ 1-5i & 10 & -6i \\ 3+5i & 6i & 50 \end{bmatrix}.$$

They satisfy $\mathbf{s}_0 \succ 0$, $\mathbf{s}_1 \succ 0$ and

$$\mathbf{s}_2 - \mathbf{s}_1 \mathbf{s}_0^{-1} \mathbf{s}_1 = \begin{bmatrix} 10 & -\frac{3}{2} - \frac{1}{2}i & -\frac{1}{2} + \frac{1}{2}i \\ -\frac{3}{2} + \frac{1}{2}i & \frac{1}{2} & 1 + \frac{1}{2}i \\ -\frac{1}{2} - \frac{1}{2}i & 1 - \frac{1}{2}i & \frac{73}{2} \end{bmatrix} \succ 0.$$

So $F(z)$ is a Hurwitz matrix polynomial.

Example 4.8. Let $F(z) \in \mathbb{C}[z]^{2 \times 2}$ of degree 4 be given as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^4 + \begin{bmatrix} 2 & 2-i \\ 2+i & 3 \end{bmatrix} z^3 + \begin{bmatrix} -58 & 5+39i \\ 9-71i & -67 \end{bmatrix} z^2 + \begin{bmatrix} -143+83i & -100+176i \\ -115-210i & -251-151i \end{bmatrix} z + \begin{bmatrix} 23+i & -2-17i \\ 1+39i & 20-5i \end{bmatrix}.$$

The related left Markov parameters are

$$\mathcal{S} = \left(\begin{bmatrix} 2 & 2-i \\ 2+i & 3 \end{bmatrix}, \begin{bmatrix} -2 & -1-i \\ -1+i & -3 \end{bmatrix}, \begin{bmatrix} 13 & 2+13i \\ 2-13i & 20 \end{bmatrix}, \begin{bmatrix} -210 & -i \\ i & -377 \end{bmatrix}, \dots \right).$$

It turns out that $H_1^{(\mathcal{S})}$ is positive definite, but $H_{1,1}^{(\mathcal{S})}$ is not. So $F(z)$ is not a Hurwitz matrix polynomial.

Given a matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$ as in [Theorem 4.4](#), observe that its infinite sequence of Hermitian right Markov parameters \mathcal{S} coincides with that of $z^{2k}F(z)$. If $F(z)$ is Hurwitz stable, then $\gamma'_+(z^{2k}F(z)) = 0$. [Theorem 3.5](#) for the matrix polynomial $z^{2k}F(z)$ in this case implies that $H_{m+k-1}^{(\mathcal{S})}, H_{1,m+k-1}^{(\mathcal{S})} \geq 0$ for all $k \in \mathbb{N}_0$. Consequently, we obtain

Corollary 4.9. Under the conditions of [Theorem 4.4](#), let $F(z)$ be a Hurwitz matrix polynomial with the related infinite sequence of right Markov parameters $(\mathbf{s}_k)_{k=0}^\infty$ (of second type when $\deg F$ is odd). Then $(\mathbf{s}_k)_{k=0}^l$ is Stieltjes positive definite (and hence extendable) for $0 < l < n$, and Stieltjes nonnegative definite extendable for $l \geq n$.

4.2. Matricial versions of the stability criteria

[Theorem 4.4](#) for scalar polynomials of even degrees coincides with the stability criterion via Markov parameters. Now, suppose that $F(z) \in \mathbb{C}[z]^{p \times p}$ is a monic matrix polynomial of odd degree $2m+1$ whose even part is regular, and whose sequence of right Markov parameters of first type is $(\mathbf{s}_k)_{k=-1}^\infty$. Analogous to the even case, we pay special attention to the case that the truncated sequence of Markov parameters

$$(\mathbf{s}_k)_{k=-1}^{2m-1} \text{ is a sequence of Hermitian matrices.} \quad (4.6)$$

We formulate the matrix analogue of the stability criterion via Markov parameters in a unified way covering both even and odd cases.

Theorem 4.10. Suppose that $F(z) \in \mathbb{C}[z]^{p \times p}$ is a monic matrix polynomial of degree $n = 2m$ or $n = 2m + 1$, and that $F_e(z)$ is regular of degree m for $n = 2m + 1$.¹ Let \mathcal{S} be the related truncated Hermitian sequence of right Markov parameters (of first type when $n = 2m + 1$). Then $F(z)$ is Hurwitz stable if and only if $H_{m-1}^{(\mathcal{S})}, H_{1,m-1}^{(\mathcal{S})} \succ 0$ and, for $n = 2m + 1$, additionally $\mathbf{s}_{-1} \succ 0$, where \mathbf{s}_{-1} is the first element of \mathcal{S} .

Proof. The even case $\deg F = 2m$ reduces to Theorem 4.4.

The odd case $\deg F = 2m + 1$: put $\mathcal{S} := (\mathbf{s}_k)_{k=-1}^\infty$, write the even and the odd parts of $F(z)$ as

$$F_e(z) := \sum_{k=0}^m z^{m-k} E_k \quad \text{and} \quad F_o(z) := \sum_{k=0}^m z^{m-k} O_k, \quad O_0 = I_p,$$

and put

$$F_{-1}(z) := (\mathbf{s}_0 + z\mathbf{s}_{-1}) F_e(z) - zF_o(z), \quad F_{0,z}(z) := zF_o(z).$$

Since $zF_o(z) = z\mathbf{s}_{-1}F_e(z) + \mathbf{s}_0F_e(z) + O(z^{m-1})$, we have $\deg F_{-1} = m - 1$. If $\tilde{\mathcal{S}}$ is the sequence of right Markov parameters of second type associated with $F(z)$, then

$$\begin{aligned} & \begin{bmatrix} I_p & \cdots & z^m I_p \end{bmatrix} \cdot \begin{bmatrix} E_m^* & \cdots & E_0^* \\ \vdots & \ddots & \\ E_0^* & & \end{bmatrix} \cdot \begin{bmatrix} \mathbf{s}_{-1} & & \\ & H_{1,m-1}^{(\mathcal{S})} & \\ & & \end{bmatrix} \cdot \begin{bmatrix} E_m & \cdots & E_0 \\ \vdots & \ddots & \\ E_0 & & \end{bmatrix} \cdot \begin{bmatrix} I_p \\ \vdots \\ u^m I_p \end{bmatrix} \\ &= \begin{bmatrix} I_p & \cdots & z^{m-1} I_p \end{bmatrix} \mathbf{B}_{F_{-1}^\vee, F_e^\vee} (F_{-1}, F_e) \begin{bmatrix} I_p \\ \vdots \\ u^{m-1} I_p \end{bmatrix} + F_e^\vee(z) \mathbf{s}_{-1} F_e(u) \\ &= \frac{1}{z-u} (F_e^\vee(z) F_{-1}(u) - F_{-1}^\vee(z) F_e(u)) + F_e^\vee(z) \mathbf{s}_{-1} F_e(u) \\ &= \frac{1}{z-u} (z F_o^\vee(z) F_e(u) - F_e^\vee(z) u F_o(u)) \\ &= \begin{bmatrix} I_p & \cdots & z^m I_p \end{bmatrix} \mathbf{B}_{F_e^\vee, F_{0,z}^\vee} (F_e, F_{0,z}) \begin{bmatrix} I_p \\ \vdots \\ u^m I_p \end{bmatrix} \\ &= \begin{bmatrix} I_p & \cdots & z^m I_p \end{bmatrix} \cdot \begin{bmatrix} O_m^* & \cdots & O_0^* \\ \vdots & \ddots & \\ O_0^* & & \end{bmatrix} H_m^{(\tilde{\mathcal{S}})} \begin{bmatrix} O_m & \cdots & O_0 \\ \vdots & \ddots & \\ O_0 & & \end{bmatrix} \cdot \begin{bmatrix} I_p \\ \vdots \\ u^m I_p \end{bmatrix}, \end{aligned}$$

where the 1-st and the last equations are due to Lemma 3.4. It follows that $H_m^{(\tilde{\mathcal{S}})}$ is Hermitian if and only if both $H_{1,m-1}^{(\mathcal{S})}$ and \mathbf{s}_{-1} are Hermitian. Moreover, $H_m^{(\tilde{\mathcal{S}})} \succ 0$ is equivalent to $H_{1,m-1}^{(\mathcal{S})} \succ 0$, $\mathbf{s}_{-1} \succ 0$.

Analogously, one can prove that $H_{m-1}^{(\mathcal{S})}$ is Hermitian (resp. positive definite) if and only if $H_{1,m-1}^{(\tilde{\mathcal{S}})}$ is Hermitian (resp. positive definite). In view of Theorem 4.4, we complete the proof. \square

Remark 4.11. In Theorem 4.10, the right Markov parameters may be substituted by the left Markov parameters; the corresponding result holds true, as is seen from Remark 2.3.

The proof of Theorem 4.10 shows that, given a stable monic matrix polynomial $F(z)$ of odd degree, its truncated sequence of right Markov parameters of second type satisfies the condition (4.4) with $\mathbf{s}_0 \neq 0_p$ if and only if the even part of $F(z)$ is regular and of the same degree as the odd part (so that $F(z)$ has well-defined right Markov parameters of first type) and (4.6) holds. In other words, a matrix polynomial whose stability can be tested via Theorem 4.4, but not via Theorem 4.10, cannot be Hurwitz stable.

Theorem 4.4 indeed provides a situation where the answer to question (Q1) from Section 2 is positive and may be written in a simple form. Based on Theorem 4.4 and [13, Theorem 7.10], we can now deduce the following connection between Hurwitz matrix polynomials and matricial Stieltjes continued fractions and thereby complete our answer to question (Q3):

Theorem 4.12. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be monic of degree n with the even part $F_e(z)$ and odd part $F_o(z)$. Assume that \mathcal{S} is the related truncated Hermitian sequence of right Markov parameters (of second type when $n = 2m + 1$). Then

¹ As is seen from Theorem 4.4, the polynomial $F(z)$ is necessarily unstable in the case $\deg F_e < m$.

- (i) in the even case $n = 2m$, $F(z)$ is a Hurwitz matrix polynomial if and only if there exists a sequence of $p \times p$ positive definite matrices $(\mathbf{c}_k)_{k=1}^n$ such that the identity

$$\frac{F_o(z)}{F_e(z)} = \frac{I_p}{z\mathbf{c}_1 + \frac{I_p}{\mathbf{c}_2 + \dots + \frac{I_p}{\mathbf{c}_{n-2} + \frac{I_p}{z\mathbf{c}_{n-1} + \mathbf{c}_n^{-1}}}}} \quad (4.7)$$

holds for large enough $z \in \mathbb{C}$.

- (ii) in the odd case $n = 2m + 1$, the following statements are equivalent:

- (a) $F(z)$ is a Hurwitz matrix polynomial.
 (b) There exists a sequence of $p \times p$ positive definite matrices $(\mathbf{c}_k)_{k=1}^n$ such that the identity

$$\frac{F_e(z)}{F_o(z)} = \frac{I_p}{\mathbf{c}_1 + \frac{I_p}{z\mathbf{c}_2 + \dots + \frac{I_p}{\mathbf{c}_{n-2} + \frac{I_p}{z\mathbf{c}_{n-1} + \mathbf{c}_n^{-1}}}}} \quad (4.8)$$

holds for large enough $z \in \mathbb{C}$.

- (c) $F_e(z)$ is regular and there exists a sequence of $p \times p$ positive definite matrices $(\mathbf{c}_k)_{k=1}^n$ such that the identity

$$\frac{F_o(z)}{F_e(z)} = \mathbf{c}_1 + \frac{I_p}{z\mathbf{c}_2 + \dots + \frac{I_p}{\mathbf{c}_{n-2} + \frac{I_p}{z\mathbf{c}_{n-1} + \mathbf{c}_n^{-1}}}}$$

holds for large enough $z \in \mathbb{C}$.

In [13], matrix polynomials satisfying (4.7) or (4.8) are called matrix Hurwitz type polynomials. In this sense, the notions “matrix Hurwitz type polynomial” and “Hurwitz matrix polynomial” are equivalent. In the scalar case when $p = 1$, the statement (i) and the equivalence between (a) and (c) of (ii) are indeed the classical stability criterion via continued fractions.

4.3. Hurwitz matrix polynomials, block-Hankel minors and quasiminors

The rest of this paper is devoted to block-Hankel minors and block-Hankel quasiminors in connection with Hurwitz matrix polynomials.

We begin with an introduction of quasideterminants, which play an important role in noncommutative algebra as determinants do in commutative algebra. The most general definitions [41, (1.1), P. 92] and [42, Definition 1.2.5] become simpler in the particular case we need — for quasideterminants of block matrices over \mathbb{C} .

Definition 4.13. Let $l \in \mathbb{N}$ and $l \leq 2$ and let $\mathbf{M} \in \mathbb{C}^{pl \times pl}$ with a block decomposition $\mathbf{M} := [M_{jk}]_{j,k=1}^l$, where $M_{jk} \in \mathbb{C}^{p \times p}$. Suppose that $\mathbf{M}_{(l;l)} := [M_{jk}]_{j,k=1}^{l-1} \in \mathbb{C}^{(l-1)p \times (l-1)p}$ is nonsingular. The quasideterminant of \mathbf{M} with index (l, l) , denoted by $|\mathbf{M}|_{ll}$, is the following expression

$$|\mathbf{M}|_{ll} := M_{ll} - [M_{l1} \quad \dots \quad M_{l,l-1}] \mathbf{M}_{(l;l)}^{-1} \begin{bmatrix} M_{1l} \\ \vdots \\ M_{l-1,l} \end{bmatrix}.$$

In fact, $|\mathbf{M}|_{ll}$ in our setting coincides with the Schur complement of $\mathbf{M}_{(l;l)}$ in \mathbf{M} .

Definition 4.14. Let

$$H := [\mathbf{s}_{j+k}]_{j,k=0}^{\infty} \quad (4.9)$$

be an infinite block Hankel matrix with $\mathbf{s}_k \in \mathbb{C}^{p \times p}$. For $l \in \mathbb{N}$ and $l \geq 2$, let

$$\tilde{H}_l := \begin{bmatrix} \mathbf{s}_{j_1+k_1} & \mathbf{s}_{j_1+k_2} & \dots & \mathbf{s}_{j_1+k_l} \\ \mathbf{s}_{j_2+k_1} & \mathbf{s}_{j_2+k_2} & \dots & \mathbf{s}_{j_2+k_l} \\ \vdots & \vdots & & \vdots \\ \mathbf{s}_{j_l+k_1} & \mathbf{s}_{j_l+k_2} & \dots & \mathbf{s}_{j_l+k_l} \end{bmatrix} \in \mathbb{C}^{pl \times pl} \quad (4.10)$$

be a submatrix of H , where $0 \leq k_1 < k_2 < \dots < k_l$ and $0 \leq j_1 < j_2 < \dots < j_l$. Then

- (i) $\det \tilde{H}_l$ is called a block-Hankel minor of order l of H ;

(ii) if $|\tilde{H}_l|_l$ is well-defined, $|\tilde{H}_l|_l$ is called a block-Hankel quasiminor of order l of H .

Moreover, if $j_l - j_{l-1} = \dots = j_2 - j_1 = 1$ and $k_l - k_{l-1} = \dots = k_2 - k_1 = 1$, then

- (i) $\det \tilde{H}_l$ is called a contiguous block-Hankel minor of order l of H ;
- (ii) if $|\tilde{H}_l|_l$ is well-defined, $|\tilde{H}_l|_l$ is called a contiguous block-Hankel quasiminor of order l of H ;
- (iii) block-Hankel minors and quasiminors that are not contiguous we call non-contiguous.

For a given matrix polynomial, let us show that large enough block-Hankel minors built from matricial Markov parameters vanish.

Proposition 4.15. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be monic of degree $n = 2m$ or $n = 2m + 1$. Assume that \mathcal{S} is the related sequence of right Markov parameters (of first type for $n = 2m + 1$ if the even part $F_e(z)$ is regular). Then all block-Hankel minors of $H_\infty^{(\mathcal{S})}$ of order $> m$ are equal to zero.

Proof. We only give a proof for $n = 2m$. Let \tilde{H}_l be any given $lp \times lp$ submatrix of H as in (4.10). Comparing both sides of (2.3) yields that

$$\begin{bmatrix} \mathbf{s}_{t+k} & \cdots & \mathbf{s}_{t+k+m-1} \\ \mathbf{s}_{t+k+1} & \cdots & \mathbf{s}_{t+k+m} \\ \vdots & & \vdots \\ \mathbf{s}_{t+k+j_l} & \cdots & \mathbf{s}_{t+k+j_l+m-1} \end{bmatrix} = \begin{bmatrix} \mathbf{s}_k & \cdots & \mathbf{s}_{k+m-1} \\ \mathbf{s}_{k+1} & \cdots & \mathbf{s}_{k+m} \\ \vdots & & \vdots \\ \mathbf{s}_{k+j_l} & \cdots & \mathbf{s}_{k+j_l+m-1} \end{bmatrix} C^t, \quad (4.11)$$

where C is as in (4.3) and $t \in \mathbb{N}_0$. Using (4.11), we have

$$\begin{bmatrix} \mathbf{s}_{k_1} & \cdots & \mathbf{s}_{k_l} \\ \mathbf{s}_{k_1+1} & \cdots & \mathbf{s}_{k_l+1} \\ \vdots & & \vdots \\ \mathbf{s}_{k_1+j_l} & \cdots & \mathbf{s}_{k_l+j_l} \end{bmatrix} = \begin{bmatrix} \mathbf{s}_0 & \cdots & \mathbf{s}_{m-1} \\ \mathbf{s}_1 & \cdots & \mathbf{s}_m \\ \vdots & & \vdots \\ \mathbf{s}_{j_l} & \cdots & \mathbf{s}_{j_l+m-1} \end{bmatrix} \cdot [C_1^{k_1} \quad C_1^{k_2} \quad \cdots \quad C_1^{k_l}],$$

where $C_1^{k_j} := C^{k_j} \cdot [I_p \quad 0_p \quad \cdots \quad 0_p]^T$ is the first column of C^{k_j} for $j = 1, \dots, l$. Then it follows that

$$\text{rank } \tilde{H}_l \leq \text{rank} \begin{bmatrix} \mathbf{s}_{k_1} & \cdots & \mathbf{s}_{k_l} \\ \mathbf{s}_{k_1+1} & \cdots & \mathbf{s}_{k_l+1} \\ \vdots & & \vdots \\ \mathbf{s}_{k_1+j_l} & \cdots & \mathbf{s}_{k_l+j_l} \end{bmatrix} \leq \text{rank} \begin{bmatrix} \mathbf{s}_0 & \cdots & \mathbf{s}_{m-1} \\ \mathbf{s}_1 & \cdots & \mathbf{s}_m \\ \vdots & & \vdots \\ \mathbf{s}_{j_l} & \cdots & \mathbf{s}_{j_l+m-1} \end{bmatrix} = mp,$$

and therefore $\det \tilde{H}_l = 0$. \square

The following theorem describes Hurwitz matrix polynomials via the block-Hankel minors and quasiminors built from right Markov parameters.

Theorem 4.16. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be monic of degree $n = 2m$ or $n = 2m + 1$. Assume that \mathcal{S} is the related Hermitian matrix sequence of right Markov parameters (of first type for $n = 2m + 1$ when the even part $F_e(z)$ is regular of degree m). Then $F(z)$ is a monic Hurwitz matrix polynomial if and only if the following statements are simultaneously true:

- (i) All contiguous block-Hankel quasiminors of $H_\infty^{(\mathcal{S})}$ of order $\leq m$ are positive definite.
- (ii) All block-Hankel minors of $H_\infty^{(\mathcal{S})}$ of order $> m$ are 0.
- (iii) If $n = 2m + 1$, the first element of \mathcal{S} satisfies $\mathbf{s}_{-1} \succ 0$.

Proof. We only give a proof for the even case $n = 2m$. The “if” implication is obvious due to Theorem 4.10.

The “only if” implication: If $F(z)$ is Hurwitz stable, (ii) and (iii) are immediate consequences of Proposition 4.15 and Theorem 4.10, respectively. Moreover, Theorem 4.10 implies $H_{m-1}^{(\mathcal{S})}, H_{1,m-1}^{(\mathcal{S})} \succ 0$, and hence $H_{k,m-1}^{(\mathcal{S})} \succ 0$ for all $k \in \mathbb{N}_0$ by (4.5). So, the inertia additivity formula for the Schur complement yields (i). \square

Remark 4.17. On account of Remark 2.3, the right Markov parameters in Proposition 4.15 and Theorem 4.16 may be replaced by the left Markov parameters; the corresponding results also hold true.

It follows from Theorem 4.16 that all contiguous block-Hankel minors of order $\leq m$ are real and positive. This motivated us to seek the block-Hankel total positivity of $H_\infty^{(\mathcal{S})}$ as simultaneous satisfaction of the following:

- (a) the conditions (i)–(ii) of Theorem 4.16 hold, and
- (b) all non-contiguous block-Hankel minors of order $\leq m$ are positive real.

However, this new property does not help in constructing a direct matrix generalization of [12, Theorem 20, Chapter XV]. More specifically, the answer to the following question:

Is Hurwitz stability of $F(z)$ equivalent to the condition (iii) of Theorem 4.16 and the block-Hankel total positivity of $H_\infty^{(\mathcal{S})}$? is unfortunately negative, as is seen from the following counterexamples.

Consider the real monic matrix polynomial

$$F(z) = \begin{bmatrix} z^4 + 3z^3 + 19z^2 + 19z + 60 & -\frac{5z^3}{2} - 14z^2 - 19z - 56 \\ -\frac{5z^3}{2} + 12z^2 + 124z + 24 & z^4 + \frac{57z^3}{4} - 9z^2 - \frac{221z}{2} - 22 \end{bmatrix}$$

of degree 4, whose right Markov parameters

$$\mathbf{s}_0 = \begin{bmatrix} 3 & -\frac{5}{2} \\ -\frac{5}{2} & \frac{57}{4} \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} 8 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{69}{4} \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 26 & \frac{11}{2} \\ \frac{11}{2} & \frac{101}{4} \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 92 & \frac{47}{2} \\ \frac{47}{2} & \frac{189}{4} \end{bmatrix}, \quad \mathbf{s}_4 = \begin{bmatrix} 338 & \frac{155}{2} \\ \frac{155}{2} & \frac{437}{4} \end{bmatrix}$$

are Hermitian (here real symmetric). Both block Hankel matrices $[\mathbf{s}_{i+j}]_{i,j=0}^1$ and $[\mathbf{s}_{i+j+1}]_{i,j=0}^1$ are positive definite, so $F(z)$ is a Hurwitz matrix polynomial. Nonetheless,

$$\begin{vmatrix} \mathbf{s}_0 & \mathbf{s}_2 \\ \mathbf{s}_1 & \mathbf{s}_3 \end{vmatrix} = -\frac{3}{4}, \quad \begin{vmatrix} \mathbf{s}_0 & \mathbf{s}_3 \\ \mathbf{s}_1 & \mathbf{s}_4 \end{vmatrix} = -\frac{581}{4}, \quad \begin{vmatrix} \mathbf{s}_1 & \mathbf{s}_3 \\ \mathbf{s}_2 & \mathbf{s}_4 \end{vmatrix} = -18.$$

Claim 4.18. Let $F(z) \in \mathbb{C}[z]^{p \times p}$ be a monic Hurwitz matrix polynomial of degree $n = 2m$ or $n = 2m + 1$. Assume that \mathcal{S} is the related Hermitian matrix sequence of left or right Markov parameters (of first type for $n = 2m + 1$ if the even part of $F(z)$ is regular of degree m). Then non-contiguous block-Hankel minors of $H_\infty^{(\mathcal{S})}$ of order $\leq m$ may have non-positive real values.

Now, let $F(z) \in \mathbb{C}[z]^{2 \times 2}$ be the following monic matrix polynomial of degree 6:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^6 + \begin{bmatrix} 8 & 3i \\ -3i & 9 \end{bmatrix} z^5 + \frac{17 - 32i}{1313} \left(\begin{bmatrix} 150 + 300i & 12i \\ 64 + 40i & 190 + 340i \end{bmatrix} z^4 + \begin{bmatrix} 587 + 1664i & -1071 + 570i \\ 1425 - 186i & 1372 + 2356i \end{bmatrix} z^3 \right. \\ & \quad \left. + \begin{bmatrix} 331 + 676i & 36i \\ 412 + 220i & 551 + 956i \end{bmatrix} z^2 + \begin{bmatrix} -89 + 2464i & -2313 + 837i \\ 3467 + 287i & 2587 + 4288i \end{bmatrix} z + \begin{bmatrix} 198 + 408i & 24i \\ 348 + 180i & 378 + 648i \end{bmatrix} \right) \end{aligned}$$

with Hermitian right Markov parameters

$$\mathbf{s}_0 = \begin{bmatrix} 8 & 3i \\ -3i & 9 \end{bmatrix}, \quad \mathbf{s}_1 = \begin{bmatrix} 29 & 3 \\ 3 & 22 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 145 & 21 - 24i \\ 21 + 24i & 100 \end{bmatrix}, \quad \mathbf{s}_3 = \begin{bmatrix} 839 & 153 - 186i \\ 153 + 186i & 592 \end{bmatrix},$$

$$\mathbf{s}_4 = \begin{bmatrix} 5173 & 1185 - 1212i \\ 1185 + 1212i & 3784 \end{bmatrix}, \quad \mathbf{s}_5 = \begin{bmatrix} 32879 & 9273 - 7530i \\ 9273 + 7530i & 24832 \end{bmatrix}.$$

Since both block Hankel matrices $[\mathbf{s}_{i+j}]_{i,j=0}^2$ and $[\mathbf{s}_{i+j+1}]_{i,j=0}^2$ are positive definite, $F(z)$ is a Hurwitz matrix polynomial. However, some non-contiguous block-Hankel minors of $[\mathbf{s}_{i+j}]_{i,j=0}^\infty$ of order $\leq \deg F = 3$ have complex values:

$$\begin{aligned} \begin{vmatrix} \mathbf{s}_0 & \mathbf{s}_2 \\ \mathbf{s}_1 & \mathbf{s}_3 \end{vmatrix} &= 3323095 - 24840i, & \begin{vmatrix} \mathbf{s}_0 & \mathbf{s}_3 \\ \mathbf{s}_1 & \mathbf{s}_4 \end{vmatrix} &= 152111099 - 2414520i, \\ \begin{vmatrix} \mathbf{s}_1 & \mathbf{s}_3 \\ \mathbf{s}_2 & \mathbf{s}_4 \end{vmatrix} &= 327769380 - 2969280i, & \begin{vmatrix} \mathbf{s}_0 & \mathbf{s}_1 & \mathbf{s}_2 \\ \mathbf{s}_2 & \mathbf{s}_3 & \mathbf{s}_4 \\ \mathbf{s}_3 & \mathbf{s}_4 & \mathbf{s}_5 \end{vmatrix} &= (5859396 - 3456i) \cdot 10^3. \end{aligned}$$

Claim 4.19. Under the conditions of Claim 4.18, non-contiguous block-Hankel minors of $H_\infty^{(\mathcal{S})}$ of order $\leq m$ may be not real.

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Appendix. Greatest common divisors of matrix polynomials

Definition A.1. A matrix polynomial $F(z) \in \mathbb{C}[z]^{p \times p}$ is called *unimodular* if $\det F(z)$ never vanishes in \mathbb{C} .

A GRCD/GLCD of two matrix polynomials is only unique up to multiplication by a unimodular matrix polynomial.

Proposition A.2 ([43, pp. 377–378]). Let $F(z), \tilde{F}(z) \in \mathbb{C}[z]^{p \times p}$ and let $F_1(z)$ be a GLCD (resp. GRCD) of $F(z)$ and $\tilde{F}(z)$. Then $F_2(z) \in \mathbb{C}[z]^{p \times p}$ is a GLCD (resp. GRCD) of $F(z)$ and $F(z)$ if and only if there exists a unimodular matrix polynomial $W(z) \in \mathbb{C}[z]^{p \times p}$ such that

$$F_1(z) = F_2(z)W(z) \quad (\text{resp. } F_1(z) = W(z)F_2(z)).$$

Lemma 6.3-3 of [43] provides a sufficient way to obtain a GRCD/GLCD. We give a completion by clarifying the necessity.

Proposition A.3. Let $F(z), F_1(z)$ and $\tilde{F}(z) \in \mathbb{C}[z]^{p \times p}$.

- (i) $F_1(z)$ is a GRCD of $F(z)$ and $\tilde{F}(z)$ if and only if there exist a unimodular matrix polynomial $U_R(z) \in \mathbb{C}[z]^{2p \times 2p}$ and a matrix polynomial $F_1(z) \in \mathbb{C}[z]^{p \times p}$ such that

$$U_R(z) \cdot \begin{bmatrix} F(z) \\ \tilde{F}(z) \end{bmatrix} = \begin{bmatrix} F_1(z) \\ 0_p \end{bmatrix}. \quad (\text{A.1})$$

- (ii) $F_1(z)$ is a GLCD of $F(z)$ and $\tilde{F}(z)$ if and only if there exists a unimodular matrix polynomial $U_L(z) \in \mathbb{C}[z]^{2p \times 2p}$ such that

$$\begin{bmatrix} F(z) & \tilde{F}(z) \end{bmatrix} \cdot U_L(z) = \begin{bmatrix} F_1(z) & 0_p \end{bmatrix}.$$

Proof. The “if” implication follows from [43, Lemma 6.3-3].

To obtain the “only if” implication of (i), transform $\begin{bmatrix} F(z) \\ \tilde{F}(z) \end{bmatrix}$ into the Hermitian form (see [43, Theorem 6.3-2]): in particular, there exist a unimodular matrix polynomial $\tilde{U}_R(z) \in \mathbb{C}[z]^{2p \times 2p}$ and some $F_2(z) \in \mathbb{C}[z]^{p \times p}$ such that

$$\tilde{U}_R(z) \cdot \begin{bmatrix} F(z) \\ \tilde{F}(z) \end{bmatrix} = \begin{bmatrix} F_2(z) \\ 0_p \end{bmatrix}. \quad (\text{A.2})$$

Then due to the “if” implication, $F_2(z)$ is a GRCD of $F(z)$ and $\tilde{F}(z)$. Suppose that $F_1(z)$ is a GRCD of $F(z)$ and $\tilde{F}(z)$. As is seen from Proposition A.2, there exists a unimodular $W(z) \in \mathbb{C}[z]^{p \times p}$ such that $W(z)F_2(z) = F_1(z)$. Substituting

$$U_R(z) := \text{diag}(W(z), I_p)\tilde{U}_R(z)$$

into (A.2) then gives (A.1).

The validity of (ii) follows analogously to the proof of (i). \square

Remark A.4. Suppose that $F(z), F_1(z)$ and $\tilde{F}(z) \in \mathbb{C}[z]^{p \times p}$. Then, by Proposition A.3, $F_1(z)$ is a GRCD of $F(z)$ and $\tilde{F}(z)$ if and only if $F_1^\vee(z)$ is a GLCD of $F^\vee(z)$ and $\tilde{F}^\vee(z)$.

The next proposition shows a relation between GRCDs or GLCDs of matrix polynomials and that of their transformations.

Proposition A.5. Let $F(z), \tilde{F}(z)$ and $F_1(z) \in \mathbb{C}[z]^{p \times p}$ and let $U(z) \in \mathbb{C}[z]^{2p \times 2p}$ be unimodular. Define $E(z), \tilde{E}(z) \in \mathbb{C}[z]^{p \times p}$ by

$$\begin{bmatrix} E(z) \\ \tilde{E}(z) \end{bmatrix} := U(z) \cdot \begin{bmatrix} F(z) \\ \tilde{F}(z) \end{bmatrix} \quad (\text{resp. } \begin{bmatrix} E(z) & \tilde{E}(z) \end{bmatrix} := \begin{bmatrix} F(z) & \tilde{F}(z) \end{bmatrix} \cdot U(z)).$$

Then $F_1(z)$ is a GRCD (resp. GLCD) of $F(z)$ and $\tilde{F}(z)$ if and only if $F_1(z)$ is a GRCD (resp. GLCD) of $E(z)$ and $\tilde{E}(z)$.

Proof. We only give a proof in the case for GRCD, which can be converted to the case for GLCD due to Remark A.4.

Suppose that $F_1(z)$ is a GRCD of $F(z)$ and $\tilde{F}(z)$. By Proposition A.3, there exists a unimodular matrix polynomial $U_R(z) \in \mathbb{C}[z]^{2p \times 2p}$ such that (A.1) holds. Let, for $z \in \mathbb{C}$,

$$\tilde{U}(z) := U_R(z)(U(z))^{-1},$$

which is unimodular. It follows from (A.1) that

$$\tilde{U}(z) \cdot \begin{bmatrix} E(z) \\ \tilde{E}(z) \end{bmatrix} = \begin{bmatrix} F_1(z) \\ 0_p \end{bmatrix}. \quad (\text{A.3})$$

So, Proposition A.3 yields that F_1 is a GRCD of E and \tilde{E} .

Conversely, suppose that F_1 is a GRCD of E and \tilde{E} . By Proposition A.3, there exists a unimodular $\tilde{U}(z) \in \mathbb{C}[z]^{2p \times 2p}$ such that (A.3) holds. Then the matrix polynomial $U_R(z) := \tilde{U}(z)U(z)$ is unimodular and satisfies

$$U_R(z) \cdot \begin{bmatrix} F(z) \\ \tilde{F}(z) \end{bmatrix} = \tilde{U}(z) \cdot \begin{bmatrix} E(z) \\ \tilde{E}(z) \end{bmatrix} = \begin{bmatrix} F_1(z) \\ 0_p \end{bmatrix},$$

which implies that $F_1(z)$ is a GRCD of $F(z)$ and $\tilde{F}(z)$. \square

References

- [1] D. Henrion, D. Arzelier, D. Peaucelle, Positive polynomial matrices and improved LMI robustness conditions, *Automatica* 39 (2003) 1479–1485.
- [2] D. Henrion, D. Arzelier, D. Peaucelle, M. Sebek, An LMI condition for robust stability of polynomial matrix polytopes, *Automatica* 37 (2001) 461–468.
- [3] D.H. Lee, J.B. Park, Y.H. Joo, A less conservative LMI condition for robust \mathcal{D} -stability of polynomial matrix polytopes— a projection approach, *IEEE Trans. Automat. Control* 56 (2011) 868–873.
- [4] D.H. Lee, J.B. Park, Y.H. Joo, K.C. Lin, Lifted versions of robust \mathcal{D} -stability and \mathcal{D} -stabilisation conditions for uncertain polytopic linear systems, *IET Control Theory Appl.* 6 (2012) 24–36.
- [5] L. Lerer, L. Rodman, M. Tismenetsky, Inertia theorem for matrix polynomials, *Linear Multilinear Algebra* 30 (1991) 157–182.
- [6] L. Lerer, M. Tismenetsky, The Bezoutian and the eigenvalue-separation problem for matrix polynomials, *Integral Equations Oper. Theory* 5 (1982) 387–444.
- [7] R.R. Bitmead, B.D.O. Anderson, The matrix Cauchy index: properties and applications, *SIAM J. Appl. Math.* 33 (1977) 655–672.
- [8] B.D.O. Anderson, R.R. Bitmead, Stability of matrix polynomials, *Internat. J. Control* 26 (1977) 235–247.
- [9] F.J. Kraus, M. Mansour, M. Sebek, Hurwitz Matrix for Polynomial Matrices, in: *Int. Series of Numerical Mathematics*, vol. 121, Birkhäuser Verlag, 1996, pp. 67–74.
- [10] R. Galindo, Stabilisation of matrix polynomials, *Internat. J. Control* 88 (2015) 1925–1932.
- [11] G. Hu, X. Hu, Stability criteria of matrix polynomials, *Internat. J. Control* 92 (12) (2019) 2973–2978.
- [12] F.R. Gantmacher, *The Theory of Matrices*, Vols. 1, 2, Chelsea Publishing Co., New York, 1959, translated by K.A. Hirsch.
- [13] A.E. Choque Rivero, On matrix Hurwitz type polynomials and their interrelations to Stieltjes positive definite sequences and orthogonal matrix polynomials, *Linear Algebra Appl.* 476 (2015) 56–84.
- [14] K. Green, T. Wagenknecht, Pseudospectra and delay differential equations, *J. Comput. Appl. Math.* 196 (2) (2006) 567–578.
- [15] E. Fridman, Tutorial on Lyapunov-based methods for time-delay systems, *Eur. J. Control* 20 (2014) 271–283.
- [16] C. Huang, Y. Qiao, L. Huang, R. Agarwal, Dynamical behaviors of a food-chain model with stage structure and time delays, *Adv. Differential Equations* 2018 (2018).
- [17] X. Long, S. Gong, New results on stability of Nicholson's blowflies equation with multiple pairs of time-varying delays, *Appl. Math. Lett.* 100 (2020) 106027.
- [18] J. Wang, C. Huang, L. Huang, Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type, *Nonlinear Anal. Hybrid Syst.* 33 (2019) 162–178.
- [19] J. Wang, X. Chen, L. Huang, The number and stability of limit cycles for planar piecewise linear systems of node-saddle type, *J. Math. Anal. Appl.* 469 (2019) 405–427.
- [20] X. Hu, Y. Cong, G. Hu, Delay-dependent stability of Runge–Kutta methods for linear delay differential–algebraic equations, *J. Comput. Appl. Math.* 363 (2020) 300–311.
- [21] P. Lancaster, M. Tismenetsky, *The Theory of Matrices with Applications*, second ed., Academic Press, New York, 1985.
- [22] D. Matignon, Stability results for fractional differential equations with applications to control processing, in: *Computational Engineering in Systems Applications*, Vol. 2, Lille, France, 1996, pp. 963–968.
- [23] C. Song, S. Fei, J. Cao, C. Huang, Robust synchronization of fractional-order uncertain chaotic systems based on output feedback sliding mode control, *Mathematics* 7 (2019) 599.
- [24] I. Gohberg, P. Lancaster, L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.
- [25] O.Y. Kushel, Unifying matrix stability concepts with a view to applications, *SIAM Rev.* 61 (4) (2019) 643–729.
- [26] S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Vector spaces of linearizations for matrix polynomials, *SIAM J. Matrix Anal. Appl.* 28 (4) (2006) 971–1004.
- [27] F. De Terán, F.M. Dopico, D.S. Mackey, Spectral equivalence of matrix polynomials and the Index Sum Theorem, *Linear Algebra Appl.* 459 (2014) 264–333.
- [28] F.M. Dopico, J. Pérez, P. Van Dooren, Block minimal bases ℓ -ifications of matrix polynomials, *Linear Algebra Appl.* 562 (2019) 163–204.
- [29] M. Fujiwara, Über die algebraischen Gleichungen, deren Wurzeln in Einem Kreise oder in einer Halbebene liegen, *Math. Z.* 24 (1926) 161–169 (in German).
- [30] H. Dym, D. Volok, Zero distribution of matrix polynomials, *Linear Algebra Appl.* 425 (2007) 714–738.
- [31] I. Gohberg, M.A. Kaashoek, L. Lerer, L. Rodman, Common multiples and common divisors of matrix polynomials, II. Vandermonde and resultant matrices, *Linear Multilinear Algebra* 12 (1982) 159–203.
- [32] B.D.O. Anderson, E. Jury, Generalized Bezoutian and Sylvester matrices in multivariable linear control, *IEEE Trans. Autom. Control* 21 (1976) 551–556.
- [33] V.M. Adamyan, I.M. Tkachenko, Solution of the Stieltjes truncated matrix moment problem, *Opuscula Math.* 25 (2005) 5–24.
- [34] V.M. Adamyan, I.M. Tkachenko, General solution of the Stieltjes truncated matrix moment problem, *Oper. Theory Adv. Appl.* 163 (2006) 1–22.
- [35] T. Andô, Truncated moment problems for operators, *Acta. Sci. Math. (Szeged)* 31 (1970) 319–334.
- [36] V.A. Bolotnikov, Degenerate Stieltjes moment problem and associated J -inner polynomials, *Z. Anal. Anwend.* 14 (1995) 441–468.
- [37] G.N. Chen, Y.J. Hu, A unified treatment for the matrix Stieltjes moment problem in both nondegenerate and degenerate case, *J. Math. Anal. Appl.* 254 (2001) 23–34.
- [38] Yu.M. Dyukarevs, B. Fritzsche, B. Kirstein, C. Mädler, On truncated matricial Stieltjes type moment problems, *Complex Anal. Oper. Theory* 4 (2010) 905–951.
- [39] Y.J. Hu, G.N. Chen, A unified treatment for the matrix Stieltjes moment problem, *Linear Algebra Appl.* 380 (2004) 227–239.
- [40] Yu.M. Dyukarev, Indeterminacy criteria for the Stieltjes matrix moment problem, *Mat. Zametki* 75 (2004) 71–88.
- [41] I. Gelfand, V. Retakh, Determinants of matrices over noncommutative rings, *Funct. Anal. Appl.* 25 (1991) 91–102.
- [42] I. Gelfand, S. Gelfand, V. Retakh, R.L. Wilson, Quasideterminants, *Adv. Math.* 193 (1) (2005) 56–141.
- [43] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ., 1980.