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Ten ways to generate the II'in and related schemes

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Abstract

For one-dimensional singularly perturbed boundary value problems there exist a remarkable variety of possibilities to generate uniformly convergent schemes. We describe the basic ideas of ten different approaches and the advantages and disadvantages of several methods.

Keywords: Singular perturbation; Boundary value problem; Discretization

0. Introduction

We are interested in numerical methods for singularly perturbed elliptic differential equations of the convection-diffusion type ($0 < \epsilon \ll 1$)

$$-\epsilon \Delta u + a \nabla u + bu = f, \quad \text{in } \Omega,$$

plus some boundary conditions. The solution of such problems is often required in practice, but, nevertheless, in several dimensions there are still many open questions and technical difficulties.

It is desirable that a method used for such problems should approximately solve the given problem with an accuracy *independent* of the value of the perturbation parameter ϵ . Traditional numerical techniques for solving singularly perturbed problems require a very fine mesh covering the whole domain and hence they are inefficient. There are some attempts to solve singularly perturbed problems using *special grids* (see [5,17,41,43,46]), but we are mainly interested in *special methods* on uniform or quasi-uniform meshes. We call a method *uniformly convergent* with respect to the perturbation parameter if

$$|||u - u_h||| \leq Ch^\gamma$$

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holds in some adapted norm $||| \cdot |||$, where $\gamma > 0$ is the convergence rate, h characterizes the mesh size and C (both here and throughout the paper) is a generic constant which is independent both of ϵ and of the mesh.

In the two-dimensional case only some uniform convergence results under restrictive assumptions are known [12,20,23,30,40]. The situation in the one-dimensional case is absolutely different: there exist a lot of different ways to generate uniformly convergent schemes, and there exist even uniformly convergent schemes of arbitrary order in some situations [16,37].

The aim of this paper consists first in showing the remarkable variety of possibilities to generate uniformly convergent schemes in the one-dimensional case and secondly in stimulating further research to achieve convergence results in higher dimensions. Concerning higher dimensions, one can state that in general it is no problem to transfer the discretization technique from the one-dimensional case to problems in higher dimensions. But the convergence proofs strongly depend on the dimension of the problem. Of course it is also possible to discuss the different methods of analysis proving uniform convergence. Such a list should include

- (A1) classical finite-difference approach (uniform consistency + uniform stability);
- (A2) the double mesh method;
- (A3) maximum principles and comparison functions;
- (A4) comparison problems with frozen coefficients;
- (A5) finite-element methodology.

We decided to concentrate on the derivation of the schemes because the variety of possibilities is much larger; further for some discretization techniques so far there do not exist proofs of uniform convergence based on an analysis typical for the discretization technique (for instance, schemes are derived via collocation but analysed using (A1)) and it is sometimes difficult to classify a proof into (Ai) (for instance, the first proof of O'Riordan and Stynes [29] used parts of (A5) and (A3), five years later they were able to give a proof using only finite-element techniques [42]).

So we will give the proofs of uniform convergence only in two cases where we believe them to be very simple and elegant. First we present in Section 3 a proof of type (A4) which is extremely short. Later we describe in Section 8 a very nice idea of O'Riordan and Stynes.

In detail we are going to study

$$Lu := -\epsilon u'' + a(x)u' = f(x), \quad \text{in } (0, 1), \quad u(0) = u(1) = 0, \quad (1)$$

under the assumption $a(x) \geq a > 0$, but, in general, the described techniques work for the more general problem

$$-\epsilon u'' + a(x)u' + b(x)u = f(x), \quad \text{in } (0, 1), \quad u(0) = u(1) = 0, \quad (2)$$

(with $a(x) \geq a > 0$ again) too. For proofs based on maximum principles, it is usual to require additionally $b(x) \geq 0$; in finite-element arguments $b(x) - \frac{1}{2}a'(x) \geq \gamma > 0$.

Let us introduce the grid

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1,$$

with $x_i = ih$ (only for simplicity we use an equidistant grid, all corresponding schemes and formulas for more general grids are a little bit longer) and denote the midpoint of the interval (x_{i-1}, x_i) by $x_{i-1/2}$. Further, by v_i we denote some approximation of the value $v(x_i)$ of a given continuous

function. Finally we use the notation (\cdot, \cdot) for the L^2 inner product, H^1 as well as H_0^1 are the usual Sobolev spaces on $(0, 1)$. The essential supremum norm on $L^\infty(0, 1)$ is denoted by $\|\cdot\|_\infty$.

The basic discretization for singularly perturbed boundary value problems was first derived by Allen and Southwell [2]. Later Il'in [22] and Scharfetter and Gummel [39] derived in principle the same scheme. Some people call the scheme the Allen–Southwell/Il'in scheme, but there are some reasons to call it Allen–Southwell/Il'in/Scharfetter–Gummel. That is a little long-winded, and because Il'in was the first to prove uniform convergence, we prefer the name Il'in scheme.

1. The original Il'in derivation

It is a well-known fact that the stability of the central difference approximation

$$-\epsilon \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + a_i \frac{u_{i+1} - u_{i-1}}{2h} = f_i, \quad u_0 = u_N = 0,$$

applied to the boundary value problem (1) requires the step size condition

$$h \leq \frac{2\epsilon}{\max |a|},$$

and, therefore, is not convenient for $\epsilon \ll 1$. Il'in introduced the *fitting factor* σ_i corresponding to the scheme

$$-\epsilon \sigma_i \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + a_i \frac{u_{i+1} - u_{i-1}}{2h} = f_i, \quad u_0 = u_N = 0. \quad (3)$$

In the next step Il'in considered the case $a \equiv \text{const.}$, $f \equiv \text{const.}$ and a constant fitting factor. Moreover, he required that the exact solution of the boundary value problem

$$u^*(x) = \frac{f}{a}x - \frac{f}{a} \frac{e^{ax/\epsilon} - 1}{e^{a/\epsilon} - 1}$$

in this case satisfies the difference equation, too. Thus we get

$$-\epsilon \sigma \frac{e^{-\rho} - 2 + e^{\rho}}{h^2} + a \frac{e^{\rho} - e^{-\rho}}{2h} = 0,$$

with $\rho = ah/\epsilon$, and therefore,

$$\sigma = \sigma(\rho) = \frac{1}{2}\rho \coth \frac{1}{2}\rho.$$

Analogously introducing $\rho_i = a_i h/\epsilon$, $\sigma_i = \sigma(\rho_i)$, we obtain the Il'in scheme

$$-\frac{1}{2}ha_i \coth\left(\frac{h}{2\epsilon}a_i\right) \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + a_i \frac{u_{i+1} - u_{i-1}}{2h} = f_i, \quad u_0 = u_N = 0. \quad (4)$$

Remark 1. To derive the scheme, one can also start from the weighted scheme

$$-\epsilon \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + a_i \left[\left(\frac{1}{2} - \alpha_i\right) \frac{u_{i+1} - u_i}{h} + \left(\frac{1}{2} + \alpha_i\right) \frac{u_i - u_{i-1}}{h} \right] = f_i. \quad (5)$$

The schemes (3) and (5) are equivalent for

$$\alpha_i = \frac{\sigma(\rho_i) - 1}{\rho_i}.$$

It is also possible to write the II'in scheme in the form

$$\alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i, \quad u_0 = u_N = 0, \quad (6)$$

with

$$\begin{aligned} \alpha_i &= -\frac{a_i}{h} \frac{\exp(\rho_i)}{\exp(\rho_i) - 1} = -\frac{a_i}{h} \frac{1}{1 - \exp(-\rho_i)}, \\ \beta_i &= \frac{a_i \exp(\rho_i) + 1}{h \exp(\rho_i) - 1} = \frac{a_i}{h} \frac{1 + \exp(-\rho_i)}{1 - \exp(-\rho_i)}, \\ \gamma_i &= -\frac{a_i}{h} \frac{1}{\exp(\rho_i) - 1} = -\frac{a_i}{h} \frac{\exp(-\rho_i)}{1 - \exp(-\rho_i)}. \end{aligned}$$

Notice that α_i , β_i , γ_i can be written in the form

$$\alpha_i = -\epsilon B(\rho_i), \quad \gamma_i = -\epsilon B(-\rho_i), \quad \beta_i = -\alpha_i - \gamma_i,$$

where

$$B(x) = \frac{x}{1 - \exp(-x)}.$$

Remark 2. If we start from the scheme (3) and introduce the so-called *necessary convergence conditions* for uniform convergence [11], we are led after almost the same calculations as in the derivation above to the II'in scheme.

Remark 3. II'in proved the optimal $O(h)$ uniform convergence result for the boundary value problem (1) using the two-grid-method (see [11]). Kellogg and Tsan [24] derived the instructive estimate

$$|u(x_i) - u_i| \leq C \left\{ \frac{h^2}{h + \epsilon} + \frac{h^2}{\epsilon} \exp\left(\frac{-a(1 - x_i)}{\epsilon}\right) \right\}$$

based on maximum principles and comparison functions.

For corresponding difference schemes in two dimensions see [12,20,23,40].

2. Compact exponentially fitted schemes

Osborne [32], Doedel [10] and Lynch and Rice [25] developed a general approach to construct systematically high-order difference approximations to ordinary differential operators. Gartland [16] extended their ideas to construct uniform schemes for the boundary value problem (2). A scheme of order $O(h^p)$ (uniformly in ϵ) is constructed to be exact on the collection of functions of the type

$$\left\{ 1, x, \dots, x^p; \exp\left(\frac{1}{\epsilon} \int a\right), x \exp\left(\frac{1}{\epsilon} \int a\right), \dots, x^{p-1} \exp\left(\frac{1}{\epsilon} \int a\right) \right\}.$$

In the simplest case ($p = 1$), one obtains the Il'in scheme. More precisely, for discretizing the boundary value problem (1), we try to find a scheme of the form

$$\alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} = f_i,$$

and require exactness for $\{1, x, \exp(a_i x/\epsilon)\}$. For the three unknowns $\alpha_i, \beta_i, \gamma_i$ we get the system

$$\alpha_i + \beta_i + \gamma_i = 0, \quad \alpha_i(-h) + \gamma_i h = a_i, \quad \alpha_i e^{-\rho_i} + \beta_i + \gamma_i e^{\rho_i} = 0.$$

Solving these equations, we are led to the Il'in scheme in the form (6).

The proof in [16] for higher-order approximations is based on uniform consistency and a strong stability result. Although Gartland announced generalization to linear convection-diffusion problems in two dimensions, so far nothing is known in this direction.

3. Comparison problems with frozen coefficients

We define v to solve the boundary value problem

$$-\epsilon v'' + a(x)v' = f(x), \quad \text{on } (x_{i-1}, x_{i+1}), \quad v(x_{i-1}) = u(x_{i-1}), \quad v(x_{i+1}) = u(x_{i+1}).$$

Then, $v(x) \equiv u(x)$ on $[x_{i-1}, x_i]$. In a next step we *freeze* the data and define some approximation w of v and so u to solve

$$-\epsilon w'' + a_i w' = f_i, \quad \text{on } (x_{i-1}, x_{i+1}), \quad w(x_{i-1}) = u_{i-1}, \quad w(x_{i+1}) = u_{i+1}.$$

We obtain the approximation

$$w(x) = \frac{f_i}{a_i}(x - x_{i-1}) + u_{i-1} + \left(u_{i+1} - u_{i-1} - \frac{f_i}{a_i}2h\right) \frac{-1 + e^{a_i(x-x_{i-1})/\epsilon}}{-1 + e^{2a_i h/\epsilon}}.$$

Defining $u_i := w(x_i)$, we get the three-point difference scheme

$$u_i = \frac{f_i}{a_i}h + u_{i-1} + \left(u_{i+1} - u_{i-1} - \frac{f_i}{a_i}2h\right) \frac{-1 + e^{\rho_i}}{-1 + e^{2\rho_i}},$$

or again (6)

$$-\frac{e^{\rho_i}}{e^{\rho_i} - 1}u_{i-1} + \frac{e^{\rho_i} + 1}{e^{\rho_i} - 1}u_i - \frac{1}{e^{\rho_i} - 1}u_{i+1} = \frac{f_i}{a_i}h.$$

Remark 4. It is possible to improve this approach by defining the approximation w to solve

$$\bar{L}w := -\epsilon w'' + \bar{a}w' = \bar{f}, \quad \text{on } (0, 1), \quad w(0) = w(1) = 0, \quad (7)$$

with some piecewise constant $O(h)$ approximation \bar{a} of a (and \bar{f} of f). Due to the error equation

$$\bar{L}(w - u) = \bar{f} - f + (a - \bar{a})u'$$

and the stability result [18]

$$\|v\|_\infty \leq C \|\bar{L}v\|_{L^1}, \quad \text{for all } v \text{ with } v(0) = v(1) = 0,$$

we get a very quick proof for the convergence result

$$\|u - w\|_{\infty} \leq Ch,$$

taking into account $\|u'\|_{L^1} \leq C$. The natural choice

$$\bar{a}(x) = \frac{1}{2}\{a(x_{i-1}) + a(x_i)\}, \quad \text{for } x \in (x_{i-1}, x_i), \quad (8)$$

leads to the *El-Mistikawy–Werle* scheme, which is uniformly $O(h^2)$ in the grid points [6].

Remark 5. If we apply the idea of Remark 4 with the choice (8) to the boundary value problem (2), we derive the El-Mistikawy–Werle scheme with *complete exponential fitting*. If we start from

$$-\epsilon u_h'' + \bar{a}u_h' + \bar{b}\bar{u}_h = \bar{f}, \quad u_h(0) = u_h(1) = 0,$$

we derive the El-Mistikawy–Werle scheme with *partial exponential fitting*:

$$-\frac{\epsilon}{h^2}(r_i^- u_{i-1} + r_i^c u_i + r_i^+ u_{i+1}) + q_i^- b_{i-1} u_{i-1} + q_i^c b_i u_i + q_i^+ b_{i+1} u_{i+1} = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1},$$

where

$$\begin{aligned} r_i^- &= \frac{\rho_i^- \exp(-\rho_i^-)}{1 - \exp(-\rho_i^-)}, & r_i^+ &= \frac{\rho_i^+}{1 - \exp(-\rho_i^+)} & r_i^c &= -(r_i^- + r_i^+), \\ q_i^- &= \frac{\frac{1}{2}(1 - r_i^-)}{\rho_i^-}, & q_i^+ &= \frac{\frac{1}{2}(r_i^+ - 1)}{\rho_i^+}, & q_i^c &= q_i^- + q_i^+, \\ \rho_i^- &= \frac{-(a_i + a_{i-1})h}{2\epsilon}, & \rho_i^+ &= \frac{-(a_i + a_{i+1})h}{2\epsilon}. \end{aligned}$$

Generalizations to the two-dimensional case require corresponding stability results. Some investigations were published in [9] but there are a lot of open problems.

4. Exact difference schemes

There exist different possibilities to derive an exact difference scheme for the boundary value problem (1). A first approach consists in solving

$$-\epsilon u'' + a(x)v' = f(x), \quad \text{on } (x_{i-1}, x_{i+1}), \quad u(x_{i-1}) = u_{i-1}, \quad u(x_{i+1}) = u_{i+1},$$

more or less directly. We start, for instance, from the general solution

$$u(x) = d_1 + d_2 \int_{x_{i-1}}^x \exp\left(\frac{1}{\epsilon} \int_{x_{i-1}}^t a(\tau) d\tau\right) dt - \frac{1}{\epsilon} \int_{x_{i-1}}^x \int_{x_{i-1}}^t f(\tau) \exp\left(\frac{1}{\epsilon} \int_{x_{i-1}}^{\tau} a(\tau_1) d\tau_1\right) d\tau dt$$

of the differential equation. Taking into account the boundary conditions, one gets explicitly $u(x) = u(x, u_{i-1}, u_{i+1})$ and so $u(x_i) = u(x_i, u_{i-1}, u_{i+1})$. The resulting identity is called *Marchuk identity*:

$$-\frac{u_{i+1}}{\int_{x_i}^{x_{i+1}} H dx} + \left(\frac{1}{\int_{x_i}^{x_{i+1}} H dx} + \frac{1}{\int_{x_{i-1}}^{x_i} H dx} \right) u_i - \frac{u_{i-1}}{\int_{x_{i-1}}^{x_i} H dx} = \frac{1}{\epsilon} \frac{\int_{x_i}^{x_{i+1}} H^* dx}{\int_{x_i}^{x_{i+1}} H dx} - \frac{1}{\epsilon} \frac{\int_{x_{i-1}}^{x_i} H^* dx}{\int_{x_{i-1}}^{x_i} H dx},$$

with

$$H = \exp\left(\int_{x_i}^x \frac{a(\xi)}{\epsilon} d\xi\right), \quad H^* = \int_{x_{i-1}}^x f(t) \exp\left(-\int_{x_{i-1}}^t \frac{a(\tau)}{\epsilon} d\tau\right) dt.$$

Approximating the integrals as simple as possible (replacing in particular a by a piecewise constant approximation), one gets the Il'in scheme.

The following approach is more elegant. We introduce the adjoint operator

$$L^*v := -\epsilon v'' - (av)'. \quad (8)$$

Let g_i be the local Greens function of L^* at x_i , defined by

$$\begin{aligned} L^*g_i &= 0, \quad x \in (x_{i-1}, x_i) \cup (x_i, x_{i+1}), \\ g_i(x_{i-1}) &= g_i(x_{i+1}) = 0, \quad \epsilon[g_i'(x_i - 0) - g_i'(x_i + 0)] = 1. \end{aligned}$$

Then, the identity

$$\int_{x_{i-1}}^{x_{i+1}} (Lu)g_i dx = \int_{x_{i-1}}^{x_{i+1}} fg_i dx$$

and integration by parts result in

$$-\epsilon g_i'(x_{i-1})u_{i-1} + u_i + \epsilon g_i'(x_{i+1})u_{i+1} = \int_{x_{i-1}}^{x_{i+1}} fg_i dx. \quad (9)$$

This is again the Marchuk identity written only in a different form. In the case of $a(x)$ being constant on (x_{i-1}, x_{i+1}) , we can explicitly calculate g_i and obtain again the Il'in scheme if we approximate

$$\int_{x_{i-1}}^{x_{i+1}} fg_i dx \approx f_i \int_{x_{i-1}}^{x_{i+1}} g_i dx.$$

It is absolutely open whether or not the ideas of this section can be applied in more dimensions.

5. Collocation

First we introduce a collocation method based on *exponential* C^1 -splines. Let us define a globally continuous exponential spline spanned by $\{1, x, \exp(a_i x/\epsilon)\}$:

$$s(x) = d_i(x - x_{i-1/2}) + \left[u_{i-1} + (u_i - u_{i-1} - d_i h) \frac{-1 + \exp(a_i(x - x_{i-1/2})/\epsilon)}{-1 + \exp(a_i h/\epsilon)} \right], \quad (10)$$

for $x \in [x_{i-1/2}, x_{i+1/2}]$. Here d_i is a free parameter.

The spline satisfies

$$s(x_{i-1/2}) = u_{i-1} \quad \text{and} \quad s(x_{i+1/2}) = u_i.$$

This is a little unusual, because u_i should approximate $u(x_i)$, but it is possible. We require the *collocation condition*

$$-\epsilon s''(x_i) + a(x_i)s'(x_i) = f(x_i),$$

and obtain $d_i = f_i/a_i$. The C^1 -condition of our spline reads

$$\frac{f_i}{a_i} + \left(u_i - u_{i-1} - \frac{f_i}{a_i}h\right) \frac{a_i}{\epsilon} \frac{e^{\rho_i}}{e^{\rho_i} - 1} = \frac{f_{i+1}}{a_{i+1}} + \left(u_{i+1} - u_i - \frac{f_{i+1}}{a_{i+1}}h\right) \frac{a_{i+1}}{\epsilon} \frac{1}{e^{\rho_{i+1}} - 1}.$$

This is a generalized Il'in scheme which reduces to the original one if we assume $a_i = a_{i+1}$, $f_i = f_{i+1}$.

A second possibility consists in the application of *quadratic* C^1 -splines with a fitted collocation condition. We use a spline spanned by $\{1, x, x^2\}$:

$$s(x) = u_i \frac{x - x_{i-1/2}}{h} - u_{i-1} \frac{x - x_{i+1/2}}{h} + d_i(x - x_{i-1/2})(x - x_{i+1/2}), \quad x \in [x_{i-1/2}, x_{i+1/2}]. \quad (11)$$

The modified collocation condition

$$-\epsilon \sigma_i s''(x_i) + a(x_i) s'(x_i) = f(x_i)$$

leads to

$$d_i = \frac{1}{2\epsilon \sigma_i} \left(a_i \frac{u_i - u_{i-1}}{h} - f_i \right).$$

The C^1 -condition for our spline takes the form (again assuming $a_i = a_{i+1}$, $f_i = f_{i+1}$ and $\sigma_i = \sigma_{i+1}$)

$$-\frac{\epsilon \sigma_i}{h^2} (u_{i+1} - 2u_i + u_{i-1}) + a_i \frac{u_{i+1} - u_{i-1}}{2h} = f_i.$$

This scheme is identical with the fitted difference scheme (3); therefore the Il'in fitting factor σ results again in the Il'in scheme, but now derived using quadratic splines and a modified collocation condition.

For further schemes based on collocation techniques see [44,45]. Note that until now the proofs of uniform convergence for collocation methods have been based on related results for the generated difference schemes and, therefore, they are relatively complicated. A uniform convergence proof based only on typical collocation arguments (integral representation based on Greens function and the residual of the approximate solution) is open. Further, nothing is known about collocation and singular perturbations in several dimensions.

6. Finite-volume methods (finite boxes)

There exist different ways to apply finite-volume techniques to discretize the boundary value problem (1). A first version of a box technique starts from a conservation form of the differential equation, namely,

$$-\epsilon (e^{-q/\epsilon} u')' = e^{-q/\epsilon} f,$$

with $q' = a$. Integration over the box $(x_{i-1/2}, x_{i+1/2})$ results in

$$-\epsilon e^{-q/\epsilon} u' \Big|_{x_{i-1/2}}^{x_{i+1/2}} = \int_{x_{i-1/2}}^{x_{i+1/2}} e^{-q/\epsilon} f \, dx.$$

Now we assume $a = \text{const.}$ on $(x_{i-1/2}, x_{i+1/2})$ or $q = a_i x$ on this interval. Further, we replace $u'(x_{i+1/2})$ by $(u_{i+1} - u_i)/h$ and approximate the integral of the right-hand side. This leads to the difference scheme

$$-\epsilon e^{-a_i x_{i+1/2}/\epsilon} \frac{u_{i+1} - u_i}{h} + \epsilon e^{-a_i x_{i-1/2}/\epsilon} \frac{u_i - u_{i-1}}{h} = -f_i \frac{\epsilon}{a_i} (e^{-a_i x_{i+1/2}/\epsilon} - e^{-a_i x_{i-1/2}/\epsilon})$$

or

$$u_{i+1} - u_i - e^{\rho_i} (u_i - u_{i-1}) = \frac{f_i}{a_i} h (1 - e^{\rho_i}),$$

which is again the II' in scheme (6).

Remark 6. This way to derive an adapted difference scheme is well known in the field of the continuity equations of semiconductor physics. In this context the resulting scheme is called the *Scharfetter–Gummel scheme*. The close relations between discretizations for the basic equations of inner electronics and discretizations of standard convection-diffusion problems were described in [34], later in [14].

A second interesting approach to discretize the boundary value problem (1) using finite-volume ideas consists in applying the box technique only to the convection term au' . We describe the basic principle for a constant a on the box $(x_{i-1/2}, x_{i+1/2})$. The approximation (in two dimensions we have to apply the Gauss theorem)

$$\int_{x_{i-1/2}}^{x_{i+1/2}} a_i u' dx \approx a_i (\lambda_i u_{i+1} + (1 - \lambda_i) u_i - (\lambda_i u_i + (1 - \lambda_i) u_{i-1}))$$

and standard discretization with respect to the other terms of the problem generate the difference scheme

$$-\frac{\epsilon}{h} (u_{i-1} - 2u_i + u_{i+1}) + a_i (\lambda_i u_{i+1} + (1 - 2\lambda_i) u_i + (\lambda_i - 1) u_{i-1}) = f_i h.$$

The choice

$$\lambda_i = \frac{1}{\rho_i} - \frac{1}{e^{\rho_i} - 1}$$

corresponds to the II' in scheme.

Details concerning the two-dimensional case including some error estimations can be found in [3,4]. The proofs are based on typical finite-element techniques. It is open whether or not it is possible to achieve uniform convergence. Miller and Wang [27] described another version of an exponentially fitted triangular box method in higher-dimensional cases and announced uniform convergence results.

7. Polynomial conforming Petrov–Galerkin finite elements

Let us introduce the weak formulation of the boundary value problem (1). Find $u \in V = H_0^1(0, 1)$ such that

$$A(u, v) = (f, v), \quad \forall v \in V, \quad (12)$$

where the bilinear form $A(\cdot, \cdot)$ is defined by

$$A(u, v) := \epsilon(u', v') + (au', v). \quad (13)$$

A classical finite-element discretization is characterized by a finite-element space $V_h \subset V$ (the method is conforming) and an approximation $u_h \in V_h$ satisfying

$$A(u_h, v_h) = (f, v_h), \quad \forall v \in V_h. \quad (14)$$

It is well known that standard finite elements lead to the same difficulties as standard finite differences. For instance, linear finite elements combined with the rectangle rule to approximate the integrals result in the difference scheme

$$-\epsilon \frac{u_{i-1} + 2u_i - u_{i+1}}{h^2} + a_{i+1/2} \frac{u_{i+1} - u_i}{h} + a_{i-1/2} \frac{u_i - u_{i-1}}{h} = \frac{1}{2} (f_{i+1/2} + f_{i-1/2}),$$

which is not suitable for $\epsilon \ll h$.

Already in the early seventies the idea was born to apply a *Petrov–Galerkin* technique, that means, to use a test space T_h which is different from V_h and to require

$$A(u_h, v_h) = (f, v_h), \quad \forall v \in T_h. \quad (15)$$

The classical choice consists in polynomial spaces V_h and T_h , for instance, linear finite elements combined with quadratic or cubic test functions. In our one-dimensional case we choose the quadratic test functions

$$\psi_i(x) = \phi_i(x) + \alpha_{i-1/2} \sigma_{i-1/2}(x) - \alpha_{i+1/2} \sigma_{i+1/2}(x),$$

with

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sigma_{i-1/2}(x) = (x - x_{i-1})(x - x_i).$$

The resulting difference scheme is (applying the rectangle quadrature rule)

$$\begin{aligned} & -\epsilon \frac{u_{i-1} + 2u_i - u_{i+1}}{h^2} + a_{i+1/2} \left(\frac{1}{2} - \alpha_{i+1/2} \right) \frac{u_{i+1} - u_i}{h} + a_{i-1/2} \left(\frac{1}{2} + \alpha_{i-1/2} \right) \frac{u_i - u_{i-1}}{h} \\ & = f_{i+1/2} \left(\frac{1}{2} - \alpha_{i+1/2} \right) + f_{i-1/2} \left(\frac{1}{2} + \alpha_{i-1/2} \right). \end{aligned}$$

This scheme generalizes the scheme of Remark 1; under some simplifications the scheme is identical with the scheme in Remark 1 and therefore the II'in scheme if the free parameter α is suitably defined.

For some two-dimensional generalizations see [19,33] and the cited references. From the theoretical point of view for these methods there exist many open questions in higher-dimensional cases concerning the choice of free parameters and optimal convergence results.

Remark 7. It is also possible to use so-called *hinged* elements instead of classical polynomial elements. See [8,28] in the two-dimensional case.

8. Exponential Petrov–Galerkin finite-element methods

Let us consider the following conforming Petrov–Galerkin method. Find some $u_h \in V_h \subset V$ such that

$$A(u_h, v_h) = (f, v_h), \quad \forall v_h \in T_h,$$

with a test space $T_h \subset V$.

We ask the question: which test space T_h is optimal? Let us assume that it is possible to define a Greens function G of the adjoint problem with respect to a given point x_0 by (in the one-dimensional case this is not a problem)

$$A(w, G) = w(x_0), \quad \forall w \in V$$

Then, the relation $G \in T_h$ implies

$$(u - u_h)(x_0) = A(u - u_h, G) = A(u, G) - A(u_h, G) = (f, G) - (f, G) = 0.$$

This means: the error $u - u_h$ is zero in the given point x_0 ! Although there are examples with $G \in T_h$ ($-u''$, linear elements, x_0 grid point), this is rather an exceptional situation.

Now, let us introduce a neighbouring bilinear form $\bar{A}(\cdot, \cdot)$ and define u_h to solve

$$\bar{A}(u_h, v_h) = (\bar{f}, v_h), \quad \forall v_h \in T_h.$$

Defining G to be the above Greens function with respect to $\bar{A}(\cdot, \cdot)$ for $G \in T_h$, we obtain

$$(u - u_h)(x_0) = \bar{A}(u - u_h, G) = (\bar{A} - A)(u, G) + (f, G) - \bar{A}(u_h, G);$$

thus,

$$(u - u_h)(x_0) = (\bar{A} - A)(u, G) + (f - \bar{f}, G). \quad (16)$$

This identity shows, that a convenient test space contains the Greens function of some adjoint approximate problem.

For our problem (12), (13) we define

$$\bar{A}(u, v) = \epsilon(u', v') + (\bar{a}u', v), \quad (17)$$

with some piecewise constant approximation \bar{a} of a , and choose a grid point x_j to define G_j . Now we introduce the test space

$$T_h = \text{span}\{\psi_k\}_{k=1, \dots, N-1}$$

and

$$-\epsilon \psi_k'' - \bar{a} \psi_k' = 0, \quad \text{on every subinterval}, \quad \psi_k(x_j) = \delta_{kj}. \quad (18)$$

Thus, $\text{supp } \psi_k = [x_{k-1}, x_{k+1}]$ and $G_j \in T_h$. Consequently, from (17) for the error in the grid points of this conforming Petrov–Galerkin technique we obtain

$$|(u - u_h)(x_j)| \leq Ch \|G_j\|_\infty (1 + \|u'\|_{L^1}).$$

The uniform boundedness of u' in the L^1 -norm and G_j in the maximum norm results in

$$|(u - u_h)(x_j)| \leq Ch. \quad (19)$$

The convergence result (19) does not depend on the choice of V_h ! Therefore, the simplest approach of using linear splines combined with the exponential adapted test functions (18) leads to the nice result (19).

Setting

$$u_h(x) = \sum_{i=1}^{N-1} u_i \phi_i(x),$$

with the hat functions ϕ_i , we generate the difference scheme

$$-\psi_i'(x_{i-1})u_{i-1} + (\psi_i'(x_{i-0}) - \psi_i'(x_{i+0}))u_i + \psi_i'(x_{i+1})u_{i+1} = \int_{x_{i-1}}^{x_{i+1}} \bar{f} \psi_i dx. \quad (20)$$

Let us approximate

$$\int_{x_{i-1}}^{x_{i+1}} \bar{f} \psi_i dx \approx f_i \int_{x_{i-1}}^{x_{i+1}} \psi_i dx,$$

and set

$$\bar{a}(x) = \frac{1}{2} \{a(x_{i-1}) + a(x_i)\}, \quad \text{on } (x_{i-1}, x_i).$$

Then, the resulting scheme is the El-Mistikawy–Werle scheme, and we have a quick proof of uniform $O(h)$ -convergence. See [29] for the proof of the uniform $O(h^2)$ convergence in the knots. If we only think locally and set

$$\bar{a}(x) = a(x_i), \quad \text{on } (x_{i-1}, x_{i+1}),$$

we derive the Il'in scheme. But this cannot be done consistently for all the intervals and so we get a proof for the Il'in scheme only in the case $a(x) = \text{const}$. Let us finally mention that the idea to use Petrov–Galerkin methods with exponential fitted elements is due to Hemker [21].

In the two-dimensional case O'Riordan and Stynes [30] were able to prove a uniform $O(h^{1/2})$ convergence result in an energy norm for a special discretization of the boundary value problem

$$-\epsilon \Delta u + a \nabla u = f, \quad \text{in } \Omega = (0, 1) \times (0, 1), \quad u = 0, \quad \text{on } \partial\Omega,$$

under the essential assumptions

- (i) $a = (a_1, a_2)$, $a_1 > 0$, $a_2 > 0$,

(ii) $a_1 = a_1(x)$, $a_2 = a_2(y)$.

Assumption (ii) allows a Bubnov–Galerkin discretization based on conforming elements (tensor products of exponentially fitted splines).

The streamline diffusion method is a well-known *nonconforming* method for handling convection-diffusion equations. The basic idea for discretizing

$$\epsilon(\nabla u, \nabla v) + (a \nabla u, v) = (f, v)$$

consists in applying test functions of the type $w_h \in T_h$ with

$$w_h := v_h + \beta a \nabla v_h, \quad v_h \in V_h,$$

and some parameter β . In general, V_h is a usual polynomial finite-element space; therefore the standard version of the streamline-diffusion method cannot result in a uniformly convergent scheme. In [31] O’Riordan and Stynes proposed an exponential streamline-diffusion method but only in the special case $a = (a_1, 0)$, $a_1 = \text{const}$. In the one-dimensional case the method reduces to the conforming method sketched above.

For the boundary value problem (2) a real *nonconforming exponential* Petrov–Galerkin technique was developed and analyzed in [1]. This method works only in the very stiff case $\epsilon < Ch^{1+\alpha}$, $\alpha > 0$. But there is some hope that nonconforming exponentially fitted methods are the adapted tool to avoid the restrictive assumption

$$a_1 = a_1(x), \quad a_2 = a_2(y)$$

in [31].

9. An explicit Galerkin method

Let us consider the boundary value problem

$$-\epsilon \Delta u + a \nabla u = f, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega,$$

in the case $a \equiv \text{const}$. and $\Omega \subset \mathbb{R}^2$. We start with a finite-element space V_h on some triangulation, in particular with polynomial Lagrange elements of degree n . Each edge of every element can be characterized as either an inflow or outflow edge depending on the sign of $a \cdot \nu$, where ν denotes a unit outer normal direction. We call a triangle *type-1 triangle*, if the triangle has one inflow side ($a \cdot \nu < 0$), *type-2 triangle*, if it has two inflow sides. We assumed $a \cdot \nu \neq 0$ on every edge of the triangulation.

Let T be a triangle of the triangulation. We assume that u_h is known on the inflow edge or edges of T . Thus our degrees of freedom reduce to

$$\frac{1}{2}(n+1)(n+2) - (n+1) = \frac{1}{2}n(n+1), \quad \text{for triangles of type 1,}$$

$$\frac{1}{2}(n+1)(n+2) - (2n+1) = \frac{1}{2}n(n-1), \quad \text{for triangles of type 2.}$$

Therefore, it is adequate to require

$$\epsilon(\nabla u_h, \nabla v_h)_T + (a \nabla u_h, v_h)_T = (f, v_h)_T, \quad \forall v_h \in P_{n-1}(T), \quad (21)$$

$l = 1$ or $l = 2$ for triangles of type 1 respectively type 2; P_{n-l} denotes the space of polynomials of degree $n - l$.

The method (21) is called *explicit* Galerkin method because it is possible to compute the approximate solution from one element to the next in accordance with the flow direction. In [13] a discontinuous version of the method is discussed, too.

Now we return to our one-dimensional boundary value problem (1) and first freeze the coefficient $a(x)$ on the subinterval (x_{i-1}, x_{i+1}) . To generate a uniformly convergent scheme we cannot work with polynomials, therefore we choose

$$u_h(x) = u_i \phi_{i+1}^l(x) + u_{i+1} \phi_{i+1}^r(x), \quad \text{on } (x_i, x_{i+1}), \quad (22)$$

with

$$\begin{aligned} -\epsilon \phi_{i+1}'' + a_i \phi_{i+1}' &= 0, \quad \text{on } (x_i, x_{i+1}), \\ \phi_{i+1}^l(x_i) &= 1, \quad \phi_{i+1}^l(x_{i+1}) = 0, \quad \phi_{i+1}^r(x_i) = 0, \quad \phi_{i+1}^r(x_{i+1}) = 1. \end{aligned}$$

That means, we use the exponentials (22) instead of linear elements, and so in (21) the test functions are, for instance, piecewise constants. Thus the method reads as follows. Find u_h in the form (22) such that

$$\epsilon \int_{x_i}^{x_{i+1}} u_h' v_h' + \int_{x_i}^{x_{i+1}} a_i u_h' v_h = \int_{x_i}^{x_{i+1}} f v_h, \quad \forall v_h \in P_0. \quad (23)$$

Due to $v_h \in P_0$ we have $\int u_h' v_h' = 0$; nevertheless, integration by parts yields

$$\int_{x_i}^{x_{i+1}} u_h' v_h' = u_h' v_h \Big|_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} u_h'' v_h,$$

and we obtain

$$\int_{x_i}^{x_{i+1}} (-\epsilon u_h'' + a_i u_h') v_h + \epsilon u_h' v_h \Big|_{x_i}^{x_{i+1}} = \int_{x_i}^{x_{i+1}} f v_h.$$

The first term is zero, therefore this relation connects the derivative of u_h in x_i , known from the interval (x_{i-1}, x_i) , to the derivative in x_{i+1} . So we get the three-point scheme

$$\epsilon \left(u_i (\phi_{i+1}^l)'(x_{i+1}) + u_{i+1} (\phi_{i+1}^r)'(x_{i+1}) - u_{i-1} (\phi_i^l)'(x_i - 0) - u_i (\phi_i^r)'(x_i - 0) \right) = f_i h.$$

This scheme is again the II' scheme.

In [13] and some further papers of these authors explicit Galerkin methods based on polynomial finite-element spaces were analysed. The results are similar to the well-known results for the streamline-diffusion method.

10. Mixed finite elements

Let us write the boundary value problem (1) in the form (again with $q' = a$)

$$\frac{du}{dx} = e^{q/\epsilon} v, \quad \frac{dv}{dx} = -\frac{f}{\epsilon} e^{-q/\epsilon}.$$

Then, integration by parts leads to the following mixed formulation. Find $u \in V = H_0^1(0, 1)$, $v \in L^2(0, 1)$ such that

$$(u', \tau) = (e^{q/\epsilon} v, \tau), \quad \forall \tau \in L^2, \quad (v, \chi') = \left(\frac{f}{\epsilon} e^{-q/\epsilon}, \chi \right), \quad \forall \chi \in V \quad (24)$$

For the discretization we choose $u_h, \chi_h \in V$ to be piecewise linear, v_h, τ_h piecewise constant and formulate the discrete problem in the form

$$(u'_h, \tau_h) = (e^{q/\epsilon} v_h, \tau_h), \quad \forall \tau_h, \quad (v_h, \chi'_h) = \left(\frac{f}{\epsilon} e^{-q/\epsilon}, \chi_h \right), \quad \forall \chi_h. \quad (25)$$

Let us denote the constant v_h on (x_i, x_{i+1}) by $v_{i+1/2}$. The first equation in (25) is equivalent to

$$u_{i+1} - u_i = v_{i+1/2} \int_{x_i}^{x_{i+1}} e^{q/\epsilon}. \quad (26)$$

The second equation in (25) yields (χ_i is the hat function related to x_i)

$$\int_0^1 \frac{f}{\epsilon} e^{-q/\epsilon} \chi_i = \int_0^1 v_h \chi'_i = v_{i-1/2} \int_{x_{i-1}}^{x_i} \chi'_i + v_{i+1/2} \int_{x_i}^{x_{i+1}} \chi'_i = -v_{i+1/2} + v_{i-1/2}. \quad (27)$$

Thus, combining (26) and (27), we are left with the scheme

$$-\frac{u_{i+1} - u_i}{\int_{x_i}^{x_{i+1}} e^{q/\epsilon}} + \frac{u_i - u_{i-1}}{\int_{x_{i-1}}^{x_i} e^{q/\epsilon}} = \int_0^1 \frac{f}{\epsilon} e^{-q/\epsilon} \chi_i. \quad (28)$$

Approximating the integrals in the adequate way, we get the II' in scheme. In higher-dimensional cases, see [7].

Until now there does not exist a uniform convergence proof based on typical mixed finite-element arguments. But Felgenhauer [15] was able to characterize some Petrov–Galerkin methods as mixed variational formulation. This new formulation seems to allow a novel uniform convergence analysis.

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