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A new nonlinear integration formula for ODEs

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Abstract

A new class of nonlinear one-step methods based on Euler's integration formula for the numerical solution of ordinary differential equations is presented. The accuracy and stability of the proposed methods is considered and their applicability to stiff problems is also discussed.

Keywords: Euler's formula; GM Euler formula; HM Euler formula; Stability; A-stable; L-stable; ODEs

1. Introduction

The class of linear one-step methods of order one is given by

$$y_{n+1} = y_n + h(\theta f_n + (1-\theta) f_{n+1}); \tag{1.1}$$

these methods are often referred to as "θ-methods" [2, p. 240], where $y' = f(x, y)$. The formula (1.1) is shown to be A-stable if and only if $\theta \leqslant \frac{1}{2}$ [2]. Further it can be shown that these methods have truncation error

$$T_{n+1} = (\theta - \tfrac{1}{2})h^2 y_n'' + (\tfrac{1}{2}\theta - \tfrac{1}{3})h^3 y_n''' + O(h^4). \tag{1.2}$$

This error is smallest when $\theta = \frac{1}{2}$ in which case the method has order 2, called the trapezoidal rule whose truncation error is $-1/12h^3 y_n'''$. Formula (1.1) can be obtained as follows. Consider the following two formulae of Euler:

$$y_{n+1} - y_n = hy_n', \tag{1.3}$$

$$y_{n+1} - y_n = hy_{n+1}'. \tag{1.4}$$

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The first formula is the forward Euler and the second is the backward Euler formula. Multiplying Eq. (1.3) by θ and Eq. (1.4) by $(1-\theta)$ and adding we obtain equation (1.1). By taking arithmetic mean of Eqs. (1.3) and (1.4) we obtain the well known trapezoidal formula

$$y_{n+1} = y_n + \tfrac{1}{2}h(y'_n + y'_{n+1}). \tag{1.5}$$

Evans and Sanugi [1] have developed equivalent formulae in the geometric mean (GM) sense which is implicit, high-order, A-stable and moreover it is also L-stable. But this formula is computer-expensive compared to the trapezoidal method. In the present investigation a method based on the harmonic mean (HM) sense is considered and its applicability to ODEs is investigated.

2. Nonlinear formulae based on Euler formulae

The formula given by Evans and Sanugi [1] which is referred to as GM Euler method is of the form

$$y_{n+1} = y_n + h\sqrt{y'_n y'_{n+1}}. \tag{2.1}$$

Now by taking the harmonic mean of the formulae (1.3) and (1.4), a nonlinear equivalent of the trapezoidal formula is obtained as

$$y_{n+1} = y_n + 2h\frac{y'_n y'_{n+1}}{y'_n + y'_{n+1}}. \tag{2.2}$$

We will refer to this formula as the HM Euler formula. In general if we multipy (1.3) by θ and (1.4) by $(1-\theta)$ and take the harmonic mean, we obtain

$$y_{n+1} - y_n = h\frac{y'_n y'_{n+1}}{(1-\theta)y'_n + \theta y'_{n+1}}. \tag{2.3}$$

Formula (2.3) contains, as special cases, forward Euler ($\theta = 1$), backward Euler ($\theta = 0$) and formula (2.2) ($\theta = \frac{1}{2}$).

3. Accuracy of the formula (2.3)

Based on the Taylor's series expansion of $y(x_{n+1})$ about x_n, which is denoted by y_{n+1}, we obtain the expansion for y'_{n+1} as

$$\begin{aligned} y'_{n+1} &= y'_n + hy''_n + \tfrac{1}{2}h^2 y'''_n + \cdots \\ &= y'_n\left[1 + \left(\frac{hy''_n}{y'_n} + \frac{h^2 y'''_n}{2y'_n} + \cdots\right)\right]. \end{aligned} \tag{3.1}$$

Using Eq. (3.1) in the expression $(1-\theta)y_n' + \theta y_{n+1}'$ we obtain

$$(1-\theta)y_n' + \theta y_{n+1}' = (1-\theta)y_n' + \theta y_n'\left[1 + \left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right)\right]$$

$$= y_n'\left[1 + \theta\left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right)\right]. \tag{3.2}$$

Substituting Eq. (3.2) in Eq. (2.3) we obtain

$$y_{n+1} = y_n + \frac{h}{y_n'} y_n'(y_n' + hy_n'' + \tfrac{1}{2}h^2 y_n''' + \cdots)$$
$$\times\left[1 + \theta\left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right)\right]^{-1}. \tag{3.3}$$

The terms in the square bracket of (3.3) are in the form $(1+x)^{-1}$, where

$$x = \theta\left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right).$$

Using binomial theorem in Eq. (3.3) we obtain

$$y_{n+1} = y_n + h[y_n' + hy_n'' + \tfrac{1}{2}h^2 y_n''' + \cdots]$$
$$\times\left\{1 - \theta\left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right) + \theta^2\left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right)^2 - \theta^3\left(\frac{hy_n''}{y_n'} + \frac{h^2 y_n'''}{2y_n'} + \cdots\right)^3 + \cdots\right\}.$$

On further simplification, we obtain

$$y_{n+1} = y_n + hy_n' + h^2(1-\theta)y_n'' + h^3\left[(1-\theta)\frac{y_n'''}{2} - \theta(1-\theta)\frac{y_n''^2}{y_n'}\right] + \cdots. \tag{3.4}$$

The Taylor series expansion of $y(x_{n+1})$ about x_n is

$$y(x_{n+1}) = y_n + hy_n' + \tfrac{1}{2}h^2 y_n'' + \tfrac{1}{6}h^3 y_n''' + \cdots.$$

Hence

$$y(x_{n+1}) - y_{n+1} = h^2(\theta - \tfrac{1}{2})y_n'' + h^3\left[\left(\frac{\theta}{2} - \frac{1}{3}\right)y_n''' - \theta(\theta-1)\frac{y_n''^2}{y_n'}\right] + \cdots. \tag{3.5}$$

It can be clearly seen that the method agrees with the Taylor series expansion up to the first order term when $\theta \neq \frac{1}{2}$. But when $\theta = \frac{1}{2}$ the second order terms also match and the principal error term becomes

$$T_{n+1} = h^3\left[-\frac{y_n'''}{12} + \frac{y_n''^2}{4y_n'}\right]. \tag{3.6}$$

Therefore the formula (2.3) has the least error when used with $\theta = \frac{1}{2}$.

4. Stability analysis

To study the stability properties of the formula (2.3) we apply the formula to the test equation $y' = \lambda y$. So we obtain the following difference equation:

$$y_{n+1} = y_n + \frac{h(\lambda y_n)(\lambda y_{n+1})}{\theta(\lambda y_{n+1}) + (1-\theta)(\lambda y_n)}. \tag{4.1}$$

When $\theta = \frac{1}{2}$ Eq. (4.1) becomes

$$y_{n+1} = y_n + 2h\lambda \frac{y_n y_{n+1}}{y_n + y_{n+1}}, \tag{4.2}$$

which can be rewritten as

$$\frac{y_{n+1}}{y_n} = 1 + 2h\lambda \frac{y_{n+1}/y_n}{1 + (y_{n+1}/y_n)}. \tag{4.3}$$

Letting $y_{n+1}/y_n = R_n$, Eq. (4.3) becomes

$$R_n = 1 + 2h\lambda \frac{R_n}{1 + R_n},$$

which can be rewritten as

$$R_n^2 - 2h\lambda R_n - 1 = 0. \tag{4.4}$$

From (4.4) we get the roots as $R_n = h\lambda \pm \sqrt{h^2\lambda^2 + 1}$. Taking only the positive sign we have $R_n = h\lambda + \sqrt{h^2\lambda^2 + 1}$. Absolute stability requires that $|y_{n+1}/y_n| = |R_n| < 1$. This condition implies

$$|h\lambda + \sqrt{h^2\lambda^2 + 1}| < 1.$$

To see for what values of $h\lambda$ this inequality is valid, we have to consider two possibilities:

(a) *$h\lambda$ is real.* Let $h\lambda = x$, where x is real. Then the inequality becomes

$$|x + \sqrt{x^2 + 1}| < 1.$$

It is easy to verify that the function $f(x) = x + \sqrt{x^2 + 1}$ satisfies the conditions

$$|f(x)| < 1 \quad \text{for } x < 0,$$

$$|f(x)| > 1 \quad \text{for } x > 0.$$

(b) *$h\lambda$ is purely imaginary.* Let $h\lambda = \mathrm{i}y$, where y is real. Then the inequality becomes

$$|\mathrm{i}y + \sqrt{(\mathrm{i}y)^2 + 1}| < 1$$

$$\Rightarrow |\mathrm{i}y + \sqrt{1 - y^2}| < 1$$

$$\Rightarrow \sqrt{y^2 + (1 - y^2)} < 1$$

$$\Rightarrow \sqrt{1} < 1$$
$$\Rightarrow \pm 1 < 1.$$

This implies that the imaginary axis of the complex plane is the boundary for the region of absolute stability of the method. This means that the region of absolute stability contains the whole of the left-hand half-plane $\operatorname{Re} h\lambda < 0$. Hence the method is A-stable.

Moreover a one-step numerical method is said to be L-stable [2] if it is A-stable and in addition when applied to the scalar test equation $y' = \lambda y$, λ a complex constant with $\operatorname{Re}\lambda < 0$, it yields $y_{n+1} = R(h\lambda)y_n$, where $|R(h\lambda)| \to 0$ as $\operatorname{Re} h\lambda \to -\infty$.

By applying Euler's method, backward Euler's method, trapezoidal method, GM Euler method and HM Euler method to the scalar test equation $y' = \lambda y$, where $\lambda < 0$, we obtain the following expressions for $R(h\lambda)$.

(1) Euler's method: $R(h\lambda) = 1 + h\lambda$.
(2) Backward Euler's method: $R(h\lambda) = 1/(1 - h\lambda)$.
(3) Trapezoidal method: $R(h\lambda) = (1 + h\lambda/2)/(1 - h\lambda/2)$.
(4) GM Euler method: $R(h\lambda) = [(h\lambda + \sqrt{h^2\lambda^2 + 4})/2]^2$.
(5) HM Euler method: $R(h\lambda) = h\lambda + \sqrt{h^2\lambda^2 + 1}$.

For the backward Euler formula and GM Euler method $R(h\lambda) \to 0$ as $h\lambda \to -\infty$. But for the trapezoidal method we have $R(h\lambda) \to -1$ as $h\lambda \to -\infty$. For the HM Euler method also $R(h\lambda) \to 0$ as $h\lambda \to -\infty$ which is obtained as follows:

$$\begin{aligned}\lim_{h\lambda \to -\infty} h\lambda + \sqrt{h^2\lambda^2 + 1} &= \lim_{x \to \infty} -x + \sqrt{x^2 + 1}\\ &= \lim_{x \to \infty} \frac{-1}{-x - \sqrt{x^2 + 1}}\\ &= 0.\end{aligned}$$

Therefore the present method is L-stable.

Hence the present method has all the properties for a method one look for, viz, high-order, A-stable and more importantly it is L-stable. Hence it can be readily applied to stiff equations. Even though the GM Euler method (2.1) [1] has all the above-mentioned properties there are some drawbacks, viz.,

(1) GM Euler formula is computer-expensive since it involves execution of the sqrt function;

(2) Moreover, some constraints have to be incorporated in the function evaluation to make the argument of the sqrt function positive. We give a detailed explanation for this observation when we consider numerical results.

5. Class of equations in which the HM Euler formula gives improved accuracy over the trapezoidal method and the GM Euler method

Now we will consider the conditions for which HM Euler method gives better results than the trapezoidal method and the GM Euler method for a class of differential equations.

The principal error term for all the three methods are given as
Trapezoidal method: $E_{\mathrm{T}} = -h^3 y_n'''/12$.
GM Euler method: $E_{\mathrm{G}} = -h^3 y_n'''/12 + h^3 y_n''^2/(8y_n')$.
HM Euler method: $E_{\mathrm{H}} = -h^3 y_n'''/12 + h^3 y_n''^2/(4y_n')$.
Suppose if we want $|E_{\mathrm{H}}| < |E_{\mathrm{T}}|$, then by substituting the related terms in the inequality we obtain

$$\left|\frac{h^3 y_n'''}{12} - \frac{h^3 y_n''^2}{4y_n'}\right| < \left|\frac{h^3 y_n'''}{12}\right|$$

or

$$-\left|\frac{h^3 y_n'''}{12}\right| < \frac{h^3 y_n'''}{12} - \frac{h^3 y_n''^2}{4y_n'} < \left|\frac{h^3 y_n'''}{12}\right|.$$

(a) Suppose if $y_n''' > 0$. The inequalities become

$$-\frac{h^3 y_n'''}{12} < \frac{h^3 y_n'''}{12} - \frac{h^3 y_n''^2}{4y_n'} < \frac{h^3 y_n'''}{12}$$

or

$$-\frac{h^3 y_n'''}{6} < -\frac{h^3 y_n''^2}{4y_n'} < 0;$$

i.e., $y_n' > 0$ and $y_n' y_n''' > \frac{3}{2} y_n''^2$.
(b) Suppose if $y_n''' < 0$. We let $y_n''' = -x$, where $x > 0$, and the inequalities become

$$-\frac{h^3 x}{12} < -\frac{h^3 x}{12} - \frac{h^3 y_n''^2}{4y_n'} < \frac{h^3 x}{12}$$

or

$$0 < -\frac{h^3 y_n''^2}{4y_n'} < \frac{h^3 x}{6}.$$

The inequality on the left suggests that $y_n' < 0$. By letting $-y_n' = z > 0$, the inequality on the right becomes

$$\frac{-h^3 y_n''^2}{-4z} < \frac{h^3 x}{6}$$

or

$$\frac{3y_n''^2}{2} < xz,$$

which implies $y_n' y_n''' > \frac{3}{2} y_n''^2$ as in the case of $y_n''' \geqslant 0$.
Therefore the conditions for $|E_{\mathrm{H}}| < |E_{\mathrm{T}}|$ to be satisfied can be incorporated in the inequalities $y_n' y_n''' > 0$ and $y_n' y_n''' > \frac{3}{2} y_n''^2$, or more simply as

$$y_n' y_n''' > \tfrac{3}{2} y_n''^2. \tag{5.1}$$

In a similar way it can be proved that the condition

$$y'_n y'''_n > \tfrac{9}{4} y''^2_n \tag{5.2}$$

is necessary for the inequality $|E_H| < |E_G|$ to be satisfied. Failure to meet these conditions mean that $|E_H| > |E_T|$ and $|E_H| > |E_G|$. Thus we have established the necessary and sufficient conditions for the HM formula to be more accurate than the trapezoidal and GM Euler method. In addition, we have the situation $E_H = 0$ and $y'_n y'''_n = 3y''^2_n$ in which case the HM Euler method is equivalent to a method of order three. Now from the differential equation we have $y' = f(x, y)$, $y'' = f_x + ff_y$ and $y''' = f_{xx} + 2ff_{xy} + f_x f_y + f^2 f_{yy} + ff_y^2$.

By substituting these quantities into Eq. (5.1) we have

$$ff_{xx} + 2f^2 f_{xy} + ff_x f_y + f^3 f_{yy} + f^2 f_y^2 > \tfrac{3}{2}[f_x^2 + f^2 f_y^2 + 2ff_x f_y].$$

Simplifying we have

$$ff_{xx} + 2f^2 f_{xy} - 2ff_x f_y + f^3 f_{yy} - \tfrac{1}{2} f^2 f_y^2 - \tfrac{3}{2} f_x^2 > 0. \tag{5.3}$$

Similarly by performing this substitution in Eq. (5.2) we obtain the condition

$$ff_{xx} + 2f^2 f_{xy} - \tfrac{7}{2} ff_x f_y + f^3 f_{yy} - \tfrac{5}{4} f^2 f_y^2 - \tfrac{9}{4} f_x^2 > 0, \tag{5.4}$$

i.e., the HM formula will produce smaller error than the GM formula when the function satisfies the condition (5.4). However we will not know whether this condition is satisfied throughout the region of integration until we have obtained the solution throughout the interval. In practice we would therefore use this method if the inequality is satisfied at the initial point. If this condition is not satisfied, other two methods are preferred.

Suppose we have $f(x, y) = f(y)$, a function of y only, then the inequality simplifies to

$$f^3 f_{yy} - \tfrac{5}{4} f^2 f_y^2 > 0$$

or

$$4ff_{yy} - 5f_y^2 > 0.$$

6. Numerical results

By using the Picard's iteration procedure, the application of the trapezoidal, GM and HM Euler formulae results in the following recurrence equations:

Trapezoidal: $y_{n+1} = y_n + \tfrac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$,

GM Euler [1]: $y_{n+1} = y_n + h\sqrt{f(x_n, y_n) f(x_{n+1}, y_{n+1})}$,

HM Euler: $y_{n+1} = y_n + 2h \dfrac{f(x_n, y_n) f(x_{n+1}, y_{n+1})}{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}$,

where y_{n+1} on the right-hand side is the last calculated value of the left-hand side which initially may be taken arbitrarily. The computation is carried out until two successive values of y_{n+1} are acceptably close to each other. While implementing the GM Euler formula in computer, efforts should be taken to make the function values inside sqrt function positive each time when we use this formula. This involves checking another condition before applying this formula. Moreover, if $f(x, y)$ is negative GM Euler formula changes as

$$y_{n+1} = y_n - h\sqrt{f(x_n, y_n) f(x_{n+1}, y_{n+1})}. \tag{6.1}$$

So this is the main drawback of this formula.

Example 1. Consider the problem of solving

$$y' = -y, \qquad y(0) = 1,$$

whose exact solution is $y(x) = \exp(-x)$. Here since $f(x, y)$ is negative, Eq. (6.1) is used for GM Euler formula. With stepsize $h = 0.1$, the errors (result obtained by the respective formulae − exact value) observed in these three formulae are shown in Table 1.

Table 1

x	Exact solution	Error (trapezoidal)	Error (GM Euler)	Error (HM Euler)
0.0	1.000000			
0.1	0.904837	$-0.755787 \cdot 10^{-4}$	$0.376105 \cdot 10^{-4}$	$0.150144 \cdot 10^{-3}$
0.2	0.818731	$-0.136733 \cdot 10^{-3}$	$0.680685 \cdot 10^{-4}$	$0.271738 \cdot 10^{-3}$
0.3	0.740818	$-0.185490 \cdot 10^{-3}$	$0.924468 \cdot 10^{-4}$	$0.368893 \cdot 10^{-3}$
0.4	0.670320	$-0.223756 \cdot 10^{-3}$	$0.111580 \cdot 10^{-3}$	$0.445068 \cdot 10^{-3}$
0.5	0.606531	$-0.253081 \cdot 10^{-3}$	$0.126183 \cdot 10^{-3}$	$0.503421 \cdot 10^{-3}$
0.6	0.548812	$-0.274718 \cdot 10^{-3}$	$0.137031 \cdot 10^{-3}$	$0.546694 \cdot 10^{-3}$
0.7	0.496585	$-0.290006 \cdot 10^{-3}$	$0.144660 \cdot 10^{-3}$	$0.577122 \cdot 10^{-3}$
0.8	0.449329	$-0.299871 \cdot 10^{-3}$	$0.149608 \cdot 10^{-3}$	$0.596851 \cdot 10^{-3}$
0.9	0.406570	$-0.305235 \cdot 10^{-3}$	$0.152290 \cdot 10^{-3}$	$0.607610 \cdot 10^{-3}$
1.0	0.367879	$-0.306875 \cdot 10^{-3}$	$0.153124 \cdot 10^{-3}$	$0.610918 \cdot 10^{-3}$

Table 2

x	Exact solution	Error (trapezoidal)	Error (GM Euler)	Error (HM Euler)
0.0	1.00000			
0.1	1.09545	$0.190496 \cdot 10^{-3}$	$0.950098 \cdot 10^{-4}$	$-0.476837 \cdot 10^{-6}$
0.2	1.18322	$0.302196 \cdot 10^{-3}$	$0.150800 \cdot 10^{-3}$	$-0.596046 \cdot 10^{-6}$
0.3	1.26491	$0.370860 \cdot 10^{-3}$	$0.185132 \cdot 10^{-3}$	$-0.715256 \cdot 10^{-6}$
0.4	1.34164	$0.414371 \cdot 10^{-3}$	$0.206947 \cdot 10^{-3}$	$-0.953674 \cdot 10^{-6}$
0.5	1.41421	$0.442147 \cdot 10^{-3}$	$0.220776 \cdot 10^{-3}$	$-0.953674 \cdot 10^{-6}$
0.6	1.48324	$0.459790 \cdot 10^{-3}$	$0.229597 \cdot 10^{-3}$	$-0.119209 \cdot 10^{-6}$
0.7	1.54919	$0.470757 \cdot 10^{-3}$	$0.235081 \cdot 10^{-3}$	$-0.119209 \cdot 10^{-5}$
0.8	1.61245	$0.477195 \cdot 10^{-3}$	$0.238299 \cdot 10^{-3}$	$-0.119209 \cdot 10^{-5}$
0.9	1.67332	$0.480294 \cdot 10^{-3}$	$0.239849 \cdot 10^{-3}$	$-0.119209 \cdot 10^{-5}$
1.0	1.73205	$0.481248 \cdot 10^{-3}$	$0.240326 \cdot 10^{-3}$	$-0.119209 \cdot 10^{-5}$

Even though the HM Euler formula gives errors which are slightly higher than the GM Euler formula we need not incorporate any additional conditions as in the GM Euler formula. Also it was observed that the number of iterations taken to converge is less for the HM Euler method compared to the GM Euler and trapezoidal methods.

Example 2. Consider the problem

$$y' = 1/y, \qquad y(0) = 1$$

whose exact solution is $y(x) = \sqrt{2x+1}$. With stepsize $h = 0.1$ the results are shown in Table 2.

The fact that the HM Euler formula produces more accurate results in this example coincides with the fact that the inequalities (5.1) and (5.2) are satisfied.

7. Conclusions

A new class of nonlinear one-step methods based on Euler's integration formula for the numerical solution of ordinary differential equations is presented. These methods are implicit, high-order, A-stable and, more importantly, L-stable for solving stiff differential equations. Accuracy and stability of the proposed methods and their applicability to ordinary differential equations is presented with supporting numerical evidence.

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