



ELSEVIER

Journal of Computational and Applied Mathematics 139 (2002) 239–251

JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

www.elsevier.com/locate/cam

A case of Toeplitz determinants and theta functions in frequency analysis

Vigdis Petersen, Haakon Waadeland*

*Department of Mathematical Sciences, Norwegian University of Science and Technology,
N-7491 Trondheim, Norway*

Received 4 December 2000; received in revised form 31 March 2001

Abstract

For a trigonometric signal an important problem is to find the unknown frequencies from a set of observations. This is called the frequency analysis problem.

One method is to construct, from measured values, a family of positive measures, depending upon one or more parameters. These define in turn an inner product, positive definite sequences of moments, and sequences of polynomials (or more generally rational functions), orthogonal on the unit circle. Asymptotic properties of the zeros then lead to the frequencies.

In recent variations of this method the moments are modified. If the new sequence also is positive definite we are back to the method just mentioned. But sometimes it is difficult to find out if it is positive definite or not.

A “strange-looking” modification, multiplication of the “moments” by R^{n^2} where $R \in (0, 1)$ was introduced very recently. This modification turned out to be rewarding in different surprising ways in all examples carried out, although at first the question of positive definiteness was unsettled. In the present paper, the positive definiteness is proved. Moreover, it is proved that for $\alpha > 2$ multiplication by $R^{|n|^\alpha}$ does not generally lead to a positive definite sequence © 2002 Elsevier Science B.V. All rights reserved.

MSC: Primary 42C05; Secondary 94A11

Keywords: Frequency analysis; Toeplitz determinant; Carathéodory function; Theta function

1. Background

For a given trigonometric signal we mean a signal of the form

$$G(t) = \sum_{j=-I}^I A_j e^{2\pi i f_j t}, \quad (1.1)$$

* Corresponding author.

E-mail addresses: vigdis@math.ntnu.no (V. Petersen), haakonwa@math.ntnu.no (H. Waadeland).

where A_j are the *amplitudes*, $A_{-j} = A_j \in \mathbb{R}$, and f_j the *frequencies*, $f_{-j} = -f_j$.

We could also accept complex amplitudes and still have a real signal, but then with phase angles.

The *frequency analysis problem* is to find the unknown frequencies from a set of observations.

Observations are made at times $m\Delta t$ for a suitable value of Δt . The signal values shall be called $x(m)$. We find what is called a *discretized form* of the signal, where we define $\omega_j := 2\pi f_j \Delta t$ (*modified frequencies*):

$$x(m) = \sum_{j=-I}^I A_j e^{i\omega_j m}, \quad m = 0, 1, 2, \dots \quad (1.2)$$

One way of solving the trigonometric frequency analysis problem goes as follows: A family of positive measures on the unit circle is constructed from the observations $x(m)$. Under certain conditions these measures give rise to inner products and to a family of moment sequences and finally to sequences of (monic) orthogonal polynomials, the *Szegő polynomials*. It then turns out that the zeros of the Szegő polynomials determine the frequencies [5,9]. The method has roots going back to Wiener and Levinson [8,11].

In these papers, the following family of measures on the unit circle was used:

$$\frac{d\Psi^{(N)}(\theta)}{d\theta} := \frac{1}{2\pi} \left| \sum_{m=0}^{N-1} x(m) e^{-im\theta} \right|^2. \quad (1.3)$$

It is constructed from the observations $x(m)$, $m = 0, 1, 2, \dots, N-1$. (This is only one out of an infinity of possible measures that can be used [7].)

Let, for a fixed N , $\{\mu_n^{(N)}\}_{-\infty}^{\infty}$ be the sequence of moments, i.e.

$$\mu_n^{(N)} = \int_{-\pi}^{\pi} e^{-in\theta} d\Psi^{(N)}(\theta), \quad n = 0, \pm 1, \pm 2, \dots$$

Then, for any fixed N the numbers $\mu_n^{(N)}/N$ are also moments. A crucial point in the use of this measure in frequency analysis is that $\Psi^{(N)}/N$ converges weakly to the discrete measure with support in the points $e^{i\omega_j}$ with masses A_j^2 . For the measure (1.3) the sequence of moments is positive definite, meaning that $\mu_{-n}^{(N)} = \bar{\mu}_n^{(N)}$ (or without conjugate since we have a real signal) and that certain Toeplitz determinants are all positive:

$$\Delta_n = \begin{vmatrix} \mu_0^{(N)} & \mu_{-1}^{(N)} & \mu_{-2}^{(N)} & \cdot & \cdot & \mu_{-n}^{(N)} \\ \mu_1^{(N)} & \mu_0^{(N)} & \mu_{-1}^{(N)} & \cdot & \cdot & \mu_{-n+1}^{(N)} \\ \mu_2^{(N)} & \mu_1^{(N)} & \mu_0^{(N)} & \cdot & \cdot & \mu_{-n+2}^{(N)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_n^{(N)} & \mu_{n-1}^{(N)} & \mu_{n-2}^{(N)} & \cdot & \cdot & \mu_0^{(N)} \end{vmatrix} > 0. \quad (1.4)$$

An equivalent condition for positive definiteness of the sequence is for the function

$$L_0^{(N)}(z) = \mu_0^{(N)}/N + 2 \sum_{n=1}^{\infty} (\mu_n^{(N)}/N) z^n \quad (1.5)$$

to be a Carathéodory function, not of the form

$$\sum_{j=-I}^I K_j^2 \frac{e^{i\omega_j} + z}{e^{i\omega_j} - z}. \quad (1.6)$$

A Carathéodory function is a function, analytic in the open unit disk, and mapping it into the open right half plane.

The Szegő polynomials may be determined in the “usual” way, i.e. by using the Gram–Schmidt process. Based upon this, different other ways are established.

From a practical point of view the Levinson algorithm [8] is often used.

Another way of computing them is by using a certain continued fraction expansion, called Perron–Carathéodory fraction (PC-fraction), introduced in [3]. The Szegő polynomials are denominators of the odd order approximants of this particular continued fraction, when it is constructed in a certain way from the signal.

For some theoretical purposes the following well known determinant formula for Szegő polynomials $\rho_n^{(N)}(z)$ is the most useful one.

$$\rho_n^{(N)}(z) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0^{(N)}/N & \mu_1^{(N)}/N & \cdot & \cdot & \cdot & \mu_n^{(N)}/N \\ \mu_1^{(N)}/N & \mu_0^{(N)}/N & \cdot & \cdot & \cdot & \mu_{n-1}^{(N)}/N \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{n-1}^{(N)}/N & \mu_{n-2}^{(N)}/N & \cdot & \cdot & \cdot & \mu_1^{(N)}/N \\ 1 & z & \cdot & \cdot & \cdot & z^n \end{vmatrix}. \quad (1.7)$$

Here Δ_{n-1} is the $n \times n$ Toeplitz determinant (1.4) in the upper left corner, see e.g. [4].

We now recall the following facts from [5]:

Let $n_0 = 2I + L$ denote the number of frequencies. Here $L = 1$ for $A_0 \neq 0$, else $L = 0$. Take $n = n_0$ and let $N \rightarrow \infty$. Then (1.7) will tend to

$$(z - 1)^L \prod_{j=-I}^I (z - e^{i\omega_j})(z - e^{-i\omega_j}) \quad (1.8)$$

uniformly on the closed unit disk.

If $n > n_0$, the limit of (1.7) normally does not exist, but there will always be a subsequence, converging uniformly on the closed disk to a monic polynomial of degree n . For any such subsequence of N -values the polynomial (1.8) is a factor. This is the key to the solution of the frequency analysis problem.

For a fixed $n > n_0$ numerator and denominator in (1.7) both tend to zero. From the weak convergence of $\psi^{(N)}(\theta)/N$ it follows:

$$\lim_{N \rightarrow \infty} \mu_n^{(N)}/N =: \mu_n = \sum_{j=-I}^I A_j^2 e^{ni\omega_j}. \quad (1.9)$$

The sequence $\{\mu_n\}$ is *not* positive definite, but n_0 -definite, meaning that the conditions for positive definiteness hold up to and including $n = n_0$, but not further [6], and

$$\mu_n = \sum_{j=-I}^I A_j^2 e^{i\omega_j n} = A_0^2 + 2 \sum_{j=1}^I A_j^2 \cos n\omega_j. \quad (1.10)$$

However, for any fixed $R \in (0, 1)$ the sequence

$$\{\mu_n R^{|n|}\}_{n=-\infty}^{\infty} \quad (1.11)$$

is positive definite, and in [6] used to present a way of solving the frequency analysis problem: With $\mu_n^{(N)}/N$ replaced by $\mu_n R^{|n|}$ in (1.7) with μ_n given by (1.10), the limit as $R \rightarrow 1$ exists for any n . For $n \geq n_0$ the limit polynomial will contain (1.8) as a factor. This process is called the R -process and leads to the unknown frequencies without the trouble of going through subsequences.

Very recently was suggested [10] to use the sequence

$$\{\mu_n R^{n^2}\}_{n=-\infty}^{\infty}, \quad 0 < R < 1, \quad (1.12)$$

i.e. to replace the elements $\mu_n^{(N)}/N$ in (1.7) by $\mu_n R^{n^2}$. It turned out that in all examples this modification still produced the frequencies. Some interesting and useful multiplicity properties were proved. The question of positive definiteness was at first an open question. In the present paper this question will be answered affirmatively.

2. The main theorem and a lemma

The method based upon using the sequence (1.12) is justified by the following theorem:

Theorem 1. *Let A_j , $j = 0, 1, 2, \dots, I$ be real numbers, all except possibly A_0 different from 0. Moreover, let ω_j , $j = 1, 2, \dots, I$ be distinct real numbers in $(0, \pi)$ and R an arbitrary number in $(0, 1)$. Then the sequence*

$$\left\{ \left(A_0^2 + 2 \sum_{j=1}^I A_j^2 \cos(n\omega_j) \right) R^{n^2} \right\}_{n=-\infty}^{\infty} \quad (2.1)$$

is positive definite.

2.1. Preparation for a proof

Using the notation from (1.10) the n th term in the sequence shall be denoted $\mu_n R^{n^2}$. The theorem will be proved if we can establish that the function

$$\mu_0 + 2 \sum_{n=1}^{\infty} (\mu_n R^{n^2}) z^n \quad (2.2)$$

is a Carathéodory function. Now, as easily seen, the function (2.2) is an entire function. It may also be written as a sum of entire functions:

$$A_0^2 \left(1 + 2 \sum_{n=1}^{\infty} R^{n^2} z^n \right) + 2 \sum_{j=1}^I A_j^2 \left(1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\omega_j) z^n \right). \quad (2.3)$$

If we can prove that for any $\omega \in [0, \pi)$ the function

$$1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\omega) z^n$$

is a Carathéodory function, then (2.2) is also a Carathéodory function. It is an entire function and is thus not of the form (1.6), since that function has poles on the unit circle.

The proof of Theorem 1 thus boils down to proving the following:

Lemma 2. For any $\omega \in [0, \pi]$ and $R \in (0, 1)$ the function

$$F_R(z) = 1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\omega) z^n \quad (2.4)$$

is a Carathéodory function, or equivalently

$$\{R^{n^2} \cos(n\omega)\}_{-\infty}^{\infty} \quad (2.5)$$

is positive definite.

Inclusion of $\omega = 0$ takes care of the case $A_0 \neq 0$ in the proofs. The proof of Theorem 1 comes in Section 4.

3. Multiplication by $R^{|n|^\alpha}$, where $\alpha > 2$

We know that for $\alpha = 1$ the sequence $\{R^{|n|^\alpha} \cos(n\omega)\}$ is positive definite [6]. For other α -values less than 2, we have no proof of positive definiteness. Numerical experiments strongly indicate that $\alpha = 2$ is a border value between the cases where $\{R^{|n|^\alpha} \cos(n\omega)\}$ is positive definite or not. The purpose of the present section is to prove that for any $\alpha > 2$ the sequence is not always positive definite. More precisely:

Proposition 3. For $\alpha > 2$ there exist (ω, R) -values in $(0, \pi) \times (0, 1)$ such that the function

$$F_R(z) = 1 + 2 \sum_{n=1}^{\infty} R^{|n|^\alpha} \cos(n\omega) z^n \quad (3.1)$$

is not a Carathéodory function, or equivalently

$$\{R^{|n|^\alpha} \cos(n\omega)\}_{-\infty}^{\infty} \quad (3.2)$$

is not positive definite.

Proof. In the proof we consider a simple example to show that the sequence (3.2) is not always positive definite.

We consider the signal (1.2) with $I = 1$, $A_0 = 0$ and $A_1 = A_{-1} = \frac{1}{\sqrt{2}}$:

$$x(m) = \frac{1}{\sqrt{2}}e^{im\omega} + \frac{1}{\sqrt{2}}e^{-im\omega} = \sqrt{2}\cos m\omega.$$

From this signal we find that the limit of the moments is $2A_1^2\cos n\omega = \cos n\omega$. We define

$$\mu_n^{(\alpha)} = \mu_{-n}^{(\alpha)} := \cos(n\omega)R^{|n|^\alpha}.$$

The sequence $\{\cos(n\omega)R^{|n|^\alpha}\}_{n=-\infty}^\infty$ is positive definite iff

$$(i) \mu_{-n}^{(\alpha)} = \mu_n^{(\alpha)},$$

$$(ii) \Delta_n^{(\alpha)} > 0.$$

The first part is immediately seen from (3.2). In the second part we choose $n = 3$ in $\Delta_{n-1}^{(\alpha)}$. In the rest of the proof we use the notation $\cos \omega =: x$. In this situation the determinant in (1.4) with the elements $\cos(n\omega)R^{|n|^\alpha}$ is

$$\begin{aligned} \Delta_2^{(\alpha)} &= \begin{vmatrix} 1 & xR & (2x^2 - 1)R^{2^\alpha} \\ xR & 1 & xR \\ (2x^2 - 1)R^{2^\alpha} & xR & 1 \end{vmatrix} \\ &= (1 - (2x^2 - 1)R^{2^\alpha})(1 - 2x^2R^2 + (2x^2 - 1)R^{2^\alpha}). \end{aligned} \quad (3.3)$$

We look at the factors for $x^2 \in (0, 1)$, $R \in (0, 1)$ and $\alpha > 2$. The first factor

$$1 - (2x^2 - 1)R^{2^\alpha}$$

is always positive.

For the second factor

$$1 - 2x^2R^2 + (2x^2 - 1)R^{2^\alpha},$$

we have

$$(1 - 2x^2R^2 + (2x^2 - 1)R^{2^\alpha})_{x^2=1} = 1 - 2R^2 + R^{2^\alpha} =: g(R).$$

Then

$$g'(R) = -4R + 2^\alpha R^{2^\alpha-1}.$$

Hence

$$g'(R) < 0 \quad \text{for } R < \left(\frac{1}{2^{\alpha-2}}\right)^{1/2^\alpha-2} = \frac{1}{2^{(\alpha-2)/(2^\alpha-2)}}$$

and

$$g'(R) > 0 \quad \text{for } R > \frac{1}{2^{(\alpha-2)/(2^\alpha-2)}}.$$

Observe that $0 < (1/2^{(\alpha-2)/(2^2-2)}) < 1$ for $\alpha > 2$. Then $g(R)$ has a minimum value for $R(\alpha) = 1/2^{(\alpha-2)/(2^2-2)}$. Since $g(1) = 0$, the minimum value is negative.

This proves that $g(R) < 0$ in a subinterval of $(0, 1)$. By continuity we then have that

$$1 - 2x^2R^2 + (2x^2 - 1)R^{2^2} < 0$$

for any fixed R in this interval and sufficiently small $(1 - x^2)$. This completes the proof of Proposition 3. To illustrate Proposition 3 we show the graphs of $\Delta_2^{(\alpha)}$ for $\alpha = 2, 3, 10$. (Figs. 1–3).

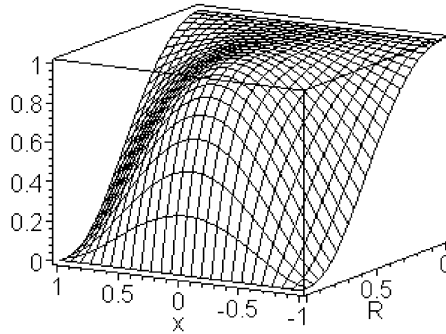


Fig. 1. Toeplitz determinant for alpha = 2.

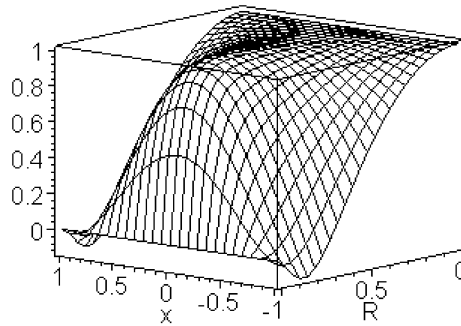


Fig. 2. Toeplitz determinant for alpha = 3.

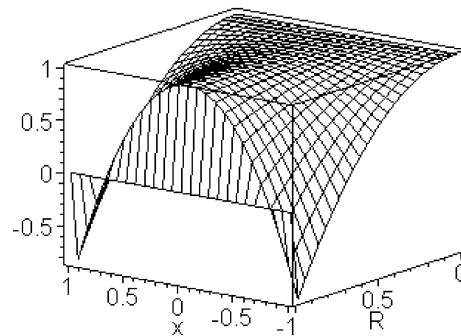


Fig. 3. Toeplitz determinant for alpha = 10.

Numerical experiments on zeros of Szegő polynomials for $\alpha = 2$ are included in [10].

4. Proof of Theorem 1

As seen in Section 2, the proof is now reduced to proving that the function $F_R(z)$ in Lemma 2 is a Carathéodory function.

Since for fixed ω and R the function F_R is an entire function and thus in particular analytic in the closed unit disk, it follows from the maximum modulus principle that F_R is a Carathéodory function if and only if

$$\min_{|z|=1} \Re F_R(z) \geq 0, \quad (4.1)$$

i.e.

$$1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\omega) \cos(n\theta) \geq 0 \quad (4.2)$$

for all $\theta \in [0, 2\pi)$. Moreover, since we are aiming at a result valid for all $\omega \in [0, \pi]$ (and then automatically for all $\omega \in [0, 2\pi)$) we may just as well, for fixed R , regard the left-hand side of (4.2) as a function of the two real variables θ and ω . The statement of the lemma is equivalent to the statement

$$\min_{(\theta, \omega) \in [0, 2\pi) \times [0, 2\pi)} \left(1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\omega) \cos(n\theta) \right) \geq 0. \quad (4.3)$$

The existence of a minimum is obvious, since the function is continuous on the compact set $[0, 2\pi] \times [0, 2\pi]$. (The inclusion of $\theta = 2\pi$ or $\omega = 2\pi$ is inessential because of the periodicity.)

The left-hand side of (4.3) may also be written

$$1 + \sum_{n=1}^{\infty} R^{n^2} \cos(n(\theta + \omega)) + \sum_{n=1}^{\infty} R^{n^2} \cos(n(\theta - \omega)). \quad (4.4)$$

Now, let u_0 be such that

$$\sum_{n=1}^{\infty} R^{n^2} \cos(nu)$$

for all real u is smallest for $u = u_0$. The existence of u_0 follows from the fact that the function is continuous on a compact set. Then the first sum in (4.4) is smallest for

$$\theta + \omega = u_0 \quad (4.5a)$$

and the second sum in (4.4) is smallest for

$$\theta - \omega = u_0. \quad (4.5b)$$

The equalities (4.5a) and (4.5b) are simultaneously true for $\omega = 0$ and $\theta = u_0$, where we so far do not know the value of u_0 . The problem of proving the lemma is now reduced to proving that

$$1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\theta) \geq 0 \quad (4.6)$$

for all $\theta \in [0, 2\pi]$ or, without loss of generality, all $\theta \in [0, \pi]$.

We shall now use a little dash of the theory of theta functions, and recall from [2, p. 2] the definitions

$$\vartheta_3(z, t) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz), \quad (4.7a)$$

$$\vartheta_4(z, t) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} (-1)^n \cos(2nz), \quad (4.7b)$$

where $q = e^{i\pi t}$ and $\Im t > 0$ (such that $|q| < 1$). In our case $q = R$, and t can be taken to be purely imaginary, $t = i\tau$, where $\tau > 0$. Actually

$$\tau = \frac{1}{\pi} \log(1/R).$$

Finally, we have for z in (4.7) the real value $\theta/2$, and thus

$$\vartheta_3(\theta/2, i\tau) = 1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\theta), \quad (4.8)$$

i.e. the left-hand side of (4.6).

A simple relation between ϑ_3 and ϑ_4 is

$$\vartheta_3(z, t) = \vartheta_4(z \pm \pi/2, t). \quad (4.9)$$

For $\vartheta_4(z, t)$ the following product representation holds:

$$\vartheta_4(z, t) = \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos(2z) + q^{4n-2}) \prod_{n=1}^{\infty} (1 - q^{2n}) \quad (4.10)$$

[2, Thm. 6, p. 42]. Hence

$$\begin{aligned} 1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\theta) &= \vartheta_4\left(\frac{\theta \pm \pi}{2}, i\tau\right) \\ &= \prod_{n=1}^{\infty} (1 + 2R^{2n-1} \cos \theta + R^{4n-2}) \times \prod_{n=1}^{\infty} (1 - R^{2n}). \end{aligned} \quad (4.11)$$

From this follows immediately that the minimal value is taken for $\theta = \pm \pi$, and we thus have

$$\begin{aligned} \min_{0 \leq \omega \leq \pi} \left(\min_{|z| \leq 1} F_R(z) \right) &= \vartheta_4(\pi, i\tau) = \vartheta_4(0, i\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n R^{n^2} \\ &= \prod_{n=1}^{\infty} (1 - R^{2n-1})^2 \prod_{n=1}^{\infty} (1 - R^{2n}). \end{aligned} \quad (4.12)$$

This is positive, since the products converge. Lemma 2 is thus proved, and hence also Theorem 1. Observe that when R increases from 0 to 1, the minimum of $F_R(z)$ decreases strictly from 1 to 0.

We conclude this section by presenting explicitly the measure $\nu(\theta)$ producing the moments (1.12). It is absolutely continuous and given by

$$\begin{aligned} \frac{d\nu(\theta)}{d\theta} &= \frac{1}{2\pi} \left[A_0^2 \left(1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\theta) \right) \right] \\ &\quad + \frac{1}{2\pi} \left[2 \sum_{j=1}^I A_j^2 \left(1 + 2 \sum_{n=1}^{\infty} R^{n^2} \cos(n\omega_j) \cos(n\theta) \right) \right], \end{aligned} \quad (4.13)$$

see e.g. [6, (3.7)]. Using the rewriting in (4.4) and the notation in (4.7) we may write the measure in the following way:

$$\frac{d\nu(\theta)}{d\theta} = \frac{1}{2\pi} [A_0^2 \vartheta_3(\theta/2, i\tau)] + \frac{1}{2\pi} \left[\sum_{j=1}^I A_j^2 (\vartheta_3((\theta + \omega)/2, i\tau) + \vartheta_3((\theta - \omega)/2, i\tau)) \right]. \quad (4.14)$$

5. A different approach

By using another property of the theta-function ϑ_4 we may find a different expression for the minimum of $F_R(z)$, i.e. for the right-hand side of (4.12). From [1, p. 569] we have the following definitions (slightly rephrased):

Let m and m_1 be non-negative numbers, such that

$$m + m_1 = 1. \quad (5.1)$$

(Then m and m_1 are both in the closed interval $[0, 1]$.) Here m is called the *parameter*, m_1 the *complementary parameter*. The *quarter-periods* K and K' are defined by

$$K(m) = K = \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}}, \quad (5.2a)$$

$$iK'(m) = iK' = i \int_0^{\pi/2} \frac{d\theta}{(1 - m_1 \sin^2 \theta)^{1/2}}. \quad (5.2b)$$

These are essential in the theory of Jacobian elliptic functions.

In [1, p. 579] we find the formula 16.38.8. In our notation from the previous section it is

$$\vartheta_4(0, i\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n R^{n^2} = (2m_1^{1/2} K/\pi)^{1/2}, \quad (5.3)$$

where

$$R = e^{-\pi K'/K} = e^{-\pi\tau}. \quad (5.4)$$

When m_1 increases from 0 to 1, then $K'(m)$ increases strictly from $\pi/2$ to ∞ , and $K(m)$ decreases strictly from ∞ to $\pi/2$. Hence when m_1 increases from 0 to 1, then R decreases strictly from 1 to 0.

Next we have

$$\begin{aligned} 2m_1^{1/2} K/\pi &= \frac{2}{\pi} \int_0^{\pi/2} \frac{m_1^{1/2} d\theta}{(1 - (1 - m_1) \sin^2 \theta)^{1/2}} \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{(\cos^2 \theta/m_1 + \sin^2 \theta)^{1/2}}. \end{aligned} \quad (5.5)$$

When m_1 increases from 0 to 1, this expression increases strictly from 0 to 1.

In conclusion we have: the sum

$$1 + 2 \sum_{n=1}^{\infty} (-1)^n R^{n^2}$$

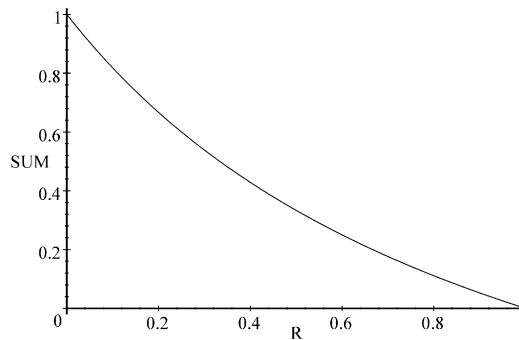
decreases strictly from 1 to 0 when R increases from 0 to 1. This is (of course) the same as we found in the previous section with another representation.

Below are illustrated the graphs of the function

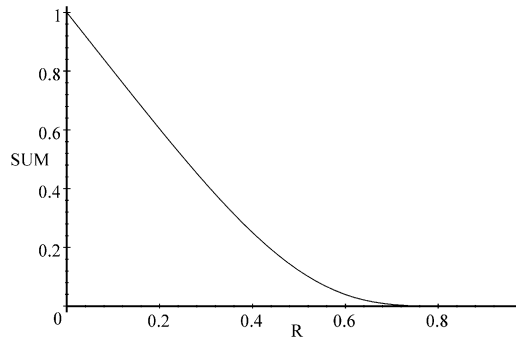
$$\text{SUM} := 1 + 2 \sum_{n=1}^{\infty} (-1)^n R^{n^2}$$

in the three cases:

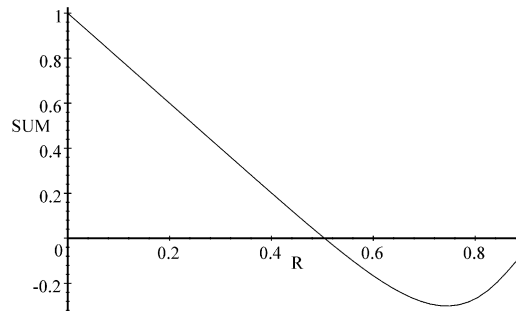
$\alpha = 1$, where the sum is equal to $(1 - R)/(1 + R)$,



$\alpha = 2$, where the sum is equal to $\vartheta_4(0, i\tau)$,



and $\alpha = 3$ (compare Fig. 2 in Section 3).



References

- [1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970.
- [2] R. Bellman, *A Brief Introduction to Theta Functions*, Holt Rinehart and Winston, New York, 1961.
- [3] W.B. Jones, O. Njåstad, W.J. Thron, Continued fractions associated with trigonometric and other strong moment problems, *Constr. Approx.* 2 (1986) 197–211.
- [4] W.B. Jones, O. Njåstad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle, *Bull. London Math. Soc.* 21 (1989) 113–152.
- [5] W.B. Jones, O. Njåstad, W.J. Thron, H. Waadeland, Szegő polynomials applied to frequency analysis, *J. Comput. Appl. Math.* 46 (1993) 217–228.
- [6] W.B. Jones, O. Njåstad, H. Waadeland, An alternative way of using Szegő polynomials in frequency analysis, in: S.C. Cooper, W.J. Thron (Eds.), *Continued Fractions and Orthogonal Functions*, Marcel Dekker, New York, 1994, pp. 141–152.
- [7] W.B. Jones, V. Petersen, Frequency analysis, Szegő polynomials and PC-continued fractions, *Acta Applicandae Mathematicae* 61 (2000) 149–174.
- [8] N. Levinson, The Wiener RMS (root mean square) error criterion in filter design and prediction, *J. Math. Phys.* 25 (1947) 261–278.

- [9] K. Pan, E.B. Saff, Asymptotics for Zeros of Szegő polynomials associated with trigonometric polynomial signals, *J. Approx. Theory* 71 (1992) 239–251.
- [10] V. Petersen, Modification of a Method Using Szegő Polynomials in Frequency Analysis: The V-process, *J. Comput. Appl. Math.* 133 (2001) 535–544.
- [11] N. Wiener, Extrapolation, interpolation and smoothing of stationary time series, Technology Press, MIT and Wiley, New York, 1949.