Journal of Computational and Applied Mathematics 228 (2009) 174–181

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Periodic solutions for a kind of Liénard equation with a deviating argument[☆]

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ARTICLE INFO

Article history:
Received 21 November 2007
Received in revised form 30 May 2008

Keywords:
Liénard equation
Deviating argument
Periodic solution
Existence
Uniqueness
Coincidence degree

ABSTRACT

In this paper, the Liénard equation with a deviating argument

$$x''(t)+f_1(t,x(t))x'(t)+f_2(x(t))(x'(t))^2+g(t,x(t-\tau(t)))=p(t)$$

is studied. By applying the coincidence degree theory, we obtain some new results on the existence and uniqueness of T-periodic solutions to this equation. Our results improve and extend some existing ones in the literature.

1. Introduction

Consider the Liénard equation with a deviating argument of the form

$$x''(t)+f_1(t,x(t))x'(t)+f_2(x(t))(x'(t))^2+g(t,x(t-\tau(t)))=p(t), \tag{1.1}$$

where $f_2,\ \tau, p: R\to R$ and $f_1, g: R\times R\to R$ are continuous functions, τ and p are T-periodic, f_1 and g are T-periodic in their first argument, and $T>0$.

As a model comes form physics, mechanics and engineering (for example, see [1–4]), Eq. (1.1) has been the object of intensive analysis by numerous authors. In particular, there have been extensive results on the existence and uniqueness of periodic solutions to Eq. (1.1) with $f_2(x)\equiv 0$ (see [5–12]).

However, for the existence and uniqueness of periodic solutions to Eq. (1.1) without $f_2(x)\equiv 0$, the results are scarce. Thus, it is worthwhile to study Eq. (1.1) in this case.

The main purpose of this paper is to establish sufficient conditions ensuring the existence and uniqueness of T-periodic solutions of Eq. (1.1) without $f_2(x)\equiv 0$. By using an illustrative example, we show that our results improve the main results obtained in [9–12].

☆ This work was supported by grant (06JJ2063, 07JJ46001) from the Scientific Research Fund of Hunan Provincial Natural Science Foundation of China, and the Scientific Research Fund of Hunan Provincial Education Department of China (08C616).

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doi:10.1016/j.cam.2008.09.006

2. Preliminary results

In this section, we give some technical yet elementary results that will serve us well in the sections to follow.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = \left(\int_0^T |x(t)|^k \mathrm{d}t\right)^{1/k}, \qquad |x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Let

$$X = \{x | x \in C^1(R,\ R),\ x(t+T) = x(t),\ \text{for all } t \in R\}$$

and

$$Y = \{x | x \in C(R,\ R),\ x(t+T) = x(t),\ \text{for all } t \in R\}$$

be two Banach spaces with the norms

$$\|x\|_X = \max\{|x|_\infty,\ |x'|_\infty\}, \quad \text{and} \quad \|x\|_Y = |x|_\infty.$$

Define a linear operator $L : D(L) \subset X \longrightarrow Y$ by setting

$$D(L) = \{x | x \in X, x'' \in C(R,\ R)\}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{2.1}$$

We also define a nonlinear operator $N : X \longrightarrow Y$ by setting

$$Nx = -f_1(t, x(t))x'(t) - f_2(x(t))(x'(t))^2 - g(t, x(t-\tau(t))) + p(t). \tag{2.2}$$

It is easy to see that

$$\text{Ker } L = R, \quad \text{and} \quad \text{Im } L = \left\{x | x \in Y,\ \int_0^T x(s)\mathrm{d}s = 0\right\}.$$

Thus the operator L is a Fredholm operator with index zero.

Define the continuous projector $P : X \longrightarrow \text{Ker } L$ and the averaging projector $Q : Y \longrightarrow Y$ by setting

$$Px(t) = x(0) = x(T)$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)\mathrm{d}s.$$

Hence, $ImP = \text{Ker } L$ and $\text{Ker } Q = \text{Im } L$. Denoting by $L_P^{-1} : \text{Im } L \longrightarrow D(L) \cap \text{Ker } P$ the inverse of $L|_{D(L)\cap \text{Ker } P}$, we have

$$L_P^{-1}y(t) = -\frac{t}{T}\int_0^T (t-s)y(s)\mathrm{d}s + \int_0^t (t-s)y(s)\mathrm{d}s. \tag{2.3}$$

It is convenient to introduce the following assumption.

(A_0) Assume that there exist nonnegative constants C_1 and C_2 such that

$$|f_1(t, x)| \le C_1, \quad \text{for all } t, x \in R$$

and

$$f_2 \in C^1(R, R), f_2'(x) \le 0, |f_2(x_1) - f_2(, x_2)| \le C_2|x_1 - x_2|, \quad f_2(0) = 0,$$

for all $x, x_1, x_2 \in R$.

In view of (2.1) and (2.2), the operator equation $Lx = \lambda Nx$ is equivalent to the following equation

$$x'' + \lambda[f_1(t, x(t))x'(t) + f_2(x(t))(x'(t))^2 + g(t, x(t-\tau(t)))] = \lambda p(t), \quad \lambda \in (0, 1). \tag{2.4}$$

For convenience of use, we introduce the Continuation Theorem [15] as follows.

Lemma 2.1. *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \longrightarrow Y$ is a Fredholm operator with index zero and $N : X \longrightarrow Y$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0,\ 1)$;*

(2) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(3) *The Brouwer degree*

$$\deg\{QN,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0,$$

then equation $Lx = Nx$ has at least one solution on $\overline{\Omega}$.

Lemma 2.2. *If $x \in C^2(R, R)$ with $x(t+T) = x(t)$, then*

$$|x'(t)|_2^2 \le \left(\frac{T}{2\pi}\right)^2 |x''(t)|_2^2. \tag{2.5}$$

Proof. Lemma 2.2 is a direct consequence of Wirtinger inequality, and see [13,14] for its proof. □

Lemma 2.3. *Suppose that there exists a constant $d > 0$ such that*
(A_1) *$x(g(t,x) - p(t)) < 0$, for all $t \in R$, $|x| \ge d$.*
If $x(t)$ is a T-periodic solution of (2.4), then

$$|x|_\infty \le d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.6}$$

Proof. Let $x(t)$ be a T-periodic solution of (2.4). Set

$$x(t_1) = \max_{t\in R} x(t), \qquad x(t_2) = \min_{t\in R} x(t), \quad \text{where } t_1, t_2 \in R,$$

then we have

$$x'(t_1) = 0, \qquad x''(t_1) \le 0, \quad \text{and} \quad x'(t_2) = 0, \qquad x''(t_2) \ge 0.$$

It follows from (2.4) that

$$g(t_1, x(t_1 - \tau(t_1))) - p(t_1) \ge 0 \quad \text{and} \quad g(t_2, x(t_2 - \tau(t_2))) - p(t_2) \le 0.$$

In view of (A_1), we obtain

$$x(t_1 - \tau(t_1)) < d \quad \text{and} \quad x(t_2 - \tau(t_2)) > -d.$$

Since $x(t - \tau(t))$ is a continuous function on R, it follows that there exists a constant $\xi \in R$ such that

$$|x(\xi - \tau(\xi))| \le d.$$

Let $\xi - \tau(\xi) = mT + t_0$, where $t_0 \in [0, T]$, and m be an integer. Then, we have

$$|x(t)| = \left|x(t_0) + \int_{t_0}^{t} x'(s)\mathrm{d}s\right| \le d + \int_{t_0}^{t} |x'(s)|\mathrm{d}s, \quad t \in [t_0,\ t_0 + T],$$

and

$$|x(t)| = |x(t-T)| = \left|x(t_0) - \int_{t-T}^{t_0} x'(s)\mathrm{d}s\right| \le d + \int_{t-T}^{t_0} |x'(s)|\mathrm{d}s, \quad t \in [t_0, t_0 + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned}
|x|_\infty &= \max_{t\in[t_0,\ t_0+T]} |x(t)| \\
&\le \max_{t\in[t_0,\ t_0+T]} \left\{d + \frac{1}{2}\left(\int_{t_0}^{t} |x'(s)|\mathrm{d}s + \int_{t-T}^{t_0} |x'(s)|\mathrm{d}s\right)\right\} \\
&= \max_{t\in[t_0,\ t_0+T]} \left\{d + \frac{1}{2}\int_{t-T}^{t} |x'(s)|\mathrm{d}s\right\} \\
&= d + \frac{1}{2}\int_0^T |x'(s)|\mathrm{d}s \\
&\le d + \frac{1}{2}\sqrt{T}|x'|_2.
\end{aligned} \tag{2.7}$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Suppose that* (A_0) *and* (A_1) *hold, and the following condition is satisfied:*
(A_2) *There exists a nonnegative constant* b *such that*

$$C_1\frac{T}{2\pi}+b\frac{T^2}{4\pi}<1,\quad \text{and}\quad |g(t,x_1)-g(t,x_2)|\le b|x_1-x_2|,\quad \text{for all } t,x_1,x_2\in R.$$

If $x(t)$ *is a* T*-periodic solution of Eq.* (1.1), *then*

$$|x'|_\infty\le\frac{1}{2}\frac{[bd+\max\{|g(t,0)|:0\le t\le T\}+|p|_\infty]T}{1-(C_1\frac{T}{2\pi}+b\frac{T^2}{4\pi})}:=D. \tag{2.8}$$

Proof. Let $x(t)$ be a T-periodic solution of Eq. (1.1). From (A_1), we can easily show that (2.7) also holds. Multiplying Eq. (1.1) by $x''(t)$ and integrating it from 0 to T, in view of (2.5) and (2.7), (A_0), (A_2) and the inequality of Schwarz, we have

$$\begin{aligned}
|x''|_2^2 &= -\int_0^T f_1(t,x(t))x'(t)x''(t)\mathrm{d}t-\int_0^T f_2(x(t))(x'(t))^2x''(t)\mathrm{d}t-\int_0^T g(t,x(t-\tau(t)))x''(t)\mathrm{d}t+\int_0^T p(t)x''(t)\mathrm{d}t\\
&= -\int_0^T f_1(t,x(t))x'(t)x''(t)\mathrm{d}t+\frac{1}{3}\int_0^T f_2'(x(t))(x'(t))^4\mathrm{d}t-\int_0^T g(t,x(t-\tau(t)))x''(t)\mathrm{d}t+\int_0^T p(t)x''(t)\mathrm{d}t\\
&\le C_1\frac{T}{2\pi}|x''|_2^2+\int_0^T[|g(t,x(t-\tau(t)))-g(t,0)|+|g(t,0)|]\cdot|x''(t)|\mathrm{d}t+\int_0^T|p(t)|\cdot|x''(t)|\mathrm{d}t\\
&\le C_1\frac{T}{2\pi}|x''|_2^2+b\int_0^T|x(t-\tau(t))|\cdot|x''(t)|\mathrm{d}t+\int_0^T|g(t,0)|\cdot|x''(t)|\mathrm{d}t+\int_0^T|p(t)|\cdot|x''(t)|\mathrm{d}t\\
&\le C_1\frac{T}{2\pi}|x''|_2^2+b|x|_\infty\sqrt{T}|x''|_2+[\max\{|g(t,0)|:0\le t\le T\}+|p|_\infty]\sqrt{T}|x''|_2\\
&\le\left(C_1\frac{T}{2\pi}+b\frac{T^2}{4\pi}\right)|x''|_2^2+[bd+\max\{|g(t,0)|:0\le t\le T\}+|p|_\infty]\sqrt{T}|x''|_2,
\end{aligned} \tag{2.9}$$

which, together with (A_2), implies that

$$|x''|_2\le\frac{[bd+\max\{|g(t,0)|:0\le t\le T\}+|p|_\infty]\sqrt{T}}{1-(C_1\frac{T}{2\pi}+b\frac{T^2}{4\pi})}. \tag{2.10}$$

Since $x(0)=x(T)$, there exists a constant $\zeta\in[0,T]$ such that

$$x'(\zeta)=0,$$

by using a similar argument as that in the proof of (2.9), we have

$$|x'(t)|_\infty\le\frac{1}{2}\sqrt{T}|x''|_2. \tag{2.11}$$

Thus, in view of (2.10) and (2.11), we get

$$|x'|_\infty\le\frac{1}{2}\frac{[bd+\max\{|g(t,0)|:0\le t\le T\}+|p|_\infty]T}{1-(C_1\frac{T}{2\pi}+b\frac{T^2}{4\pi})}:=D.$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Suppose that* (A_1) *holds, and the following condition is satisfied:*
(A_3) *Suppose that* (A_0) *hold,* $f_1(t,x)\equiv f_1(t)$ *for all* $t,x\in R$, $g(t,x)$ *is a strictly monotone decreasing function in* x, *and there exists a nonnegative constant* b *such that*

$$C_1\frac{T}{2\pi}+C_2D^2\frac{T^2}{4\pi}+2DC_2(d+TD)\frac{T}{2\pi}+b\frac{T^2}{4\pi}<1,$$

and

$$|g(t,x_1)-g(t,x_2)|\le b|x_1-x_2|,\quad \text{for all } t,x_1,x_2\in R,$$

then Eq. (1.1) *has at most one* T*-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of Eq. (1.1). Set $Z(t) = x_1(t) - x_2(t)$, we obtain

$$\begin{aligned} &Z''(t) + (f_1(t)x_1'(t) - f_1(t)x_2'(t)) + (f_2(x_1(t))(x_1'(t))^2 - f_2(x_2(t))(x_2'(t))^2) \\ &\quad + (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t)))) = 0. \end{aligned} \tag{2.12}$$

Set

$$Z(\bar{t}_1) = \max_{t\in R} Z(t), \qquad Z(\bar{t}_2) = \min_{t\in R} Z(t), \quad \text{where } \bar{t}_1, \bar{t}_2 \in R.$$

Then, we have

$$Z'(\bar{t}_1) = x_1'(\bar{t}_1) - x_2'(\bar{t}_1) = 0, \qquad Z''(\bar{t}_1) \le 0, \tag{2.13}$$

and

$$Z'(\bar{t}_2) = x_1'(\bar{t}_2) - x_2'(\bar{t}_2) = 0, \qquad Z''(\bar{t}_2) \ge 0. \tag{2.14}$$

Now, we prove that there exists a constant $\bar{\xi} \in R$ such that

$$Z(\bar{\xi}) = 0. \tag{2.15}$$

Contrarily, one of the following cases occurs:
(a) $Z(t) = x_1(t) - x_2(t) > 0$, for all $t \in R$.
(b) $Z(t) = x_1(t) - x_2(t) < 0$, for all $t \in R$.
If (a) holds, in view of (A_3), (2.12) and $x_1'(\bar{t}_1) = x_2'(\bar{t}_1)$, we get

$$\begin{aligned} Z''(\bar{t}_1) &= -(f_1(\bar{t}_1)x_1'(\bar{t}_1) - f_1(\bar{t}_1)x_2'(\bar{t}_1)) - (f_2(x_1(\bar{t}_1))(x_1'(\bar{t}_1))^2 \\ &\quad - f_2(x_2(\bar{t}_1))(x_2'(\bar{t}_1))^2) - (g(\bar{t}_1, x_1(\bar{t}_1 - \tau(\bar{t}_1))) - g(\bar{t}_1, x_2(\bar{t}_1 - \tau(\bar{t}_1)))) \\ &= -(x_1'(\bar{t}_1))^2(f_2(x_1(\bar{t}_1)) - f_2(x_2(\bar{t}_1))) - (g(\bar{t}_1, x_1(\bar{t}_1 - \tau(\bar{t}_1))) - g(\bar{t}_1, x_2(\bar{t}_1 - \tau(\bar{t}_1)))) \\ &\quad > 0, \end{aligned} \tag{2.16}$$

which contradicts (2.13). So we have that (2.15) is true.
If (b) holds, in view of (A_3), (2.12) and $x_1'(\bar{t}_2) = x_2'(\bar{t}_2)$, we obtain

$$\begin{aligned} Z''(\bar{t}_2) &= -(f_1(\bar{t}_2)x_1'(\bar{t}_2) - f_1(\bar{t}_2)x_2'(\bar{t}_2)) - (f_2(x_1(\bar{t}_2))(x_1'(\bar{t}_2))^2 \\ &\quad - f_2(x_2(\bar{t}_2))(x_2'(\bar{t}_2))^2) - (g(\bar{t}_2, x_1(\bar{t}_2 - \tau(\bar{t}_2))) - g(\bar{t}_2, x_2(\bar{t}_2 - \tau(\bar{t}_2)))) \\ &= -(x_1'(\bar{t}_2))^2(f_2(x_1(\bar{t}_2)) - f_2(x_2(\bar{t}_2))) - (g(\bar{t}_2, x_1(\bar{t}_2 - \tau(\bar{t}_2))) - g(\bar{t}_2, x_2(\bar{t}_2 - \tau(\bar{t}_2)))) \\ &\quad < 0, \end{aligned} \tag{2.17}$$

which contradicts (2.14). Thus, (2.15) is true.
Let $\bar{\xi} = nT + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0, T]$ and n be an integer. Then,

$$Z(\widetilde{\gamma}) = 0,$$

by arguments similar to those used in the proof of (2.9), we obtain

$$|Z|_\infty \le \frac{1}{2}\sqrt{T}|Z'|_2. \tag{2.18}$$

Multiplying (2.12) by $Z''(t)$ and integrating it from 0 to T, from (A_1), (A_3), (2.5), (2.8) and (2.18) and Schwarz inequality, we get

$$\begin{aligned} |Z''|_2^2 &= -\int_0^T (f_1(t)x_1'(t) - f(t)x_2'(t))Z''(t)\mathrm{d}t - \int_0^T (f_2(x_1(t))(x_1'(t))^2 \\ &\quad - f_2(x_2(t))(x_2'(t))^2)Z''(t)\mathrm{d}t - \int_0^T (g(t, x_1(t-\tau(t))) - g(t, x_2(t-\tau(t))))Z''(t)\mathrm{d}t \\ &\le \int_0^T |f_1(t)||x_1'(t) - x_2'(t)||Z''(t)|\mathrm{d}t + \int_0^T |f_2(x_1(t))||(x_1'(t))^2 - (x_2'(t))^2||Z''(t)|\mathrm{d}t \\ &\quad + \int_0^T |f_2(x_1(t)) - f_2(x_2(t))|(x_2'(t))^2|Z''(t)|\mathrm{d}t + b\int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z''(t)|\mathrm{d}t \\ &\le \int_0^T C_1|x_1'(t) - x_2'(t)||Z''(t)|\mathrm{d}t + C_2|x_1|_\infty \int_0^T |x_1'(t) + x_2'(t)||x_1'(t) - x_2'(t)||Z''(t)|\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T C_2|x_1(t)-x_2(t)|(x_2'(t))^2|Z''(t)|\mathrm{d}t + b\int_0^T |x_1(t-\tau(t))-x_2(t-\tau(t))||Z''(t)|\mathrm{d}t\\
&\le C_1|Z'|_2|Z''|_2 + |x_1|_\infty 2DC_2|Z'|_2|Z''|_2 + C_2D^2|Z|_\infty\sqrt{T}|Z''|_2 + b|Z|_\infty\sqrt{T}|Z''|_2\\
&\le \left(C_1\frac{T}{2\pi} + C_2D^2\frac{T^2}{4\pi} + b\frac{T^2}{4\pi}\right)|Z''|_2^2 + |x_1|_\infty 2DC_2\frac{T}{2\pi}|Z''|_2^2.
\end{aligned}
\tag{2.19}
$$

In view of (2.7), we have

$$
|x|_\infty \le d + \frac{1}{2}\int_0^T |x'(s)|\mathrm{d}s \le d + T|x'|_\infty \le d + TD,
$$

which, together with (2.19), implies that

$$
|Z''|_2^2 \le \left[C_1\frac{T}{2\pi} + C_2D^2\frac{T^2}{4\pi} + 2DC_2(d+TD)\frac{T}{2\pi} + b\frac{T^2}{4\pi}\right]|Z''|_2^2.
\tag{2.20}
$$

Since $Z(t), Z'(t)$ and $Z''(t)$ are T-periodic and continuous functions, in view of (A_3), (2.15) and (2.19), we have

$$
Z(t) \equiv Z'(t) \equiv Z''(t) \equiv 0, \quad \text{for all } t \in R.
$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in R$. Therefore, Eq. (1.1) has at most one T-periodic solution. The proof of Lemma 2.5 is now completed. □

3. Main results

Theorem 3.1. *Suppose that (A_1) and (A_3) hold, then Eq. (1.1) has a unique T-periodic solution.*

Proof. By Lemma 2.5, it is easy to see that Eq. (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of Eq. (2.4) is bounded.

Let $x(t)$ be a T-periodic solution of Eq. (2.4). Multiplying $x''(t)$ and Eq. (2.4) and then integrating it from 0 to T, in view of (A_1), (A_3), (2.5) and (2.6) and the inequality of Schwarz, we have

$$
\begin{aligned}
|x''|_2^2 &= -\lambda\int_0^T f_1(t)x'(t)x''(t)\mathrm{d}t - \lambda\int_0^T f_2(x(t))(x'(t))^2x''(t)\mathrm{d}t - \lambda\int_0^T g(t,x(t-\tau(t)))x''(t)\mathrm{d}t + \lambda\int_0^T p(t)x''(t)\mathrm{d}t\\
&= -\lambda\int_0^T f_1(t)x'(t)x''(t)\mathrm{d}t + \lambda\frac{1}{3}\int_0^T f_2'(x(t))(x'(t))^4\mathrm{d}t - \lambda\int_0^T g(t,x(t-\tau(t)))x''(t)\mathrm{d}t + \lambda\int_0^T p(t)x''(t)\mathrm{d}t\\
&\le C_1\frac{T}{2\pi}|x''|_2^2 + \int_0^T [|g(t,x(t-\tau(t))) - g(t,0)| + |g(t,0)|]\cdot|x''(t)|\mathrm{d}t + \int_0^T |p(t)|\cdot|x''(t)|\mathrm{d}t\\
&\le \left(C_1\frac{T}{2\pi} + b\frac{T^2}{4\pi}\right)|x''|_2^2 + [bd + \max\{|g(t,0)| : 0 \le t \le T\} + |p|_\infty]\sqrt{T}|x''|_2,
\end{aligned}
\tag{3.1}
$$

which, together with (A_3), implies that there exist positive constants D_1 and D_2 such that

$$
|x''|_2 < D_1,
\tag{3.2}
$$

and

$$
|x'|_2 < D_2, \quad |x|_\infty < D_2.
\tag{3.3}
$$

Since $x(0) = x(T)$, there exists a constant $\bar{\zeta} \in [0, T]$ such that

$$
x'(\bar{\zeta}) = 0,
$$

and

$$
|x'(t)| = |x'(\bar{\zeta}) + \int_{\bar{\zeta}}^t x''(s)\mathrm{d}s| \le \sqrt{T}|x''|_2 < \sqrt{T}D_1, \quad \text{for all } t \in [0,\ T].
\tag{3.4}
$$

Therefore, in view of (3.3) and (3.4), there exists a positive constant $M_1 > \sqrt{T}D_1 + D_2$ such that

$$
\|x\|_X = \max\{|x|_\infty, |x'|_\infty\} \le |x|_\infty + |x'|_\infty < M_1.
$$

If $x \in \Omega_1 = \{x | x \in \operatorname{Ker} L \cap X, \ \text{ and } Nx \in \operatorname{Im} L\}$, then there exists a constant M_2 such that

$$x(t) \equiv M_2, \quad \text{and} \quad \int_0^T [g(t, M_2) - p(t)]\mathrm{d}t = 0. \tag{3.5}$$

Thus,

$$|x(t)| \equiv |M_2| < d, \quad \text{for all } x(t) \in \Omega_1. \tag{3.6}$$

Let

$$M = M_1 + d + 1, \qquad \Omega = \{x | x \in X, |x|_\infty < M, |x'|_\infty < M\}.$$

It is easy to see from (2.2) and (2.3) that N is L-compact on $\overline{\Omega}$. We have from (3.5) and (3.6) and the fact $M > \max\{M_1, \ d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define a continuous function $H(x, \mu)$ by setting

$$H(x, \mu) = (1 - \mu)x - \mu \cdot \frac{1}{T}\int_0^T [g(t, x) - p(t)]\mathrm{d}t, \quad \mu \in [0\ 1].$$

It follows from (A_1) that

$$xH(x, \mu) \neq 0, \quad \text{for all } x \in \partial\Omega \cap \operatorname{Ker} L.$$

Hence, using the homotopy invariance theorem, we have

$$\begin{aligned}\deg\{QN,\ \Omega \cap \operatorname{Ker} L,\ 0\} &= \deg\left\{-\frac{1}{T}\int_0^T [g(t, x) - p(t)]\mathrm{d}t, \Omega \cap \operatorname{Ker} L,\ 0\right\}\\ &= \deg\{x,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0.\end{aligned}$$

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved. □

Remark 3.1. In the results of [9–12], we have found certain errors as following:

(1) In (2.4) of Liu [9], $|x|_2$ should be replaced by $|x|_\infty$.
(2) In (16) and (35) of Liu [10], $|x|_2$ should be replaced by $|x|_\infty$. Moreover, the same errors should be corrected in the proofs whenever necessary.
(3) In (2.14) of Zhou [11], $|x|_2$ should be replaced by $|x|_\infty$.
(4) In lines 17–18 of page 3 in Gao [12],

$$y_1(t^*) - y_2(t^*) > 0$$

does not imply

$$y_1(t^* - \tau(t^*)) - y_2(t^* - \tau(t^*)) > 0.$$

Thus, the authors of [12] can not show $v''(t^*) > 0$. So, Theorem 3.1 in [12] is incorrect.

In the proof of Theorem 3.1, one can find that the above-mentioned major mistakes have been corrected. This implies that Theorem 3.1 improved some results of the literature to a certain extent.

4. An example

Example 4.1. Let $g(t, x) = -\frac{2}{6\pi}x$, for all $t, x \in R$. Then the Liénard equation

$$x''(t) + \frac{1}{8}(\sin 4t)x'(t) - \frac{1}{8}(\arctan x(t))(x'(t))^2 + g(t, x(t - \sin^2 t)) = \frac{1}{6\pi}\mathrm{e}^{-\cos^2 t} \tag{4.1}$$

has a unique $\frac{\pi}{2}$-periodic solution.

Proof. By (4.1), we have $d = 1,\ b = \frac{2}{6\pi}, C_1 = C_2 = \frac{1}{8}, \tau(t) = \sin^2 t,\ T = \frac{\pi}{2}$ and $p(t) = \frac{1}{6\pi}\mathrm{e}^{-\cos^2 t}$, then

$$\frac{1}{2}\frac{[bd + \max\{|g(t, 0)| : 0 \le t \le T\} + |p|_\infty]T}{1 - (C_2\frac{T}{2\pi} + b\frac{T^2}{4\pi})} := D = \frac{1}{2}\frac{[\frac{2}{6\pi} + \frac{1}{6\pi}] \times \frac{\pi}{2}}{1 - \frac{1}{32} - \frac{1}{48}} = \frac{12}{91},$$

$$\begin{aligned}&C_1\frac{T}{2\pi} + C_2D^2\frac{T^2}{4\pi} + 2DC_2(d + TD)\frac{T}{2\pi} + b\frac{T^2}{4\pi}\\ &\quad = \frac{1}{32} + \frac{1}{8}\cdot\left(\frac{12}{91}\right)^2\cdot\frac{\pi}{16} + 2\cdot\frac{12}{91}\cdot\frac{1}{8}\cdot\left(1 + \frac{\pi}{2}\cdot\frac{12}{91}\right)\cdot\frac{1}{4} + \frac{1}{48} < 1.\end{aligned}$$

It is obvious that the assumptions (A_1) and (A_3) hold. Hence, by Theorem 3.1, Eq. (4.1) has a unique $\frac{\pi}{2}$-periodic solution. □

Remark 4.1. Eq. (4.1) is a very simple version of Liénard equation. Since $f_2(x) = \frac{1}{8}(\arctan x)$, all the results in [5–12,14,15] and the references therein can not be applicable to Eq. (4.1) to obtain the existence and uniqueness of $\frac{\pi}{2}$-periodic solutions. This implies that the results of this paper are essentially new.

Acknowledgement

The author would like to express his sincere appreciation to the reviewer for his/her helpful comments in improving the presentation and quality of the paper.

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