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Laguerre polynomials as Jensen polynomials of Laguerre–Pólya entire functions

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ABSTRACT

We prove that the only Jensen polynomials associated with an entire function in the Laguerre–Pólya class that are orthogonal are the Laguerre polynomials.

1. Introduction

A real entire function $\psi(x)$ is said to belong to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty}(1+x/x_k)e^{-x/x_k},$$

where $c, \beta, x_k \in \mathbb{R}, \alpha \geq 0, m \in \mathbb{N} \cup \{0\}, \sum x_k^{-2} < \infty$. Similarly, the real entire function $\phi(x)$ is said to be of type I in the Laguerre–Pólya class, written $\varphi \in \mathcal{LPI}$, if $\phi(x)$ or $\phi(-x)$ can be represented in the form

$$\phi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\infty}(1+x/x_k),$$

with $c \in \mathbb{R}, \sigma \geq 0, m \in \mathbb{N} \cup \{0\}, x_k > 0, \sum 1/x_k < \infty$. The class $\mathcal{LP}$ consists of entire functions which are uniform limits on the compact sets of the complex plane of polynomials with only real zeros and the functions in $\mathcal{LPI}$ are locally uniform limits of polynomials whose zeros are real and are either all positive, or all negative.

The classes $\mathcal{LP}$ and $\mathcal{LPI}$ were first studied in [1] in the nineteenth century and in the beginning of the twentieth century. This topic attracted the interest of great masters of the Classical Analysis, such as Hurwitz, Jensen, Pólya, Schur, Obrechkoff, Tchakaloff, de Bruijn, Schoenberg, Edrei (see [2–8] and the references cited therein). The main reason for this interest was the fact that they are closely related to the Riemann Hypothesis (see [3,7,9]). Unfortunately, the expectations that the hypothesis could be attacked by the theory on $\mathcal{LP}$ developed by these celebrated mathematicians could not be justified for many reasons. Nevertheless, the interest in the theory and applications of the Laguerre–Pólya functions is still alive.

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In this paper we provide a very short and beautiful proof of the fact that the zeros of the Bessel function

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\alpha+1)} \frac{(z/2)^{2k}}{k!}$$

are all real when $\alpha > -1$. In order to prove this we use a fundamental fact that relates a function in $\mathcal{LP}$ and the so-called Jensen polynomials. It turns out that the Jensen polynomials associated with the Bessel function, properly normalized, are exactly the Laguerre polynomials. Thus, the natural question arises if there exist other entire special functions whose Jensen polynomials are orthogonal. We show that, unfortunately, this is an exceptional case, proving that the unique orthogonal polynomials that are simultaneously Jensen ones, are the Laguerre polynomials. One of the four simple proofs we provide is based on a result of Chihara on orthogonal polynomials with Brenke generating functions. At the end of the paper we state an interesting open question about the so-called Brenke polynomials.

2. Jensen polynomials

Since the class $\mathcal{LP}$ is the complement, in the sense of the local uniform convergence, of polynomials with only real zeros, it is of interest whether, given an entire function $\varphi(x)$, represented in its Maclaurin series

$$\varphi(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}, \tag{2.1}$$

there exist sequences of polynomials which possess real zeros only and converge locally uniformly to $\varphi(z)$. The first natural candidates are the nth partial sums because they do converge to φ. Nevertheless, it is quite easy to find functions in the Laguerre–Pólya class whose partial sums have nonreal zeros. As an example, we mention the exponential function and its partial sums $S_n(z) = \sum_{k=0}^{n} z^k/k!$. It is known that S_n have either one or no real zeros depending on the parity of n (see Problem 74, Chapter 4 in [10]).

It turns out that there is a universal sequence of real polynomials which can be constructed immediately from the Maclaurin series of an entire function, such that every polynomial from the sequence possesses only real zeros whenever the entire function is in the Laguerre–Pólya class. With every entire function $\varphi(x)$ with Maclaurin expansion (2.1) we associate the corresponding Jensen polynomials

$$g_n(\varphi; x) = g_n(x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j.$$

Jensen himself established the following fundamental fact in [11]:

Theorem A. *The function $\varphi(x)$ belongs to $\mathcal{LP}$ if and only if all the polynomials $g_n(\varphi; x)$, $n = 1, 2, \ldots,$ have only real zeros. Moreover, the sequence $g_n(\varphi; z/n)$ converges locally uniformly to $\varphi(z)$.*

We refer the reader to [12,2,13] as well as to Problem 162, Chapter 4 in [10] for proofs of Jensen's theorem. Thus, if the zeros of all $g_n(\varphi; z)$ are real, then $\varphi \in \mathcal{LP}$ and vice versa. Pólya and Schur [14] showed that the class $\mathcal{LP}$ is closed under differentiation. It follows immediately from the fact that

$$g_{n,k}(\varphi; x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_{k+j} x^j, \quad n = 0, 1, \ldots$$

are the Jensen polynomials associated with $\varphi^{(k)}(x)$, that is $g_{n,k}(\varphi; x) = g_n(\varphi^{(k)}; x)$, and they are constant multiples of the polynomials $g_n^{(k)}(x)$,

$$g_n^{(k)}(\varphi; x) = \frac{n!}{(n-k)!}\, g_{n-k,k}(\varphi; x),$$

which can be rewritten as $g_n^{(k)}(\varphi; x) = (n!/(n-k)!) g_{n-k}(\varphi^{(k)}; x)$. This immediately yields the result of Pólya and Schur.

Another sequence of polynomials related to $\varphi(x)$ consists of the reciprocal of the Jensen polynomials,

$$A_n(\varphi; x) = x^n g_n(1/x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^{n-j}.$$

The polynomials $A_n(\varphi; x)$ are called Appell polynomials associated with $\varphi(x)$. Observe that, generally, a sequence of polynomials $A_n(x)$ is called an Appell one if there are nonzero constants d_n such that $A_n'(x) = d_n A_{n-1}(x)$ for every $n \in \mathbb{N}$. Obviously $A_n(\varphi; x)$ satisfy this property.

We recall the following characteristic properties of Jensen polynomials:

Lemma 1. *Let (2.1) be a formal power series with $\gamma_k \neq 0$ for all $k = 0, 1, \ldots$ and let $g_n(x) = g_n(\varphi; x)$ be its associated Jensen polynomials. Then the following statements are equivalent:*

(i) *The polynomials $g_n(x)$, $n \geq 0$, are generated by*

$$e^t \varphi(xt) = \sum_{n=0}^{\infty} g_n(x) \frac{t^n}{n!}; \tag{2.2}$$

(ii) *The sequence $\{A_n(x)\}_{n\geq 0}$ defined by $A_n(x) = x^n g_n(1/x)$ is an Appell one;*

(iii) *The polynomials $g_n(x)$ satisfy the differential–recurrence relation*

$$x g_n'(x) = n g_n(x) - n g_{n-1}(x), \quad n \geq 1; \tag{2.3}$$

(iv) *The polynomials $g_n(x)$, $n \geq 0$, possess a multiplication formula of the form*

$$g_n(xy) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} g_k(x). \tag{2.4}$$

The property (i) was stated in [2, Proposition 1]. It may also be derived by the use of the rearrangement technique for obtaining generating functions. It was shown in [15, Lemma 2.1] that these four properties are equivalent.

3. Reality of zeros of $J_\alpha(z)$

In what follows we use Theorem A to prove that all the zeros of the Bessel function $J_\alpha(z)$ are all real when $\alpha > -1$. In order to do this, we shall show that $(z/2)^{-\alpha} J_\alpha(z)$ belongs to $\mathcal{LP}$ and since the latter function is an even one, it suffices to prove that $\widetilde{J}_\alpha(z) := z^{-\alpha/4} J_\alpha(2\sqrt{z})$,

$$\widetilde{J}_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\alpha+1)} \frac{z^k}{k!},$$

is in $\mathcal{LP}I$. Let us write the Jensen polynomials of $\widetilde{J}_\alpha(z)$,

$$g_n(\widetilde{J}_\alpha; z) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\Gamma(k+\alpha+1)} z^k.$$

Recall that the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{n} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_k$ is the Pochhammer symbol given by $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k \geq 1$. Then the functional relation $\Gamma(z+1) = z\Gamma(z)$ for the Gamma function and straightforward manipulation with the binomial coefficients yield

$$g_n(\widetilde{J}_\alpha; z) = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} z^k = \frac{1}{\Gamma(\alpha+1)} {}_1F_1(-n; \alpha+1; z).$$

On the other hand the Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x).$$

Since all the zeros of $L_n^{(\alpha)}(x)$ are real for every $n \in \mathbb{N}$, when $\alpha > -1$, then $\widetilde{J}_\alpha(z)$ belongs to the Laguerre–Pólya class and its zeros are real for these values of the parameter α. Moreover the fact that $g_n(\varphi; z/n)$ converge locally uniformly to $\varphi(z)$ immediately implies that

$$\frac{n!}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}\left(\frac{x}{n}\right) \longrightarrow x^{-\alpha/4} J_\alpha(2\sqrt{x}) \quad \text{as } n \to \infty,$$

uniformly on every compact subset of the complex plane.

4. The only Jensen polynomials that are orthogonal are the Laguerre polynomials

Having in mind the beauty of the above proof, it is quite challenging to know if there are other entire functions, essentially different from the Bessel one, whose Jensen polynomials form an orthogonal sequence. We prove that the answer of this natural question is, unfortunately, negative. We state the following:

Theorem 1. *The only Jensen polynomials that are orthogonal are the Laguerre polynomials.*

Such characterization takes into account the fact that polynomial sets which are obtainable from one another by a linear change of variable are assumed equivalent.

Next, we use the four equivalent properties given in Lemma 1 that characterize Jensen polynomials to provide four straightforward proofs for Theorem 1.

Proof 1. Polynomial sequences $\{B_n(x)\}_{n\geq 0}$, having generating functions of the form

$$B(t)\varphi(xt) = \sum_{n=0}^{\infty} B_n(x)t^n, \tag{4.1}$$

where $B(t) = \sum_{n=0}^{\infty} b_n t^n$, $b_0 \neq 0$, were first studied in [16]. Chihara [17] determined explicitly all orthogonal polynomial sets generated by (4.1). He found eight families. The only one corresponding to $B(t) = \mathrm{e}^t$ is the Laguerre polynomial set.

Proof 2. Toscano [18] proved that $\{P_n(x)\}_{n\geq 0}$ is an orthogonal polynomial set and $\{x^n P_n(1/x)\}_{n\geq 0}$ is an Appell polynomial sequence if and only if $P_n(x)$ coincide with the Laguerre polynomials.

Proof 3. Al-Salam and Chihara [19] proved that the classical orthogonal polynomials of Jacobi, Laguerre and Hermite are characterized as the only orthogonal polynomials with a differentiation formula of the form

$$\pi(x)P_n'(x) = (\alpha_n x + \beta_n)\,P_n(x) + \delta_n P_{n-1}(x),$$

where $\pi(x) = ax^2 + bx + c$. They showed in particular that the only orthogonal polynomials satisfying (2.3) are the Laguerre ones.

Proof 4. Feldheim [20] proved that the only orthogonal polynomials which satisfy the multiplication formula (2.4) are those of Laguerre.

5. Concluding remarks

Remark 1. The Brenke polynomials $\{B_n(x)\}_{n\geq 0}$ generated by (4.1) are simple in structure since

$$B_n(B, \varphi; x) := B_n(x) = \sum_{k=0}^{n} b_{n-k}\frac{\gamma_k}{k!}x^k, \quad n = 0, 1, 2, \ldots.$$

The leading coefficient of $B_n(x)$ is $b_0\gamma_n/n!$ so $B_n(x)$ is of exact degree n. Observe that the Jensen polynomials are special case of Brenke's polynomials corresponding to $b_n = 1/n!$, that is, to $B(t) = \mathrm{e}^t$. Here a natural question arises: Do there exist other sequences $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ for which the sequence $\{B_n(B, \varphi; c_n x)\}_{n\geq 0}$ converges locally uniformly to $\varphi(x)$? Notice that for Jensen polynomials $c_n = 1/n$.

Remark 2. Chihara [17] determined the eight couples (B, φ) for which the corresponding polynomials defined by (4.1) are orthogonal. In this paper we show that, when $B(t) = \mathrm{e}^t$ and φ is the properly normalized Bessel function, the zeros of the latter are real. The question is then: What about the other formal power series given in [17]? Do they define entire functions in the Laguerre–Pólya class? In view of Hurwitz's theorem on zeros of local uniform limits of analytic functions, it is sufficient to find, among the orthogonal sets characterized by Chihara, the sequences of Brenke polynomials $\{B_n(B, \varphi; c_n x)\}_{n\geq 0}$ which converge locally uniformly to $\varphi(x)$ for a suitable $\{c_n\}_{n\geq 0}$.

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